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CATEGORICAL, FUNCTORIAL AND ALGEBRAIC ASPECTS OF THE TYPE-FREE LAMBDA CALCULUS

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0. Introduction

From the set-theoretical point of view, the type-free λ-calculus initiated by A. Church around 1930 may be labelled as a deductive system fit for examining sets with the property

$$A = A^A. \tag{1}$$

Unfortunately, (1) is satisfied only if A is a one-element set and if we agree to identify the unique element in A with the unique function $A \to A$. The problem of finding non-trivial models of the type-free λ-calculus turned out to be difficult and was solved by D. Scott in 1969. Even the question what should be meant by a "model" of the type-free λ-calculus requires some consideration. In this paper we shall outline a certain new approach to the syntax and the semantics of the type-free λ-calculus. In Sections 2 and 3 some modifications of the classical syntax of the type-free λ-calculus are described. In Sections 4, 5 and 6 we "categorize" the syntax of the type-free λ-calculus: we construct some categories from λ-terms and we introduce the concept of a Church algebraic theory. In Section 7 "models" in the style of Wadsworth [5] are discussed; we give a new characterization of these "models", which is independent of the syntax of the type-free λ-calculus. In Section 8 a method of "functorializing" the semantics of the type-free λ-calculus is described; "models" of the type-free λ-calculus are identified with certain functors defined on Church algebraic theories. In Sections 9, 10, the new concepts of a hyperalgebra and hyperhomomorphism are introduced and discussed; it is shown that certain "models" of the type-free λ-calculus can be treated as hyperalgebras.

1. Preliminaries

1.1. We shall use the following symbols:

$?$ is the symbol of a variable,

$\boldsymbol{N}$ is the set of all non-negative integers $\{0, 1, 2, \ldots\}$,

$\boldsymbol{N}^+$ is the set of all positive integers $\{1, 2, 3, \ldots\}$,

$\boldsymbol{n}$ is the set $\{1, 2, \ldots, n\}$, and $\boldsymbol{n+1}$ is the set $\{1, 2, \ldots, n+1\}$.

If A is a set, then $\operatorname{card} A$ is the cardinal number of A. By a family $(a_t \colon t \in T)$ of elements of A we mean the function $t \mapsto a_t$ from the set T into the set A. If $T = \{*\}$ is a one-element set, then we shall identify $(a_t \colon t \in T)$ with a_*. If A and B are sets, then the set of all functions from B to A will be denoted by A^B. If f is a function from B to C, then

$$A^f\colon A^C \to A^B, \qquad f^A\colon B^A \to C^A$$

are functions defined by

$$A^f(g\colon C \to A) = g\circ f, \qquad f^A(h\colon A \to B) = f\circ h,$$

respectively, where $\circ$ is the composition of functions. To avoid superfluous notational complications we shall identify the sets A^n, where n may be considered as a von Neumann number, with sets defined inductively as follows:

$$A^0 = \{0\}, \qquad A^1 = A, \qquad A^{n+1} = A^n \times A \qquad \text{for} \qquad n \in \boldsymbol{N}^+.$$

If f is a function from A to B, then $f^n\colon A^n \to B^n$ may be identified with the function given by

$$f^n(a_1, \ldots, a_n) = \big(f(a_1), \ldots, f(a_n)\big);$$

in particular, $f^0\colon A^0 \to B^0$ is given by $f^0(0) = 0$.

The symbol $!^n(A)$ $(n \in \boldsymbol{N})$ will denote the constant function from A^n to $A^0 = \{0\}$.

The symbol $\mathrm{pr}_i^n(A)$ $(n \in \boldsymbol{N}^+, i \in \boldsymbol{n})$ will denote the ith projection from A^n onto A: if $(a_1, \ldots, a_n) \in A^n$, then $\mathrm{pr}_i^n(A)(a_1, \ldots, a_n) = a_i$, in particular, $\mathrm{pr}_1^1(A) = \mathrm{id}_A$ is the identity function on A. If $(f_i\colon A^m \to A \colon i \in \boldsymbol{n})$ is a family of functions, then $\langle f_i\colon i \in \boldsymbol{n}\rangle$ will denote the function from A^m into A^n defined as

$$\langle f_i\colon i \in \boldsymbol{n}\rangle(x) = \big(f_1(x), \ldots, f_n(x)\big) \qquad \text{for all } x \in A^m.$$

If A is a set, then $\lambda_A[?]$ will denote the mapping assigning to any function $f\colon B\times A \to A$ the function $\lambda_A[f] = g\colon B \to A^A$ defined by

$$g(b)(a) = f(b, a) \qquad \text{for all } b \in B \text{ and all } a \in A,$$

and to any function $f\colon A \to A$ the function $\lambda_A[f] = g\colon A^0 \to A^A$ defined by $g(0) = f$.

1.2. Let $f\colon A \to B$ and $g\colon C \to D$ be functions. We shall say that the *composition* $g\circ f$ *is defined* iff the set of values of f is contained in the domain of g; if that is the case, then $g\circ f$ is a function from A to D. Let us note that, contrary to our convention, the composition $g\circ f$ is usually considered only in the case of $B = C$.

1.3. For all unexplained terms concerning category theory we refer the reader to MacLane [3]. If K is a category, then $\operatorname{Ob} K$ will denote the class of all objects of K, and $\operatorname{Ar} K$ will denote the class of all arrows of K. If $f\colon A \to B$ is an arrow, then $\mathrm{dom}(f)$ will denote its domain A, and $\mathrm{cod}(f)$ will denote its codomain B. If $A, B \in \operatorname{Ob} K$, then $K(A, B)$ will denote the set ("hom-set") of all arrows with domain A and codomain B. The composition of arrows $f\colon A \to B$ and $g\colon B \to C$ will be dnoted by $gf\colon A \to C$ or sometimes by $g\cdot f\colon A \to C$. The opposite category of K will be denoted by K^{op}.

The category of sets and functions is denoted by Set. The symbol

$$K(?_1, ?_2)\colon K^{\mathrm{op}} \times K \to \mathrm{Set}$$

denotes the hom-bifunctor.

If $(f_i\colon A \to B_i \colon i \in \boldsymbol{n})$ is a family of arrows in K and B is a categorical product $B_1 \times \ldots \times B_n$ with product projections $\mathrm{pr}_i\colon B \to B_i$, then the symbol $\langle f_i\colon i \in \boldsymbol{n}\rangle$ or $\langle f_1, \ldots, f_n\rangle$ will denote the unique arrow $h\colon A \to B$ such that $\mathrm{pr}_i\cdot h = f_i$ for all $i \in \boldsymbol{n}$.

1.4. A *congruence* on a category K is an equivalence relation R on $\operatorname{Ar} K$ satisfying the following conditions:

(c_1) if $fR\,f'$, then $\mathrm{dom}(f) = \mathrm{dom}(f')$ and $\mathrm{cod}(f) = \mathrm{cod}(f')$,

(c_2) if $f, f' \in K(A, B)$, $g, g' \in K(B, C)$, fRf' and gRg', then $gfRg'f'$.

If R is a congruence on K, then the quotient category K/R has the same objects as K and $(K/R)(A, B) = K(A, B)/R_{A,B}$, where $R_{A,B}$ is the restriction of R to $K(A, B)$; it follows from (c_2) that the composition of arrows in K induces the composition of arrows in K/R.

1.5. An *algebraic theory* (cf. Lawvere [2]) is a triple $\boldsymbol{T} = (T, [?], P)$ such that

(a_1) T is a category, $[?]$ is a bijection $n \mapsto [n]$ with domain $\boldsymbol{N}$ and codomain $\operatorname{Ob} T$, and $P = (\mathrm{pr}_i^n\colon n \in \boldsymbol{N}^+, i \in \boldsymbol{n})$ is a family of arrows of T,

(a_2) $\mathrm{pr}_i^n \in \boldsymbol{T}([n], [1])$ for all $n \in \boldsymbol{N}^+$ and all $i \in \boldsymbol{n}$,

(a_3) the object $[n]$ is the product of n copies of $[1]$ for all $n \in \boldsymbol{N}$, and $(\mathrm{pr}_i^n\colon i \in \boldsymbol{n})$ is a family of product projections for all $n \in \boldsymbol{N}^+$.

It follows from (a_3) that the object $[0]$ is the product of the empty family of objects, i.e. $[0]$ is a terminal object in T; in other words, for any $n \in \boldsymbol{N}$ there is a unique arrow in T from $[n]$ to $[0]$. This arrow will be denoted by $!^n\colon [n] \to [0]$.

An algebraic theory $\boldsymbol{T}$ can be considered as a category with selected products $[m]\times[n] = [m+n]$ and selected product projections

$$\mathrm{pr}_1^{[m]\times[n]}\colon [m+n]\to[m],$$
$$\mathrm{pr}_2^{[m]\times[n]}\colon [m+n]\to[n]$$

defined by

$$\mathrm{pr}_1^{[m]\times[n]} = \begin{cases}\langle \mathrm{pr}_i^{m+n}\colon i\in \boldsymbol{m}\rangle & \text{for } m\in \boldsymbol{N}^+,\ n\in\boldsymbol{N},\\ !^n & \text{for } m = 0,\ n\in\boldsymbol{N};\end{cases}$$
$$\mathrm{pr}_2^{[m]\times[n]} = \begin{cases}\langle \mathrm{pr}_{m+i}^{m+n}\colon i\in \boldsymbol{n}\rangle & \text{for } n\in \boldsymbol{N}^+,\ m\in\boldsymbol{N},\\ !^m & \text{for } n = 0,\ m\in\boldsymbol{N}.\end{cases}$$

In the sequel we shall use the shorter notation

$$\mathrm{pr}_1^{[m]\times[n]} = \mathrm{pr}_{m,n}, \qquad \mathrm{pr}_2^{[m]\times[n]} = \mathrm{pr}^{m,n}.$$

The selected products $[m]\times[n]$ give rise to the product-bifunctor $?_1\times ?_2\colon T\times T\to T$, which is defined on arrows in the following way: if $f\colon [m]\to[n]$, $g\colon [k]\to[j]$ are arrows of T, then $f\times g\colon [m+k]\to[n+j]$ is the unique arrow of T such that

$$\mathrm{pr}_{n,j}\cdot(f\times g) = f\cdot\mathrm{pr}_{m,k} \quad\text{and}\quad \mathrm{pr}^{n,j}\cdot(f\times g) = g\cdot\mathrm{pr}^{m,k}.$$

In particular, for $g = \mathrm{id}_{[1]}$ we obtain the following formula, which will repeatedly be used in this paper:

$$(0)\qquad f\times\mathrm{id}_{[1]} = \begin{cases}\langle \mathrm{pr}_1^n\cdot f\cdot\mathrm{pr}_{m,1},\ldots,\mathrm{pr}_n^n\cdot f\cdot\mathrm{pr}_{m,1},\mathrm{pr}_{m+1}^{m+1}\rangle\\ \qquad\qquad \text{for}\quad f\in T([m],[n]),\ m\in\boldsymbol{N},\ n\in\boldsymbol{N}^+,\\ \mathrm{pr}_{m+1}^{m+1}\quad\text{for}\quad f = !^m,\ m\in\boldsymbol{N}.\end{cases}$$

1.6. Let $\boldsymbol{T} = (T,[?],P)$ be an algebraic theory. By an *algebraic congruence* on $\boldsymbol{T}$ we shall mean a congruence R on T such that

(c_3) if $f,f'\in T([n],[m])$ and $\mathrm{pr}_i^m f\,R\,\mathrm{pr}_i^m f'$ for all $i\in\boldsymbol{m}$, then fRf'.

It is easy to verify that if R is an algebraic congruence on $\boldsymbol{T}$, then the triple $\boldsymbol{T}/R = (T/R,[?],P/R)$, where

$$P/R = (\{f\colon f\,R\,\mathrm{pr}_i^n\}\colon n\in\boldsymbol{N}^+,\ i\in\boldsymbol{n}),$$

is an algebraic theory. This algebraic theory will be called the *quotient algebraic theory*.

It follows from (c_2) and (c_3) that an algebraic congruence on $\boldsymbol{T}$ is completely determined by its restriction to the set $\bigcup_{n\in N} T([n],[1])$. In fact, if R is an algebraic congruence on $\boldsymbol{T}$ and if $f,f'\in T([n],[m])$, then fRf' iff $\mathrm{pr}_i^m\cdot fR\mathrm{pr}_i^m\cdot f'$ for all $i\in\boldsymbol{m}$. Therefore algebraic congruences on $\boldsymbol{T}$ may be identified with restricted algebraic congruences defined in the following way:

1.7. A *restricted algebraic congruence* on $\boldsymbol{T} = (T,[?],P)$ is an equivalence relation on $\bigcup_{n\in N} T([n],[1])$ satisfying the following conditions:

(c_1') if fRf', then $\mathrm{dom}(f) = \mathrm{dom}(f')$,

(c_2') if $\mathrm{dom}(g) = [m]$, gRg' and f_iRf_i' for all $i\in\boldsymbol{m}$, then

$$g\cdot\langle f_i\colon i\in\boldsymbol{m}\rangle\,R\,g'\cdot\langle f_i'\colon i\in\boldsymbol{m}\rangle.$$

1.8. Let $\boldsymbol{T} = (T,[?],P)$ be an algebraic theory and let K be a category with finite products (including a terminal object). We shall say that a functor $G\colon \boldsymbol{T}\to K$ is a *p-functor* iff $G([0])$ is a terminal object in K and $(G(\mathrm{pr}_i^n)\colon i\in\boldsymbol{n})$ is the family of product projections in K for all $n\in\boldsymbol{N}^+$. It is easy to verify that G is a p-functor iff G preserves finite products.

We shall say that a functor $G\colon \boldsymbol{T}\to\mathrm{Set}$ is an *sp-functor* iff

$$G([n]) = A^n \quad\text{for all } n\in\boldsymbol{N},\ \text{where } A = G([1])$$

and

$$G(\mathrm{pr}_i^n) = \mathrm{pr}_i^n(A) \quad\text{for all } n\in\boldsymbol{N}^+,\ i\in\boldsymbol{n}$$

(i.e. G preserves specified finite products).

1.9. A *cartesian closed category* is a category K equipped with the following adjunctions (cf. MacLane [3], p. 95):

(1) there is a right adjoint functor $1\colon \boldsymbol{1}\to K$ to the unique functor

$$K\to\boldsymbol{1}$$

($\boldsymbol{1}$ is the category with one object and one arrow),

(2) there is a right adjoint functor $?_1\times ?_2\colon K\times K\to K$ to the diagonal functor

$$K\to K\times K$$

(the diagonal functor is given by $f\mapsto(f,f)$);

(3) for each $A\in\mathrm{Ob}K$, there is a right adjoint functor $(?)^A\colon K\to K$ to the functor

$$?\times A\colon K\to K.$$

The functor $(?)^A$ is called an *exponentiation by* A and the counit of the adjunction (3) is denoted by ev_A.

For any $B\in\mathrm{Ob}K$ the arrow $\mathrm{ev}_{A,B}\colon B^A\times A\to B$ is the component of the counit ev_A.

If $f\colon C\times A\to B$ is an arrow in a cartesian closed category K, then $\lambda_{A,B}[f]$ will denote a unique arrow $h\colon C\to B^A$ in K such that the following diagram commutes:

$$\begin{tikzcd} B^A\times A \arrow[r, "\mathrm{ev}_{A,B}"] & B \\ C\times A \arrow[u, "h\times\mathrm{id}_A"] \arrow[ur, "f"'] & \end{tikzcd}$$

Let us note that in the cartesian closed category Set the exponentiation by A is the same as the covariant hom-functor $(?)^A = \mathrm{Set}(A, ?)$.

An example of a cartesian closed category is the category Clatt of all complete lattices and continuous functions (cf. Scott [4]).

2. General lambda-theories

2.1. We shall use λ-terms in a modified form, using different symbols for free and bound variables. Let $(x_i\colon i \in \boldsymbol{N}^+)$ and $(\xi_i\colon i \in \boldsymbol{N}^+)$ be two families such that if $i \neq j$, then $\mathrm{card}\{x_i, x_j, \xi_i, \xi_j\} = 4$ (x_i is a *free variable*, ξ_i is a *bound variable*), and let C be a set of constant symbols (elements of C are different from free and bound variables, C can be empty). The set $\mathrm{Exp}[C]$ of λ-terms (more precisely λ^C-*terms*) is defined by induction as follows:

(i) each constant symbol and each free variable is an element of $\mathrm{Exp}[C]$;

(ii) if M and N are elements of $\mathrm{Exp}[C]$, then $(MN) \in \mathrm{Exp}[C]$;

(iii) if M is an element of $\mathrm{Exp}[C]$ and $j = \min\{k\colon \xi_k$ does not occur in $M\}$, then $\lambda\xi_j\cdot(x_i/\xi_j)M$ is an element of $\mathrm{Exp}[C]$, where $(x_i/\xi_j)M$ is the result of substituting ξ_j for x_i in M, for each free variable x_i.

Remark. The λ-terms defined above are in a one-to-one correspondence with equivalence classes of terms defined in the classical way (cf. Barendregt [1], p. 1096), where we identify terms differing only in the names of their bound variables (see Barendregt [1], p. 1097).

The set $\mathrm{Exp}[\emptyset]$ will also be denoted by Exp. The set $\{x_i\colon i \in \boldsymbol{N}^+\}$ of all free variables will be denoted by V. If M is a λ-term, then $\mathrm{BV}(M)$ will denote the set of all bound variables occurring in M, and $\mathrm{FV}(M)$ will denote the set of all free variables occurring in M.

2.2. We shall use the following notion of simultaneous substitution for λ-terms: if $(x_{j_i}\colon i \in \boldsymbol{n})$ is a family of free variables such that $i \neq k$ implies $x_{j_i} \neq x_{j_k}$, and if $(N_i\colon i \in \boldsymbol{n})$ is a family of λ-terms, then the result of *simultaneous substitution* of N_i for x_{j_i} in a λ-term M is the term, denoted by

$$(*) \qquad [x_{j_i}/N_i\colon i \in \boldsymbol{n}]M \quad \text{or} \quad [x_{j_1}/N_1, \ldots, x_{j_n}/N_n]M,$$

obtained from M by replacing x_{j_i} by N_i for each $i \in \boldsymbol{n}$, with a suitable change of bound variables. More precisely, $(*)$ is defined by induction as follows:

(i) $[x_{j_i}/N_i\colon i \in \boldsymbol{n}]c = c$ for each $c \in C$,

$$[x_{j_i}/N_i\colon i \in \boldsymbol{n}]x_p = \begin{cases} N_k & \text{if } p = j_k, \\ x_p & \text{if } p \text{ is different from } j_1, \ldots, j_n, \end{cases}$$

(ii) $[x_{j_i}/N_i\colon i \in \boldsymbol{n}](MN) = ([x_{j_i}/N_i\colon i \in \boldsymbol{n}]M[x_{j_i}/N_i\colon i \in \boldsymbol{n}]N)$,

(iii[a]) if $p = j_r$ for a certain r in $\boldsymbol{n}$, then

$$[x_{j_i}/N_i\colon i \in \boldsymbol{n}]\lambda\xi_q\cdot(x_p/\xi_q)M = \lambda\xi_k\cdot(x_m/\xi_k)[x_{j_i}/N_i'\colon i \in \boldsymbol{n}]M,$$

where

$$N_i' = \begin{cases} N_i & \text{for } i \neq r, \\ x_m & \text{for } i = r \end{cases}$$

and

$$m = \min\{i\colon x_i \notin \mathrm{FV}(M) \cup \bigcup_{j \neq r} \mathrm{FV}(N_j)\},$$

$$k = \min\{j\colon \xi_j \notin \mathrm{BV}([x_{j_i}/N_i'\colon i \in \boldsymbol{n}]M)\},$$

(iii[b]) if p is different from $j_1, \ldots, j_n$, then

$$[x_{j_i}/N_i\colon i \in \boldsymbol{n}]\lambda\xi_q\cdot(x_p/\xi_q)M = \lambda\xi_k\cdot(x_m/\xi_k)[x_{s_i}/N_i''\colon i \in \boldsymbol{n}+\boldsymbol{1}]M,$$

where

$$x_{s_i} = \begin{cases} x_{j_i} & \text{for } i \in \boldsymbol{n}, \\ x_p & \text{for } i = n+1; \end{cases} \qquad N_i'' = \begin{cases} N_i & \text{for } i \in \boldsymbol{n}, \\ x_m & \text{for } i = n+1, \end{cases}$$

and

$$m = \min\{i\colon x_i \notin \mathrm{FV}(M) \cup \bigcup_{j \in n} \mathrm{FV}(N_j)\},$$

$$k = \min\{j\colon \xi_j \notin \mathrm{BV}([x_{s_i}/N_i''\colon i \in \boldsymbol{n}+\boldsymbol{1}]M).$$

A *general lambda-theory* (shortly lambda-theory) is an ordered pair (C, E), where C is a set of constant symbols and E is an equivalence relation on the set $\mathrm{Exp}[C]$, called a *conversion* on $\mathrm{Exp}[C]$, satisfying the following conditions:

(β) $\quad (\lambda\xi_j\cdot(x_i/\xi_j)MN)\ E\ [x_i/N]M$

for all $M \in \mathrm{Exp}[C]$, $i \in \boldsymbol{N}^+$ and $j = \min\{k\colon \xi_k \notin \mathrm{BV}(M)\}$;

(τ) $\quad$ if $M\ E\ N$, then $(MP)\ E\ (NP)$, and $(PM)\ E\ (PN)$, and

$$\lambda\xi_n\cdot(x_i/\xi_n)M\ E\ \lambda\xi_m\cdot(x_i/\xi_m)N$$

for all $P \in \mathrm{Exp}[C]$, $i \in \boldsymbol{N}^+$ and

$$n = \min\{k\colon \xi_k \notin \mathrm{BV}(M)\}, \qquad m = \min\{k\colon \xi_k \notin \mathrm{BV}(N)\}.$$

Remark. The notion of a λ-theory in the sense of Barendregt [1] is a particular case of the notion of a general lambda-theory.

Let **con** denote smallest conversion on $\mathrm{Exp} = \mathrm{Exp}[\emptyset]$, and let $\mathbf{con}_\eta$ denote the smallest conversion on Exp satisfying the condition

(η) $\quad \lambda\xi_j\cdot(M\xi_j)\ \mathbf{con}_\eta\ M$

for all $M \in \mathrm{Exp}$ and $j = \min\{k\colon \xi_k \notin \mathrm{BV}(M)\}$, The general lambda-theory $(\emptyset, \mathbf{con})$ is called the *pure type-free $\lambda\beta$-calculus* and the general lambda-theory $(\emptyset, \mathbf{con}_\eta)$ is called the *pure type-free $\lambda\beta\eta$-calculus*. Another example of a lambda-theory is the ordered pair $(\{\Omega\}, Q)$, where Ω is a fixed constant and Q is the smallest conversion satisfying:

(Ω_1) $\qquad \lambda\xi_1 \cdot \Omega\, Q\, \Omega$

and

(Ω_2) $\qquad (\Omega M)\, Q\, \Omega \quad$ for all $M \in \mathrm{Exp}[\{\Omega\}]$

(cf. Barendregt [1], p. 1126).

The above examples of lambda-theories give rise to the concept of an equational lambda-theory. An equational lambda-theory is an ordered triple (C, E, Θ), where $E \subseteq \mathrm{Exp}[C] \times \mathrm{Exp}[C]$ and Θ is the smallest conversion on $\mathrm{Exp}[C]$ satisfying $E \subseteq \Theta$.

3. Labelled λ-terms

We shall now introduce the notion of a labelled λ-term, which is needed for the construction of algebraic theories from λ-terms. For any λ-term M, we define the *rank* of M to be 0 if $\mathrm{FV}(M) = \emptyset$ and $\max\{i\colon x_i \in \mathrm{FV}(M)\}$ otherwise. The rank of M will be denoted by $\mathrm{rn}(M)$. A *labelled λ^C-term* is an ordered pair (M, n), where M is a λ^C-term and $n \geqslant \mathrm{rn}(M)$. The number n will be called the *index* of (M, n). The set $\{(M, n)\colon M \in \mathrm{Exp}[C]$ and $\mathrm{rn}(M) \leqslant n \in \boldsymbol{N}\}$ of all labelled λ^C-terms will be denoted by $\mathrm{Exp}^*[C]$.

3.1. PROPOSITION. *The set* $\mathrm{Exp}^*[C]$ *is equal to the set* $\mathfrak{M}[C]$ *defined by induction as follows:*

(i) $(c, n) \in \mathfrak{M}[C]$ *for all* $c \in C$ *and all* $n \in \boldsymbol{N}$,
$(x_i, n) \in \mathfrak{M}[C]$ *for all* $n \in \boldsymbol{N}^+$ *and all* $i \in \boldsymbol{n}$;

(ii) *if* (M, n) *and* (N, n) *are in* $\mathfrak{M}[C]$, *then* $\big((MN), n\big) \in \mathfrak{M}[C]$;

(iii) *if* $(M, n+1) \in \mathfrak{M}[C]$ *and* $j = \min\{k\colon \xi_k$ *does not occur in* $M\}$, *then* $\big(\lambda\xi_j \cdot (x_{n+1}/\xi_j)M, n\big) \in \mathfrak{M}[C]$.

Sketch of proof. For any λ-term M, let the degree of M, denoted by $\deg(M)$, be the total number of occurrences of symbols λ and (in M. By induction on the degree of a λ-term we can prove that

(a) $(x_i/x_j)M \in \mathrm{Exp}[C]$ and $\deg\big((x_i/x_j)M\big) = \deg(M)$ for all M in $\mathrm{Exp}[C]$,

and by induction on the length of an expression M we can prove that

(b) if x_j does not occur in M, then $(x_i/\xi_m)M = (x_j/\xi_m)(x_i/x_j)M$, where $(x_i/x_j)M$ denotes the result of substituting x_j for x_i in M.

Using (a) and (b), we infer by induction on the degree of a λ-term that $\mathrm{Exp}^*[C] \subseteq \mathfrak{M}[C]$. The inclusion $\mathfrak{M}[C] \subseteq \mathrm{Exp}^*[C]$ is immediate.

3.2. PROPOSITION. *If* $(M, n+1)$ *is a labelled λ-term and* $\big((N_i, m)\colon i \in \boldsymbol{n}\big)$ *is a family of labelled λ-terms, then*

$$[x_1/N_1, \ldots, x_n/N_n]\lambda\xi_j \cdot (x_{n+1}/\xi_j)M = \lambda\xi_k \cdot (x_{m+1}/\xi_k)[x_1/N_1, \ldots, x_n/N_n, x_{n+1}/x_{m+1}]M,$$

where

$$j = \min\{s\colon \xi_s \notin \mathrm{BV}(M)\}$$

and

$$k = \min\{s\colon \xi_s \notin \mathrm{BV}([x_1/N_1, \ldots, x_n/N_n, x_{n+1}/x_{m+1}]M)\}.$$

Using 3.1 and 3.2, we prove by induction on $\mathfrak{M}[C]$ the following composition rule for simultaneous substitution:

3.3. PROPOSITION. *If* (M, n) *is a labelled λ-term and* $\big((P_i, k)\colon i \in \boldsymbol{m}\big)$ *and* $\big((N_i, m)\colon i \in \boldsymbol{n}\big)$ *are families of labelled λ-terms, then*

$$[x_i/P_i\colon i \in \boldsymbol{m}][x_i/N_i\colon i \in \boldsymbol{n}]M = \big[x_j/[x_i/P_i\colon i \in \boldsymbol{m}]N_j\colon j \in \boldsymbol{n}\big]M.$$

4. Algebraic theories constructed from λ-terms

4.1. Let $\boldsymbol{T}[C] = (T[C], [?], P[C])$ be the following algebraic theory: the objects of $T[C]$ are non-negative integers, i.e. $\mathrm{Ob}\,T[C] = \boldsymbol{N}$, $[n] = n$ for all $n \in \boldsymbol{N}$; the arrows from $[n]$ to $[m]$ $(m \geqslant 1)$ are m-tuples of λ^C-terms with index n, i.e.

$$T[C]([n], [m]) = \Big\{\big((M_i, n)\colon i \in \boldsymbol{m}\big)\colon \bigvee_{i \in \boldsymbol{m}} (M_i, n) \in \mathrm{Exp}^*[C]\Big\};$$

the (unique) arrow from $[n]$ to $[0]$ is $(n, 0)$, i.e. $!^n = (n, 0)$ (for technical reasons we shall assume that the set C is disjoint with $\boldsymbol{N}$, in this case all hom-sets $T[C]([n], [m])$ are disjoint), the composition of $f = \big((M_i, n)\colon i \in \boldsymbol{m}\big)\colon [n] \to [m]$ and $g = \big((N_j, m)\colon j \in \boldsymbol{s}\big)\colon [m] \to [s]$ is defined as

$$gf = \big(([x_i/M_i\colon i \in \boldsymbol{m}]N_j, n)\colon j \in \boldsymbol{s}\big)\colon [n] \to [s] \quad \text{for } m \geqslant 1;$$

and the composition of $!^n\colon [n] \to [0]$ and $h = \big((N_j', 0)\colon j \in \boldsymbol{s}\big)\colon [0] \to [s]$ is given by

$$h!^n = \big((N_j', n)\colon j \in \boldsymbol{s}\big)\colon [n] \to [s];$$

the family of projections $P[C] = (\mathrm{pr}_i^n\colon n \in \boldsymbol{N}^+, i \in \boldsymbol{n})$ is defined as

$$\mathrm{pr}_i^n = (x_i, n)\colon [n] \to [1] \quad \text{for } n \geqslant 1,\ i \in \boldsymbol{n}.$$

By 3.3 the composition is associative and by the definition of simultaneous substitution the family $P[C]$ is, in fact, the family of projections, i.e. condition 1.5(a_3) holds for $P[C]$.

4.2. Now we shall introduce a concept of a lambda-congruence. A *lambda-congruence* on $\bigcup_{n=0}^{\infty} T[C]([n],[1])$ or on $\mathrm{Exp}^*[C]$ is a restricted algebraic congruence on $\boldsymbol{T}[C]$ (cf. 1.7) satisfying the following conditions:

(i) if $(M, n+1) \sim (N, n+1)$, then

$$\big(\lambda\xi_j\cdot(x_{n+1}/\xi_j)M,\, n\big) \sim \big(\lambda\xi_m\cdot(x_{n+1}/\xi_m)N,\, n\big),$$

where

$$j = \min\{k\colon\ \xi_k \notin \mathrm{BV}(M)\} \quad \text{and} \quad m = \min\{k\colon\ \xi_k \notin \mathrm{BV}(N)\},$$

(ii) for each $(M, n+1) \in \mathrm{Exp}^*[C]$

$$\Big(\big(\lambda\xi_j\cdot(x_{n+1}/\xi_j)Mx_{n+1}\big),\ n+1\Big) \sim (M, n+1),$$

where $j = \min\{k\colon\ \xi_k \notin \mathrm{BV}(M)\}$.

By an algebraic theory constructed from λ-terms we shall mean the algebraic theory $\boldsymbol{T}[C]$ and the quotient algebraic theory $\boldsymbol{T}[C]/\!\sim$, where $\sim$ is a lambda-congruence.

The following two theorems establish the corespondence (equivalence from the proof-theoretical point of view) between quotient algebraic theories $\boldsymbol{T}[C]/\!\sim$ and general lambda-theories:

4.3. Theorem. *If (C, E) is a general lambda-theory, then the binary relation $\sim$ defined on* $\mathrm{Exp}^*[C]$ *as follows*:

(i) $(M, n) \sim (N, m)$ *iff* $M\,E\,N$ *and* $m = n \geqslant \max\{\mathrm{rn}(M), \mathrm{rn}(N)\}$,

is a lambda-congruence on $\mathrm{Exp}^*[C]$.

Conversely, if $\sim$ is a lambda-congruence on $\mathrm{Exp}^*[C]$, *then the pair (C, E), where E is the binary relation defined on* $\mathrm{Exp}[C]$ *as follows*:

(ii) $M\,E\,N$ *iff there is an $n \in \boldsymbol{N}$ such that* $(M, n) \sim (N, n)$,

is a general lambda-theory.

4.4. Theorem. *If (C, E, Θ) is an equational lambda-theory, then the lambda-congruence defined by formula 4.3 (i) for $E = \Theta$ is the smallest lambda-congruence on* $\mathrm{Exp}^*[C]$ *satisfying the following condition*:

(iii) *if* $(M, N) \in E$, *then* $(M, n) \sim (N, n)$ *for all* $n \geqslant \max\{\mathrm{rn}(M), \mathrm{rn}(N)\}$.

5. Algebraic theories with application and abstraction

We shall now describe properties of algebraic theories $\boldsymbol{T}[C]$ and $\boldsymbol{T}[C]/\!\sim$ in a more categorial way.

5.1. An *algebraic theory with application and abstraction* is an ordered triple $\mathscr{T} = \big(\boldsymbol{T}, \varepsilon, (?)^*\big)$, where $\boldsymbol{T}$ is an algebraic theory, $\varepsilon\colon [2] \to [1]$ is a distinguished arrow of T, and $(?)^*$ is a mapping assigning to each arrow $f\colon [n+1] \to [1]$ $(n \in \boldsymbol{N})$ of T an arrow $h\colon [n] \to [1]$ of T (we shall denote the value of $(?)^*$ for f by $(f)^*$). The arrow ε is called *application* and the mapping $(?)^*$ is called *abstraction*. In an obvious way the triple

$$\mathscr{T}[C] = \big(\boldsymbol{T}[C], \varepsilon, (?)^*\big),$$

where

(i) $\varepsilon = \big((x_1x_2), 2\big)$,

(ii) $\big((M, n+1)\big)^* = \big(\lambda\xi_j\cdot(x_{n+1}/\xi_j)M, n\big)$ for $j = \min\{k\colon\ \xi_k \notin \mathrm{BV}(M)\}$,

is an algebraic theory with application and abstraction.

5.2. A *lambda-algebraic theory* is an algebraic theory with application and abstraction satisfying the following condition:

(ρ) $(f)^*\cdot g = \big(f\cdot(g\times \mathrm{id}_{[1]})\big)^*$ for all $f \in T([n+1],[1])$, $g \in T([n],[m])$, $n \in \boldsymbol{N}$, $m \in \boldsymbol{N}$ (for $\times$ see 1.5).

By virtue of 1.5 (0) condition (ρ) is equivalent to the conjunction of the following two conditions:

(ρ_1) $(f)^*\cdot\langle f_i\colon\ i \in \boldsymbol{n}\rangle = (f\cdot\langle f_1\cdot\mathrm{pr}_{m,1}, \ldots, f_n\cdot\mathrm{pr}_{m,1},\ \mathrm{pr}^{m+1}_{m+1}\rangle)^*$ for $f_i \in T([m],[1])$, $m \in \boldsymbol{N}$, $n \in \boldsymbol{N}^+$, and $f \in T([n],[1])$;

(ρ_2) $(f)^*\cdot\, !^n = (f\cdot\mathrm{pr}^{n+1}_{n+1})^*$ for $f \in T([1],[1])$, $n \in \boldsymbol{N}$;

hence by virtue of 3.2 we have the following proposition:

5.3. Proposition. *The algebraic theory with application and abstraction $\mathscr{T}[C] = \big(\boldsymbol{T}[C], \varepsilon, (?)^*\big)$, where ε and $(?)^*$ are given by* 5.1 (i), (ii), *is a lambda-algebraic theory.*

5.4. A *Church algebraic theory* is a lambda-algebraic theory $\big(\boldsymbol{T}, \varepsilon, (?)^*\big)$ satisfying the following condition:

($\hat\beta$) $\varepsilon\cdot\big((f)^*\times \mathrm{id}_{[1]}\big) = f$ for all $f \in T([n],[1])$, $n \in \boldsymbol{N}^+$ (cf. 1.5).

By virtue of 1.5 (0) condition ($\hat\beta$) is equivalent to the conjunction of the following two conditions:

(β_1) $\varepsilon\cdot\langle (f)^*\cdot\, !^1, \mathrm{pr}^1_1\rangle = f$ for f in $T([1],[1])$;

(β_2) $\varepsilon\cdot\langle (f)^*\cdot\langle \mathrm{pr}^{n+1}_i\colon\ i \in \boldsymbol{n}\rangle, \mathrm{pr}^{n+1}_{n+1}\rangle = f$ for f in $T([n+1],[1])$, $n \in \boldsymbol{N}^+$;

hence by virtue of 5.3 and 4.2 we have the following:

5.5. Proposition. *If $\sim$ is a lambda-congruence on* $\mathrm{Exp}^*[C]$, *then*

$$\mathscr{T}[C]/\!\sim\ = \big(\boldsymbol{T}[C]/\!\sim,\ \varepsilon/\!\sim,\ (?)^*/\!\sim\big),$$

where

(i) $\varepsilon/\sim\, = ((x_1x_2), 2)/\sim$;

(ii) $((M, n+1)/\sim)^*/\sim\, = (\lambda\xi_j\cdot(x_{n+1}/\xi_j)M, n)/\sim$ *for* $j = \min\{k\colon\ \xi_k \notin \mathrm{BV}(M)\}$,

is a Church algebraic theory (here $(N, m)/\sim$ *denotes an equivalence class with respect to* $\sim$).

5.6. The pure type-free $\lambda\beta\eta$-calculus leads to a stronger version of the concept of a Church algebraic theory. By an *algebraic theory of type* λ-$\beta\eta$ (cf. Obtułowicz [3[a]]) we mean an algebraic theory with application and abstraction $(\boldsymbol{T}, \varepsilon, (?)^*)$ satisfying $(\hat{\beta})$ and the following condition:

$(\hat{\eta})$ $\quad (\varepsilon\cdot(h\times \mathrm{id}_{[1]}))^* = h \quad$ for all $h \in T([n], [1])$, $n \in \boldsymbol{N}$ (cf. 1.5).

By virtue of 1.5 (0) condition $(\hat{\eta})$ is equivalent to the conjunction of the following two conditions:

(η_1) $\quad (\varepsilon\cdot\langle h\cdot !^1, \mathrm{pr}_1^1\rangle)^* = h \quad$ for all h in $T([0], [1])$;

(η_2) $\quad (\varepsilon\cdot\langle h\cdot\langle \mathrm{pr}_i^{n+1}\colon i \in \boldsymbol{n}\rangle, \mathrm{pr}_{n+1}^{n+1}\rangle)^* = h \quad$ for all $n \in \boldsymbol{N}^+$ and all h in $T([n], [1])$;

hence the concept of an algebraic theory of type λ-$\beta\eta$ characterizes general lambda-theories (C, E) with E satisfying condition (η) in 2.2, where E is put instead of $\mathbf{con}_\eta$, in a similar way as Church algebraic theories characterize all general lambda-theories (cf. 4.3, 4.4 and 6.4).

5.7. THEOREM. *If* $\mathcal{T} = (\boldsymbol{T}, \varepsilon, (?)^*)$ *is an algebraic theory of type* λ-$\beta\eta$, *then* T *is a cartesian closed category with the exponentiation* (cf. 1.9) *satisfying*

$$[1]^{[1]} = [1].$$

If $\boldsymbol{T} = (T, [?], P)$ *is an algebraic theory, where* T *is a cartesian closed category with exponentiation satisfying* $[1] = [1]^{[1]}$, *then the structure of a cartesian closed category of* T *induces application and abstraction such that* $\boldsymbol{T}$ *with these data is an algebraic theory of type* λ-$\beta\eta$.

Remark. For any algebraic theory with application and abstraction satisfying $(\hat{\beta})$, condition $(\hat{\eta})$ implies condition (ρ), but the converse is, in general, not true (cf. the remark in 7.7).

6. Interpretation of labelled λ-terms in algebraic theories with application and abstraction

There is a natural question: does the concept of a Church algebraic theory characterize general lambda-theories completely? After Theorems 4.3, 4.4 and Proposition 5.5, this question reduces to the following question: is each Church algebraic theory isomorphic to some algebraic theory $\boldsymbol{T}[C]/\sim$ constructed from λ-terms? A positive answer to this last question is contained in Theorem 6.4.

6.1. PROPOSITION. *Let* $\mathcal{T} = (\boldsymbol{T}, \varepsilon, (?)^*)$ *be an algebraic theory with application and abstraction. For any set* C *and any function* f *from* C *to* $T([C], [1])$, *there is a unique function* h *from* $\mathrm{Exp}^*[C]$ *to* $\bigcup_{n\in N} T([n], [1])$ *such that the following conditions hold:*

(i) $h(c, n) = f(c)\cdot !^n$ *for* $n \in \boldsymbol{N}$, $c \in C$, *and* $h(x_i, n) = \mathrm{pr}_i^n$ *for* $n \in \boldsymbol{N}^+$, $i \in \boldsymbol{n}$;

(ii) $h((MN), n) = \varepsilon\cdot\langle h(M, n), h(N, n)\rangle$;

(iii) $h(\lambda\xi_j\cdot(x_{n+1}/\xi_j)M, n) = (h(M, n+1))^*$.

The function h defined in the above proposition will be called an *interpretation of labelled* λ*-terms in* $\mathcal{T}$ (more precisely: h will be called the *interpretation induced by* f).

6.2. PROPOSITION. *Let* $\mathcal{T} = (\boldsymbol{T}, \varepsilon, (?)^*)$ *be a lambda-algebraic theory and let* C *be a set. If* h *is an interpretation of labelled* λ*-terms in* $\mathcal{T}$, *then*

$$h([x_i/N_i\colon\ i \in \boldsymbol{n}]M, m) = h(M, n)\cdot\langle h(N_i, m)\colon\ i \in \boldsymbol{n}\rangle$$

for all $n \in \boldsymbol{N}^+$, $(M, n) \in \mathrm{Exp}^*[C]$, $(N_i, m) \in \mathrm{Exp}^*[C]$.

6.3. PROPOSITION. *Let* $\mathcal{T} = (\boldsymbol{T}, \varepsilon, (?)^*)$ *be a Church algebraic theory, let* C *be a set, and let* h *be an interpretation of labelled* λ^C*-terms in* $\mathcal{T}$. *The binary relation* $\sim$ *defined on* $\mathrm{Exp}^*[C]$ *as follows:*

(i) $\quad (M, n) \sim (N, m) \quad$ *iff* $\quad h(M, n) = h(N, m)$

is a lambda-congruence on $\mathrm{Exp}^*[C]$.

The proof follows by Proposition 6.2.

6.4. THEOREM. *For each Church algebraic theory* $\mathcal{T} = (\boldsymbol{T}, \varepsilon, (?)^*)$ *there exist a set* C *and a lambda-congruence* $\sim$ *on* $\mathrm{Exp}^*[C]$ *such that the category* T *is isomorphic with* $T[C]/\sim$.

Sketch of proof. Let $C = T([0], [1])$ and let h be the interpretation of labelled λ-terms in $\mathcal{T}$ induced by the identity function on $T([0], [1])$. We define $\sim$ by condition 6.3 (i). Since for each arrow $g\colon [n] \to [1]$ in T we have

$$h(((\ldots((g'x_1)x_2)\ldots)x_n), n) = \varepsilon\cdot\langle\ldots\varepsilon\cdot\langle\varepsilon\cdot\langle g'!^n, \mathrm{pr}_1^n\rangle, \mathrm{pr}_2^n\rangle\ldots\mathrm{pr}_n^n\rangle = g,$$

where $g'\colon [0] \to [1]$ is the result of abstraction $(?)^*$ applied n times to g, we conclude that h is a surjection; hence by 6.3 the mapping

$$I\colon \bigcup_{m,n\in N} T[C]/\sim([m], [n]) \to \bigcup_{m,n\in N} T([m], [n])$$

given by

$$I\big(((M_i, n)/\sim:\ i \in \boldsymbol{n})\big) = \langle h(M_i, n):\ i \in \boldsymbol{n}\rangle$$

is the arrow function of the functor from $T[C]/\sim$ to T which is an isomorphism of categories.

6.5. We shall introduce the following definitions: Let $\mathcal{T} = (\boldsymbol{T}, \varepsilon, (?)^*)$ and $\mathcal{T}' = (\boldsymbol{T}', \varepsilon', (?)^0)$ be Church algebraic theories. A *morphism of Church algebraic theories* is a triple $(H, \mathcal{T}, \mathcal{T}')$, where $H\colon T \to T'$ is a functor satisfying the following conditions:

$$H([0]) = [0]',$$

$$H(\mathrm{pr}_i^n) = \mathrm{pr}_i'^n \quad \text{for all } n \in \boldsymbol{N}^+ \text{ and all } i \in \boldsymbol{n},$$

$$H(\varepsilon) = \varepsilon',$$

$$H\big((f)^*\big) = \big(H(f)\big)^0 \quad \text{for all arrows } f\colon [n+1] \to [1] \text{ of } T \text{ and all } n \in \boldsymbol{N}.$$

The category CHT, called the *doctrine of Church algebraic theories*, has as objects all Church algebraic theories and as arrows from $\mathcal{T}$ to $\mathcal{T}'$ all morphisms of Church algebraic theories $(H, \mathcal{T}, \mathcal{T}')$; the composition of arrows in CHT is the composition of functors.

6.6. THEOREM. *The forgetful functor $U\colon \mathrm{CHT} \to \mathrm{Set}$ defined by*

$$U(\mathcal{T}) = T([0], [1])$$

has a left adjoint $F\colon \mathrm{Set} \to \mathrm{CHT}$ with an object function defined by

$$F(C) = \big(\boldsymbol{T}[C]/\sim, \varepsilon/\sim, (?)^*/\sim\big),$$

where $\sim$ is the smallest lambda-congruence on $\mathrm{Exp}^*[C]$ *and $\varepsilon/\sim$, $(?)^*/\sim$ are defined as in* 5.5.

7. Functional interpretation of λ-terms

7.1. Let A and C be sets. An ordered pair $(\mu\colon A \to C, \nu\colon C \to A)$ of functions is called a *normal pair* iff the following conditions hold:

$$C \subseteq A^A \quad \text{and} \quad \mu \circ \nu = \mathrm{id}_C.$$

The following notion is implicitly contained in a paper of Wadsworth [5]: An ordered pair $(\mu\colon A \to C, \nu\colon C \to A)$ of functions will be called an *interpretable pair* iff it is a normal pair and there is a function $h\colon \mathrm{Exp} \to A^{(A^V)}$ satisfying the following conditions:

(i) $h(x_i)(v) = v(x_i)$ for all $v \in A^V$ and all $i \in \boldsymbol{N}^+$,

(ii) $h\big((MN)\big)(v) = \big(\mu(h(M)(v))\big)\big(h(N)(v)\big)$ for all $v \in A^V$ and all M, $N \in \mathrm{Exp}$,

(iii) $h\big(\lambda\xi_j\cdot(x_i/\xi_j)M\big)(v) = \nu\big(\lambda_A[h(M)\circ\ulcorner x_i\urcorner](v)\big)$ for all $v \in A^V$, $M \in \mathrm{Exp}$, $i \in \boldsymbol{N}^+$, where $\ulcorner x_i\urcorner\colon A^A \times A \to A^V$ is the function defined as follows:

$$\ulcorner x_i\urcorner(v, a)(x_n) = \begin{cases} a & \text{if } n = i,\\ v(x_n) & \text{if } n \neq i.\end{cases}$$

For any interpretable pair (μ, ν) the function h from Exp to $A^{(A^V)}$ satisfying conditions 7.1 (i)–(iii) is unique: we shall call it the *explicit interpretation of λ-terms* and denote it by $[\![?]\!]_\nu^\mu$ (the value of $[\![?]\!]_\nu^\mu$ for M is $[\![M]\!]_\nu^\mu$).

7.2. THEOREM. *If (μ, ν) is an interpretable pair and M, N are elements of* Exp *such that M* **con** *N, then $[\![M]\!]_\nu^\mu = [\![N]\!]_\nu^\mu$.*

This theorem shows that an interpretable pair together with the explicit interpretation of λ-terms may be considered as a model of the pure type-free λβ-calculus.

7.3. The examples of interpretable pairs due to D. Scott are

(A) the homeomorphism $\Phi\colon D_\infty \to [D_\infty \to D_\infty]$ and its converse (Barendregt [1], p. 1110),

(B) the functions fun: $\mathcal{P}_\omega \to [\mathcal{P}_\omega \to \mathcal{P}_\omega]$, graph: $[\mathcal{P}_\omega \to \mathcal{P}_\omega] \to \mathcal{P}_\omega$ (Barendregt [1], p. 1106).

7.4. Let $(\mu\colon A \to C, \nu\colon C \to A)$ be a normal pair and let ε_ν^μ be the function from A^2 to A defined by

$$\varepsilon_\nu^\mu(a_1, a_2) = \mu(a_1)(a_2) \quad \text{for all } a_1, a_2 \in A.$$

We define by induction a family $(E_k^n\colon n \in \boldsymbol{N}, k \in \boldsymbol{N})$ of sets

$$E_0^0 = \emptyset, \quad E_0^1 = \{\mathrm{id}_A\}, \quad E_0^2 = \{\mathrm{pr}_1^2(A), \mathrm{pr}_2^2(A), \varepsilon_\nu^\mu\},$$
$$E_0^n = \{\mathrm{pr}_i^n(A)\colon i \in \boldsymbol{n}\} \quad \text{for } n > 2,$$

and

$$E_{k+1}^n = E_k^n \cup \{\varepsilon_\nu^\mu \circ \langle f_1, f_2\rangle\colon f_1, f_2 \in E_k^n\} \cup \{g\colon \exists f \in E_k^{n+1} \text{ the composition } \nu\circ\lambda_A[f] \text{ is defined and } g = \nu\circ\lambda_A[f]\}$$

(cf. convention 1.2).

$A^n(\mu, \nu)$ will denote the set $\bigcup_{k \in \boldsymbol{N}} E_k^n$.

The family $A(\mu, \nu) = \big(A^n(\mu, \nu)\colon n \in \boldsymbol{N}\big)$ gives rise to the following definition: a *regular pair* is a normal pair $(\mu\colon A \to C, \nu\colon C \to A)$ satisfying the following condition:

(Ab) for each $n \in \boldsymbol{N}^+$ and each $f \in A^n(\mu, \nu)$ the composition $\nu\circ\lambda_A[f]$ is defined.

7.5. Proposition. *For each regular pair* $(\mu\colon A\to C,\ \nu\colon C\to A)$ *there is a unique function* $J^\mu_\nu\colon \operatorname{Exp}^*[\emptyset]\to\bigcup_{n\in N}A^n(\mu,\nu)$ *satisfying the following conditions*:

(i) $J^\mu_\nu(x_i, n)=\mathrm{pr}^n_i(A)$ *for all* $n\in N^+$ *and all* $i\in \boldsymbol{n}$;

(ii) $J^\mu_\nu\big((MN), n\big)=\varepsilon^\mu_\nu\circ\langle J^\mu_\nu(M,n), J^\mu_\nu(N,n)\rangle$;

(iii) $J^\mu_\nu\big(\lambda\xi_j\cdot(x_{n+1}/\xi_j)M, n\big)=\nu\circ\lambda_A[J^\mu_\nu(M,n+1)]$.

The function J^μ_ν in Proposition 7.5 will be called a *functional interpretation of λ-terms*. Immediately from Proposition 7.5 we have the following characterization of interpretable pairs, which does not involve the notion of an interpretation of λ-terms:

7.6. Theorem. *A pair* (μ,ν) *is an interpretable pair iff it is a regular pair.*

7.7. For any interpretable pair $(\mu\colon A\to C,\ \nu\colon C\to A)$ the family $A(\mu,\nu)$ gives rise to the category T^μ_ν having as objects all sets A^n $(n\in N)$ and as arrows $f\colon A^m\to A^n$ all functions of the form $\langle f_i\colon i\in\boldsymbol{n}\rangle$, where $f_i\in A^m(\mu,\nu)$ for all $i\in\boldsymbol{n}$; the composition of arrows in T^μ_ν is the composition of functions.

The triple $\boldsymbol{T}^\mu_\nu=(T^\mu_\nu, [?], P)$, where $[n]=A^n$ and $P=\big(\mathrm{pr}^n_i(A)\colon n\in N^+, i\in\boldsymbol{n}\big)$, is an algebraic theory (note that $m\neq n$ implies $A^m\neq A^n$ even in the case of $\operatorname{card}A=1$).

The triple $\mathcal{T}^\mu_\nu=\big(\boldsymbol{T}^\mu_\nu, \varepsilon^\mu_\nu, (?)^*\big)$, where ε^μ_ν is defined in 7.4 and $(?)^*$ is defined by

$$(f)^*=\nu\circ\lambda_A[f]\quad\text{ for all } f\in T^\mu_\nu([n+1],[1]),\ n\in N,$$

is a Church algebraic theory. We shall call $\mathcal{T}^\mu_\nu$ the *Church algebraic theory constructed from the interpretable pair* (μ,ν).

Remark. The Church algebraic theory constructed from the interpretable pair (fun, graph) (cf. 7.3 (B)) is not an algebraic theory of type λ-$\beta\eta$.

7.8. There is another characterization of interpretable pairs. Let $(\mu\colon A\to C,\ \nu\colon C\to A)$ be a normal pair, and let z^μ_ν be the function

$$\varepsilon^\mu_\nu\circ\langle\varepsilon^\mu_\nu\circ\langle\mathrm{pr}^3_1(A),\mathrm{pr}^3_3(A)\rangle,\ \varepsilon^\mu_\nu\circ\langle\mathrm{pr}^3_2(A),\ \mathrm{pr}^3_3(A)\rangle\rangle.$$

We shall call (μ,ν) a *combinatorial pair* iff all compositions in the following expressions are defined (cf. convention 1.2):

$$\nu\circ\lambda_A[\mathrm{id}_A],\qquad \nu\circ\lambda_A\big[\nu\circ\lambda_A[\mathrm{pr}^2_1(A)]\big],\qquad \nu\circ\lambda_A\big[\nu\circ\lambda_A\big[\nu\circ\lambda_A[z^\mu_\nu]\big]\big].$$

Let

$$I^\mu_\nu=\nu\circ\lambda_A[\mathrm{id}_A],\qquad K^\mu_\nu=\nu\circ\lambda_A\big[\nu\circ\lambda_A[\mathrm{pr}^2_1(A)]\big],$$

$$S^\mu_\nu=\nu\circ\lambda_A\big[\nu\circ\lambda_A\big[\nu\circ\lambda_A[z^\mu_\nu]\big]\big].$$

The family $(R^n_k\colon n\in N,\ k\in N)$ is defined by induction as follows:

$$R^0_0=\{I^\mu_\nu, K^\mu_\nu, S^\mu_\nu\},\qquad R^1_0=\{I^\mu_\nu\circ !^1(A), K^\mu_\nu\circ !^1(A), S^\mu_\nu\circ !^1(A), \mathrm{id}_A\},$$

$$R^2_0=\{I^\mu_\nu\circ !^2(A), K^\mu_\nu\circ !^2(A), S^\mu_\nu\circ !^2(A), \varepsilon^\mu_\nu, \mathrm{pr}^2_1(A), \mathrm{pr}^2_2(A)\},$$

$$R^n_0=\{I^\mu_\nu\circ !^n(A), K^\mu_\nu\circ !^n(A), S^\mu_\nu\circ !^n(A)\}\cup\{\mathrm{pr}^n_i(A)\colon i\in\boldsymbol{n}\}\ \text{ for all } n>2,$$

$$R^n_{k+1}=R^n_k\cup\{\varepsilon^\mu_\nu\circ\langle f_1,f_2\rangle\colon f_1,f_2\in R^n_k\}.$$

$\mathscr{C}^n(\mu,\nu)$ will denote the set $\bigcup_{k\in N}R^n_k$.

7.9. Theorem. *A normal pair* (μ,ν) *is an interpretable pair iff it is a combinatorial pair; moreover,*

$$\mathscr{C}^n(\mu,\nu)=A^n(\mu,\nu)\quad \textit{for all } n\in N.$$

7.10. The functional interpretation of λ-terms gives rise to the concept of a homomorphism of interpretable pairs.

Let $(\mu\colon A\to C,\ \nu\colon C\to A)$ and $(\mu'\colon B\to D,\ \nu'\colon D\to B)$ be two interpretable pairs. A *lambda-homomorphism of* (μ,ν) *into* (μ',ν') is a function $f\colon A\to B$ such that for any labelled λ-term $(M,n)\in\operatorname{Exp}^*[\emptyset]$ the following diagram is commutative:

$$\begin{tikzcd} A^n \arrow[r, "J^\mu_\nu(M{,}n)"] \arrow[d, "f^n"'] & A \arrow[d, "f"] \\ B^n \arrow[r, "J^{\mu'}_{\nu'}(M{,}n)"'] & B \end{tikzcd}$$

7.11. Theorem. *Let* $(\mu\colon A\to C,\ \nu\colon C\to A)$ *and* $(\mu'\colon B\to D,\ \nu'\colon D\to B)$ *be two interpretable pairs. A function* $f\colon A\to B$ *is a lambda-homomorphism of* (μ,ν) *into* (μ',ν') *iff the following conditions hold*:

$$\varepsilon^{\mu'}_{\nu'}\circ f^2=f\circ\varepsilon^\mu_\nu,\qquad f\circ I^\mu_\nu=I^{\mu'}_{\nu'},\qquad f\circ K^\mu_\nu=K^{\mu'}_{\nu'},\qquad f\circ S^\mu_\nu=S^{\mu'}_{\nu'}.$$

8. Functorial semantics of the type-free λ-calculus

8.1. Let $G\colon \boldsymbol{T}[\emptyset]/\!\sim_0\to\mathrm{Set}$ be an sp-functor (cf. 1.8), where $\sim_0$ is the smallest lambda-congruence on $\operatorname{Exp}^*[\emptyset]$, and let $G([1])=A$. We shall say that a normal pair $(\mu\colon A\to C,\ \nu\colon C\to A)$ is *associated* with the functor G iff the following conditions are satisfied:

(i) $G(\varepsilon/\!\sim_0)=\varepsilon^\mu_\nu\colon A^2\to A$,

(ii) $G\big((f)^*/\!\sim_0\big)=\nu\circ\lambda_A[G(f)]$ for all $f\in(\boldsymbol{T}[\emptyset]/\!\sim_0)([n+1],[1])$, $n\in N$ (for $\varepsilon/\!\sim_0$ and $(?)^*/\!\sim_0$ see 5.5 (i), (ii)).

8.2. PROPOSITION. *If (μ, ν) and (μ', ν') are normal pairs associated with the functor G, then $\mu = \mu'$, $\nu = \nu'$.*

Proof. Since $\varepsilon_\nu^\mu = G(\varepsilon/\sim_0) = \varepsilon_{\nu'}^{\mu'}$, we have for each $a \in A$

$$\mu(a) = \lambda_A[\varepsilon_\nu^\mu](a) = \lambda_A[G(\varepsilon/\sim_0)](a) = \lambda_A[\varepsilon_{\nu'}^{\mu'}](a) = \mu'(a).$$

Moreover,

$$\nu \circ \mu = \nu \circ \lambda_A[\varepsilon_\nu^\mu] = G\big((\varepsilon/\sim_0)^*/\sim_0\big) = \nu' \circ \lambda_A[\varepsilon_{\nu'}^{\mu'}] = \nu' \circ \mu',$$

and hence $\nu = \nu'$, because μ is surjective.

8.3. PROPOSITION. *If (μ, ν) is a normal pair associated with some functor G, then (μ, ν) is a regular pair.*

8.4. We shall now introduce two categories.

The category Int has as objects all interpretable pairs and as arrows $f^\vee: (\mu, \nu) \to (\mu', \nu')$ all triples $f^\vee = \big(f, (\mu, \nu), (\mu', \nu')\big)$, where f is a lambda-homomorphism of (μ, ν) into (μ', ν'); the composition of arrows in Int is the composition of functions.

By a *$\lambda\beta$-functorial* model we shall mean a sp-functor $G: T[\emptyset]/\sim_0 \to \mathrm{Set}$ such that there is a normal pair associated with it.

The category $\mathrm{Fun}_{\lambda\beta}$ has as objects all $\lambda\beta$-functorial models and as arrows $f: G \to G'$ all natural transformations $G \dot{\to} G'$; the composition of arrows is the composition of natural transformations.

Using the definition of functional interpretations of λ-terms, we may define the functor $H: \mathrm{Int} \to \mathrm{Fun}_{\lambda\beta}$, called the *identification functor*, in the following way:

The object function of H assigns to each interpretable pair (μ, ν) a functor $H_{\mu,\nu}: T[\emptyset]/\sim_0 \to \mathrm{Set}$ defined as follows:

$$H_{\mu,\nu}\big(\langle(M_i, n)/\sim_0:\ i \in \boldsymbol{m}\rangle\big) = \langle J_\nu^\mu(M_i, n):\ i \in \boldsymbol{m}\rangle.$$

The arrow function of H assigns to each $f^\vee = \big(f, (\mu, \nu), (\mu', \nu')\big)$ the natural transformation

$$H_{f^\vee} = \big(f^n: H_{\mu,\nu}([n]) \to H_{\mu',\nu'}([n]):\ n \in \boldsymbol{N}\big).$$

8.5. THEOREM. *The identification functor $H: \mathrm{Int} \to \mathrm{Fun}_{\lambda\beta}$ is an isomorphism of categories.*

The proof follows by Propositions 8.2 and 8.3.

8.6. We shall now consider other "models" of the type-free λ-calculus.

A *pre-λ-object* means an ordered pair $\mathfrak{A} = (Y, g)$, where Y is a set, called a *support* of $\mathfrak{A}$, and g is a function from Exp to $Y^{(Y^V)}$, called a *structure* of $\mathfrak{A}$.

Let $\mathcal{T} = \big(\boldsymbol{T}, \varepsilon, (?)^*\big)$ be an algebraic theory with application and abstraction, let $h: \mathrm{Exp}^*[\emptyset] \to \bigcup_{n \in N} T([n], [1])$ be the interpretation of labelled λ-terms in $\mathcal{T}$ (cf. 6.1), and let G be an sp-functor defined on T. We shall say that a pre-λ-object $\mathfrak{A} = (Y, g)$ is *associated* with the functor G iff the following conditions are satisfied:

(i) $G([1]) = Y$,

(ii) $g(M)(v) = \begin{cases} G\big(h(M, n)\big)\big(v(x_1), \ldots, v(x_n)\big) & \text{if } \mathrm{rn}(M) = n \neq 0, \\ G\big(h(M, 0)\big)(0) & \text{if } \mathrm{rn}(M) = 0 \end{cases}$

for all $x \in Y^V$.

Let $\mathcal{A} = (\mathfrak{M}, \lambda^*)$ be a λ-algebra [a weakly extensional λ-algebra] (cf. Barendregt [1], pp. 1098, 1099), where $\mathfrak{M} = (X, \cdot)$ is a combinatory algebra and λ^* means an assignment $A \mapsto \lambda^* x \cdot A$. We shall say that a pre-λ-object $\mathfrak{A} = (Y, g)$ is *induced by $\mathcal{A}$* iff the following conditions are satisfied:

(a) $X = Y$,

(b) $g(M)(v) = [\![\hat{M}]\!]_v^{\mathfrak{M}}$ for all $v \in X^V$, $M \in \mathrm{Exp}$, where $[\![\hat{M}]\!]_v^{\mathfrak{M}}$ is defined in Barendregt [1], p. 1098, and $\hat{M}$ is a λ-term defined in the classical way (cf. Barendregt [1], p. 1096), chosen from the equivalence class corresponding to M (cf. Remark in 2.1).

8.7. THEOREM. *If a pre-λ-object $\mathfrak{A}$ is induced by some λ-algebra, then there is an sp-functor G defined on $T[\emptyset]/\sim_0$ such that $\mathfrak{A}$ is associated with G, where $\sim_0$ is defined in* 8.1.

8.8. Let $\mathcal{T} = \big(\boldsymbol{T}, \varepsilon, (?)^*\big)$ be an algebraic theory with application and abstraction. We introduce the following definitions:

(1) a *weak functorial model of $\mathcal{T}$ in* Set is an sp-functor G from T to Set,

(2) an *ordinary functorial model of $\mathcal{T}$ in* Set is an sp-functor G from T to Set satisfying the following condition:

(i) if $G(f) = G(g)$ and $f, g \in T([n+1], [1])$, then $G\big((f)^*\big) = \big(G(g)^*\big)$,

(3) a *strong functorial model of $\mathcal{T}$ in* Set is an sp-functor G from T to Set satisfyng the following condition:

(ii) there is a normal pair $(\mu: A \to C, \nu: C \to A)$ such that

(a) $G([1]) = A$,

(b) $G(\varepsilon) = \varepsilon_\nu^\mu$,

(c) $G\big((f)^*\big) = \nu \circ \lambda_A[f]$ for all $f \in T([n+1], [1])$, $n \in \boldsymbol{N}$.

The distinction between "weak", "ordinary", and "strong" corresponds to the different definitions of interpretation of λ-terms in a "model" of the type-free λ-calculus (it should be stressed that λ-algebras, weakly extensional λ-algebras and interpretable pairs differ essentially in the interpretation of λ-terms). In fact, the class of all interpretable pairs may be identified with the class of all strong functorial models of $\mathscr{T}[\emptyset]/\sim_0 = (\boldsymbol{T}[\emptyset]/\sim_0, \varepsilon/\sim_0, (\,?\,)^*/\sim_0)$ in Set, and the class of all pre-λ-objects induced by λ-algebras may be identified with a subclass of the class of all weak functorial models of $\mathscr{T}[\emptyset]/\sim_0$ in Set (cf. Theorems 8.5 and 8.7). Similarly, the class of all pre-λ-objects induced by weakly extensional λ-algebras may be identified with a subclass of the class of all ordinary functorial models of $\mathscr{T}[\emptyset]/\sim_0$ in Set.

8.9. In definitions 8.8 (1), (2), (3) one may replace Set by an arbitrary cartesian closed category K. This yields the following notions:

(1) a *weak functorial model of* $\mathscr{T}$ *in* K is a functor $G\colon T \to K$ which preserves finite products,

(2) an *ordinary functorial model of* $\mathscr{T}$ *in* K is a weak functorial model G of $\mathscr{T}$ in K satisfying condition 8.8 (2) (i),

(3) a *strong functorial model of* $\mathscr{T}$ *in* K is a weak functorial model G of $\mathscr{T}$ in K satisfying the following condition:

(i) there is an arrow $k\colon A \to A^A$ in K such that

(a) $G([1]) = A$,

(b) $G(\varepsilon) = \mathrm{ev}_{A,A}\langle k\cdot G(\mathrm{pr}_1^2), G(\mathrm{pr}_1^2)\rangle$,

(c) $k\cdot G((f)^*) = \lambda_{A,A}[G(f)]$ for all $f \in T([n+1], [1])$, $n \in \boldsymbol{N}$.

For example, the Scott models D_∞ and $\mathscr{P}_\omega$ (cf. 7.3) give rise to a strong functorial models of $\mathscr{T}[\emptyset]/\sim_0$ in the category Clatt of all complete lattices and continuous functions.

9. Hyperalgebras and hyperoperations

9.1. We shall consider a certain generalization of the notion of an abstract algebra and a homomorphism of algebras. To simplify the notation we shall omit parentheses in the following way: the set $A^{(B^C)}$ will be denoted by A^{B^C}, the function $A^{(f^D)}$ will be denoted by A^{f^D}, etc.

Let A be a set and let $p, n, q \in \boldsymbol{N}$. A *hyperoperation* on A of the type (p, n, q) is a function of the form

$$\omega\colon (A^{A^p})^n \to A^{A^q},$$

i.e. a function ω which assigns to each n-tuple of functions

$$\varphi_1\colon A^p \to A, \quad \ldots, \quad \varphi_n\colon A^p \to A$$

a function

$$\omega(\varphi_1, \ldots, \varphi_n)\colon A^q \to A.$$

A *type of hyperalgebras* is a quadruple $\tau = (T, P, N, Q)$, where T is a set and P, N, Q are functions from T to $\boldsymbol{N}$. A *hyperalgebra* of type τ is a pair

$$\mathfrak{A} = (A, (\omega_t^{\mathfrak{A}}\colon\ t \in T)),$$

where A is a set (the underlying set of $\mathfrak{A}$), and $(\omega_t^{\mathfrak{A}}\colon t \in T)$ is a family of hyperoperations

$$\omega_t^{\mathfrak{A}}\colon (A^{A^{P(t)}})^{N(t)} \to A^{A^{Q(t)}}.$$

Let $\mathfrak{A}$ and $\mathfrak{B} = (B, (\omega_t^{\mathfrak{B}}\colon t \in T))$ be hyperalgebras of the same type τ. A hyperhomomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ is a function $f\colon A \to B$ such that for every t in T and for $p = P(t)$, $n = N(t)$, $q = Q(t)$ the following diagram is commutative:

$$\begin{tikzcd}
& (A^{A^p})^n \arrow[r, "\omega_t^{\mathfrak{A}}"] & A^{A^q} \arrow[dr, "f^{A^q}"] & \\
(A^{B^p})^n \arrow[ur, "(A^{f^p})^n"] \arrow[dr, "(f^{B^p})^n"'] & & & B^{A^q} \\
& (B^{B^p})^n \arrow[r, "\omega_t^{\mathfrak{B}}"'] & B^{B^q} \arrow[ur, "B^{f^q}"'] &
\end{tikzcd}$$

In other words, $f\colon A \to B$ is a hyperhomomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ iff for any t in T and any functions

$$\varphi_1\colon B^p \to A, \quad \ldots, \quad \varphi_n\colon B^p \to A$$

the following diagram is commutative:

$$\begin{tikzcd}
A^q \arrow[r, "\omega_t^{\mathfrak{A}}(\varphi_1 \circ f^p, \ldots, \varphi_n \circ f^p)"] \arrow[d, "f^q"'] & A \arrow[d, "f"] \\
B^q \arrow[r, "\omega_t^{\mathfrak{B}}(f\circ\varphi_1, \ldots, f\circ\varphi_n)"'] & B
\end{tikzcd},$$

i.e.

$$f \circ \omega_t^{\mathfrak{A}}(\varphi_1 \circ f^p, \ldots, \varphi_n \circ f^p) = \omega_t^{\mathfrak{B}}(f\circ\varphi_1, \ldots, f\circ\varphi_n)\circ f^q.$$

Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} = (C, (\omega_t^{\mathfrak{C}}\colon t \in T))$ be hyperalgebras of the same type τ. It is easy to verify that if $f\colon A \to B$ is a hyperhomomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ and $g\colon B \to C$ is a hyperhomomorphism of $\mathfrak{B}$ into $\mathfrak{C}$, then $g\circ f\colon A \to C$ is a hyperhomomorphism of $\mathfrak{A}$ into $\mathfrak{C}$.

By Hyp_τ we shall mean the category whose objects are all hyperalgebras of type τ, and whose arrows are all hyperhomomorphisms between hyperalgebras.

We have the forgetful functor

$$U\colon \mathrm{Hyp}_\tau \to \mathrm{Set}$$

with the object function given by

$$U(A, (\omega_t^{\mathfrak{A}}\colon\ t \in T)) = A.$$

9.2. The following observation explains the meaning of the definition of a hyperhomomorphism. Let $\tau = (T, P, N, Q)$ and let t be a fixed element in T. Consider the bifunctors

$$F_t\colon \mathrm{Hyp}_\tau^{\mathrm{op}} \times \mathrm{Hyp}_\tau \to \mathrm{Set}, \qquad G_t\colon \mathrm{Hyp}_\tau^{\mathrm{op}} \times \mathrm{Hyp}_\tau \to \mathrm{Set}$$

defined as follows:

$$F_t(?_1, ?_2) = \Bigl(\mathrm{Set}\bigl((U(?_1))^p, U(?_2)\bigr)\Bigr)^n;$$

$$G_t(?_1, ?_2) = \mathrm{Set}\bigl((U(?_1))^q, U(?_2)\bigr),$$

where $p = P(t)$, $n = N(t)$, $q = Q(t)$. The family $(\omega_t^{\mathfrak{A}}\colon \mathfrak{A} \in \mathrm{Ob}\,\mathrm{Hyp}_\tau)$ is a dinatural transformation (in the sense of MacLane [3], p. 214) of F_t into G_t.

9.3. EXAMPLES. (A) Let T be a set and let $N\colon T \to \boldsymbol{N}$ be a function. An algebra of type (T, N) is a pair $\mathfrak{A} = \bigl(A, (\omega_t^{\mathfrak{A}}\colon t \in T)\bigr)$, where A is a set and ω_t is a $N(t)$-ary operation on A. An algebra of type (T, N) is (under the identification $A^{A^0} = A$) the same as a hyperalgebra of type $(T, \boldsymbol{0}, N, \boldsymbol{0})$, where $\boldsymbol{0}\colon T \to N$ is the function defined by $\boldsymbol{0}(t) = 0$ for all $t \in T$. It is easy to verify that a function $f\colon A \to B$ is a hyperhomomorphism between two hyperalgebras of type $(T, \boldsymbol{0}, N, \boldsymbol{0})$ iff it is a homomorphism between corresponding algebras. In other words, the category $\mathrm{Alg}_{(T,N)}$ of all algebras of the type (T, N) is isomorphic with the category $\mathrm{Hyp}_{(T,\boldsymbol{0},N,\boldsymbol{0})}$.

(B) Let (μ, ν) be a regular pair with the underlying set A (for the definition of a regular pair see 7.4). The function μ is a hyperoperation

$$\mu\colon (A^{A^0})^1 \to A^{A^1}$$

of type $(0, 1, 1)$ on A. For any $(M, n) \in \mathrm{Exp}^*[\emptyset]$ (see 3.0) let

$$c_{(M,n)}\colon (A^{A^1})^0 \to A^{A^n}$$

be a hyperoperation of type $(1, 0, n)$ on A defined by

$$c_{(M,n)}(0) = J_\nu^\mu(M, n),$$

where J_ν^μ is the functional interpretation of λ-terms defined in 7.5. Let $T = \{0\} \cup \mathrm{Exp}^*[\emptyset]$ and let $P\colon T \to \boldsymbol{N}$, $N\colon T \to \boldsymbol{N}$, $Q\colon T \to \boldsymbol{N}$ be functions defined as follows:

$$P(t) = \begin{cases} 0 & \text{for } t = 0, \\ 1 & \text{for } t \in \mathrm{Exp}^*[\emptyset], \end{cases} \qquad N(t) = \begin{cases} 1 & \text{for } t = 0, \\ 0 & \text{for } t \in \mathrm{Exp}^*[\emptyset], \end{cases}$$

$$Q(t) = \begin{cases} 1 & \text{for } t = 0, \\ n & \text{for } t = (M, n) \in \mathrm{Exp}^*[\emptyset]. \end{cases}$$

The pair $\bigl(A, (\omega_t^{(\mu,\nu)}\colon t \in T)\bigr)$, where

$$\omega_t^{(\mu,\nu)} = \begin{cases} \mu & \text{for } t = 0, \\ c_{(M,n)} & \text{for } t = (M, n), \end{cases}$$

is a hyperalgebra of type (T, P, N, Q).

Let (μ', ν') be another regular pair with the underlying set B. It is easy to verify that a function $f\colon A \to B$ is a hyperhomomorphism of a hyperalgebra $\bigl(A, (\omega_t^{(\mu,\nu)}\colon t \in T)\bigr)$ into a hyperalgebra $\bigl(B, (\omega_t^{(\mu',\nu')}\colon t \in T)\bigr)$ iff $f\colon A \to B$ is a lambda-homomorphism in the sense of the definition in 7.10.

10. Partial hyperalgebras and hyperalgebras in cartesian closed categoriesl

10.1. By a *partial hyperoperation* of type (p, n, q) on a set A we shal mean a partial function

$$\omega\colon (A^{A^p})^n \multimap A^{A^q},$$

i.e. a function $\omega\colon X \to A^{A^q}$ defined on some subset X of $(A^{A^p})^n$. If $\tau = (T, P, N, Q)$ is a type of hyperalgebras, then a *partial hyperalgebra* of type τ is a pair $\mathfrak{A} = \bigl(A, (\omega_t^{\mathfrak{A}}\colon t \in T)\bigr)$, where A is a set and, for any t in T, $\omega_t^{\mathfrak{A}}$ is a partial hyperoperation of type $\bigl(P(t), N(t), Q(t)\bigr)$ on A. The notion of a hyperhomomorphism of hyperalgebras may also be generalized to the case of partial hyperalgebras. Just as in the case of homomorphisms and partial algebras, we obtain some non-equivalent variants of the notion of a hyperhomomorphism.

10.2. EXAMPLES. (C) If (μ, ν) is a regular pair with the underlying set A, then the function ν is a partial hyperoperation

$$\nu\colon (A^{A^1})^1 \multimap A^{A^0}$$

of type $(1, 1, 0)$ on A.

(D) If (μ, ν) is a normal pair with the underlying set A (for the definition of a normal pair see 7.1), then the construction of sets $A^n(\mu, \nu)$ described in 7.4 gives rise to a partial hyperalgebra with the underlying set A and partial hyperoperations

$ap_n\colon (A^{A^n})^2 \multimap A^{A^n}$ of type $(n, 2, n)$ defined by
$$ap_n(f, g) = \varepsilon_\nu^\mu \circ \langle f, g\rangle,$$

$ab_n\colon (A^{A^{n+1}})^1 \multimap A^{A^n}$ of type $(n+1, 1, n)$ defined by
$$ab_n(f) = \nu \circ \lambda_A[f].$$

$c_i^n\colon (A^{A^1})^0 \multimap A^{A^n}$ of type $(1, 0, n)$ defined by $c_i^n(0) = \mathrm{pr}_i^n(A)$, where $n \in \boldsymbol{N}^+$, $i \in \boldsymbol{n}$.

10.3. It is easy to generalize the notion of a hyperalgebra and to define a *hyperalgebra in a cartesian closed category*. For example, the Scott model 7.3 (A) gives rise to the following hyperalgebra $\mathfrak{A}$ in the cartesian category Clatt of complete lattices and continuous functions: the underlying object of $\mathfrak{A}$ is D_∞, and the hyperoperations are the following arrows in Clatt:

$$ap_n\colon [D_\infty^n \to D_\infty]^2 \to [D_\infty^n \to D_\infty] \text{ defined by } ap_n(f, g) = \varepsilon^{\Phi}_{\Phi^{-1}} \circ \langle f, g\rangle,$$

$$ab_n\colon [D_\infty^{n+1} \to D_\infty] \to [D_\infty^n \to D_\infty] \text{ defined by } ab_n(f) = \Phi^{-1} \circ \lambda_{D_\infty^n, D_\infty}[f],$$

$c_i^n\colon [D_\infty^1 \to D_\infty]^0 \to [D_\infty^n \to D_\infty]$ defined by $c_i^n(\perp) = \mathrm{pr}_i^n(D_\infty)$, where $[A \to B]$ means the lattice of all continuous functions from the lattice A to the lattice B.

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SOME PROBLEMS OF BCK-ALGEBRAS AND GRISS TYPE ALGEBRAS

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The notions of BCK-algebras and Griss algebras were formulated first in 1966 (see [2], [3]). For example, BCK-algebras are obtained as unified theory generalizing some elementary and common properties of set-difference in set theory and implication in propositional calculi.

We know the following simple relations in set theory:

$$(A-B)-(A-C) \subset C-B,$$

$$A-(A-B) \subset B.$$

In propositional calculi, these relations are denoted by

$$(p \to q) \to \big((q \to r) \to (p \to r)\big),$$

$$p \to \big((p \to q) \to q\big).$$

From these relationships, we have a new class of algebras as follows:

Definition 1. Let X be a set with a binary operation $*$ and a constant 0. X is called a *BCK-algebra* if it satisfies the following conditions:

(1) $(x*y)*(x*z) \leqslant z*y$,

(2) $x*(x*y) \leqslant y$,

(3) $x \leqslant x$,

(4) $0 \leqslant x$,

(5) $x \leqslant y$, $y \leqslant x$ implies $x = y$,

(6) $x \leqslant y$ if and only if $x*y = 0$.

We introduced another class of algebras which are called Griss algebras. The notion is an algebraic formulation of negationless logic considered by G. F. C. Griss [1].