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ON THE SPECTRAL THEORY OF PSEUDO-DIFFERENTIAL ELLIPTIC BOUNDARY PROBLEMS

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1. Introduction

1.1. The spectral theory for boundary value problems for pseudo-differential operators seems to have been studied rather sparingly up to now. Our own motivation came from a study of differential Douglis–Nirenberg elliptic problems, where the pseudo-differential boundary problems appear by a natural reduction. This work (partly joint with G. Geymonat) is briefly described in Section 2 below, with an improvement due to new results in Section 4. In the major part of the present section, we recall some of the preceding results in the spectral theory for differential and pseudo-differential problems of a single order, first for manifolds without boundary and next in the case with boundary. Section 3 concerns a resolvent construction for ps.d.o. boundary problems, and Section 4 presents our study of remainder estimates, first for manifolds without boundary (Section 4.1), and then with boundary (Section 4.2).

Sections 1, 2, 3 and 4.1 have the character of a survey (with a few improvements of earlier results), whereas Section 4.2 contains complete proofs of new results.

1.2. The case without boundary. Let P be a classical pseudo-differential operator of order l acting in a q-dimensional complex hermitian vector bundle $\tilde{E}$ over an n-dimensional compact C^∞ manifold Σ without boundary. That P is classical means that in each local coordinate system $U\times \boldsymbol{R}^n$ for $T^*(\Sigma)$, the symbol of $p(x,\xi)$ is a $q\times q$-matrix of C^∞ functions on $U\times \boldsymbol{R}^n$, of the form $p(x,\xi)\sim \sum_{j=0}^{\infty} p_{l-j}(x,\xi)$, where the terms $p_{l-j}(x,\xi)$ are homogeneous in ξ of degree $l-j$. The top order term we denote by $p^0(x,\xi)$ and call the principal symbol of P (it has a meaning on $T^*(\Sigma)\setminus 0$), and we assume throughout that $\det p^0(x, \xi) \neq 0$ for $\xi \neq 0$, i.e., P is elliptic. Recall that an operator $S = \mathrm{Op}(s(x, \xi))$ is defined from a symbol $s(x, \xi)$ on $U \times \boldsymbol{R}^n$ by

$$(1.1) \qquad \mathrm{Op}(s(x, \xi))u(x) = (2\pi)^{-n} \int e^{i\langle x-y, \xi\rangle} s(x, \xi) u(y)\, dy\, d\xi;$$

P is locally defined in this way, modulo smoothing operators.

In [27], Seeley studied the resolvent of such operators P (under certain extra hypotheses on p^0), describing the kernel and showing that complex powers of P can again be defined as classical ps.d.o.s. In particular, for the case where P is positive selfadjoint and $l > 0$, so that the spectrum of P consists of a sequence of positive eigenvalues $\lambda_j(P)$ going to ∞ (counted with multiplicities), it was concluded in [27] that the number $N(t; P)$ of eigenvalues in $[0, t]$ satisfies

$$(1.2) \qquad N(t; P) = \tilde{c}_P t^{n/l} + o(t^{n/l}) \quad \text{for } t \to \infty;$$

where

$$(1.3) \quad \tilde{c}_P = \int_\Sigma c_P(x)\, dx, \quad \text{with} \quad c_P(x) = \frac{1}{n(2\pi)^n} \int_{|\xi|=1} \mathrm{tr}[p^0(x, \xi)^{-n/l}]\, d\omega.$$

Remainder estimates (i.e. improvements of the term $o(t^{n/l})$) were known first for the case of *differential* operators. Bypassing a long historical development, we just mention some comprehensive treatments. The works of Agmon and Kannai [3] and Hörmander [22], based on studies of the resolvent of P, imply that when $\dim \tilde{E} = 1$, (1.2) can be improved to

$$(1.4) \qquad N(t; P) = \tilde{c}_P t^{n/l} + O(t^{(n-\delta)/l}) \quad \text{for } t \to \infty,$$

for any $\delta < 1/2$, and there is a similar estimate of the *spectral function* $e(t; x, y)$ when $\delta < 1/2$:

$$(1.5) \qquad e(t; x, x) = c_P(x) t^{n/l} + O(t^{(n-\delta)/l}) \quad \text{for } t \to \infty,$$

uniformly in $x \in \Sigma$. The proofs seem to generalize easily to systems (i.e. the case where $\dim \tilde{E} > 1$). The latter is not the case for the next improvement of (1.4)–(1.5), that were shown to hold with $\delta = 1$ by Hörmander [23]; based on a study of the hyperbolic problem $D_t u(x, t) - P^{1/l} u(x, t) = f(x, t)$, which was solved to a high accuracy by the introduction of an adapted phase function $\varphi(x, y, \xi)$ instead of $\langle x-y, \xi\rangle$ (cf. (1.1)). This method works for *scalar ps.d.o.s.* P, and for systems P where the eigenvalues of $p^0(x, \xi)$ are simple. A generalization to certain systems where the eigenvalues of $p^0(x, \xi)$ have *constant multiplicity* was given by Demay [7′] (cf. also Chazarain [7], Petkov [24], and Duistermaat–Guillemin [8]).

There remains the general ps.d.o. systems. For such, the author has recently obtained the following result ([19] and Section 4.1 below): For all $\delta < 1/2$,

$$(1.6) \qquad \mathrm{tr}\, e(t; x, x) = c_P(x) t^{n/l} + O(t^{(n-\delta)/l}) \quad \text{for } t \to \infty,$$

uniformly for $x \in \Sigma$; and consequently (1.4) holds with $\delta < 1/2$ for general (positive elliptic) ps.d.o. systems P. Using Seeley's complex powers, we found as a corollary that in the case where P is selfadjoint, not lower bounded, the numbers $N^\pm(t; P)$ of positive, resp. negative eigenvalues in $[-t, t]$, satisfy, for all $\delta < 1/2$,

$$(1.7) \qquad N^\pm(t; P) = \tilde{c}_P^\pm t^{n/l} + O(t^{(n-\delta)/l}) \quad \text{for } t \to \infty;$$

where the $\tilde{c}_P^\pm$ are determined from p^0 (cf. (4.13) below); a similar formula holds for the spectral function. This is new also for differential operators.

1.3. The case with boundary. Now let $\bar{\Omega}$ be an n-dimensional compact C^∞ manifold with boundary Γ and interior $\Omega = \bar{\Omega} \setminus \Gamma$; we assume that $\bar{\Omega}$ is smoothly imbedded in Σ. We also assume that in a neighborhood Σ' of Γ in Σ, coordinates $x = (x', x_n)$ are chosen such that $\Sigma' \simeq \Gamma \times]-1, 1[$, with $\Gamma \simeq \Gamma \times \{0\}$ and $\Sigma' \cap \Omega \simeq \Gamma \times]0, 1[$. The trace operators $u \mapsto (D_{x_n}^k u)|_\Gamma$ are denoted by γ_k. Let $E = \tilde{E}|_{\bar{\Omega}}$. Then one can study *realizations* of P in Ω. Note first that P gives rise to an operator P_Ω on functions on Ω by the definition

$$(1.8) \qquad P_\Omega = r^+ P e^+,$$

where r^+ is the restriction operator ($r^+\colon u \mapsto u|_\Omega$), and e^+ is the "extension by zero" operator from Ω to Σ. Differential operators are local, so for these, the index Ω is usually omitted, but for ps.d.o.s. it is important. For instance, for two ps.d.o.s. S and S',

$$(1.9) \qquad L(S, S') = (SS')_\Omega - S_\Omega S'_\Omega$$

is zero when S is a differential operator but otherwise generally non-zero (it is a singular Green operator, cf. [5]). Let $l > 0$; then a realization P_T of P is an unbounded operator in $L^2(E)$ acting like P_Ω and with domain defined by a boundary condition $Tu = 0$. (A more general kind of realization is included later.)

For realizations of *differential* operators there is an ample literature on the spectral asymptotics. For selfadjoint positive realizations P_T, the most general results are those of Agmon [2] and Brüning [6]. Agmon showed for the spectral function of P_T:

$$(1.10) \qquad |e(t; x, x) - c_P(x) t^{n/l}| \leqslant \mathrm{const}\, t^{(n-\delta)/l} \mathrm{dist}(x, \Gamma)^{-\delta},$$

for each $\delta < 1/2$, and hence (with $\delta < 1/2$)

$$N(t;\, P_T) = c_P t^{n/l} + O(t^{(n-\delta)/l}) \quad \text{for } t \to \infty, \tag{1.11}$$

where

$$c_P = \int_\Omega c_P(x)\,dx; \tag{1.12}$$

the proof was given for scalar operators, but generalizes easily to *systems*. Brüning showed (using Hörmander [23]), that (1.10) is valid with $\delta = 1$ for *scalar* operators P, and hence

$$N(t;\, P_T) = c_P t^{n/l} + O(t^{(n-1)/l}\log t) \quad \text{for } t \to \infty. \tag{1.13}$$

(Still better results are known for special operators, cf. Seeley [28].) Let us also remark that we showed in [13] that when P_T is a selfadjoint elliptic, not lower bounded realization of a strongly elliptic differential system P, then the *negative* eigenvalues satisfy

$$N^-(t;\, P_T) \leqslant c^- t^{(n-1)/l} + o(t^{(n-1)/l}) \quad \text{for } t \to \infty; \tag{1.14}$$

here c^- is explicitly determined, and $o(t^{(n-1)/l})$ can sometimes be improved to $O(t^{(n-2)/l})$ (in fact always to $O(t^{(n-3/2+\varepsilon)/l})$, in view of the recent work [19]). The positive eigenvalues in the not lower bounded case were treated by Pham The Lai [25] with a result like (1.11) for $\delta < 1/2$. He also treated non-selfadjoint problems; we shall not go into the various contributions to this aspect.

Realizations (in $L^2(E)$) of *pseudo-differential* operators are not quite so easily defined. For a systematic study it is convenient to have more hypotheses on P near Γ (e.g. the transmission property of Boutet de Monvel [5]). But also without this, when P is selfadjoint positive on $\tilde{E}$, one can always define the Dirichlet realization P_γ as the Friedrichs extension of $P_\Omega|_{C_0^\infty(\Omega,E)}$; it has its domain in $H_0^{l/2}(E)$, so the spectrum is a sequence of eigenvalues going to ∞. We showed in [15] the principal asymptotic estimate

$$N(t;\, P_\gamma) = c_P t^{n/l} + o(t^{n/l}) \quad \text{for } t \to \infty, \tag{1.15}$$

for a class of operators arising from Douglis–Nirenberg elliptic problems (cf. Section 2 below); and indicated in [16] how (1.15) is obtained in general. The method of [15] was based on a constructive analysis of the resolvent $R_\lambda = (P_\gamma - \lambda I)^{-1}$ in the framework of the Boutet de Monvel theory (cf. Section 3 below). Now there are other methods to prove (1.15); a second one is given in Section 4 below; a third one would be to use very heavily that (1.15) has a certain stability under perturbations of P, so that one may approximate P near Γ by differential operators (this idea came up in a conversation with R. Beals).

The *resolvent construction* should be of interest for other purposes (trace, index, domains of complex powers, etc.). A construction for fairly general realizations was presented in [18] (with estimates in Sobolev spaces), and S. Rempel is coming out with another study (primarily for operators with symbols that are independent of the normal coordinate x_n near Γ, as far as we know), apparently directed at the mentioned applications. Some remarks on our construction will be given in Section 3.

In the study of *remainder estimates* like (1.11), perturbations of $p^0(x, \xi)$ are too crude, so P itself has to be nice (we require that it has the transmission property); however, perturbations of the boundary condition are still admissible when $\delta < 1$, so one can concentrate on P_γ for many questions. The resolvent is of the form, for $\lambda \in \boldsymbol{C} \setminus \overline{\boldsymbol{R}}_+$,

$$R_\lambda = (P_\gamma - \lambda I)^{-1} = (P - \lambda I)_\Omega^{-1} + G_\lambda, \tag{1.16}$$

where G_λ is a singular Green operator depending on λ. Following Agmon [2], we study R_λ in a region

$$V_\delta = \{\lambda \in \boldsymbol{C} \mid |\lambda| \geqslant 1,\ \operatorname{Re}\lambda \leqslant 0 \text{ or } |\operatorname{Im}\lambda| \geqslant c|\lambda|^{1-\delta/l}\}. \tag{1.17}$$

We had at first planned to generalize the complete analysis of [15], [18] (carried out essentially for "sectors" V_0) to regions V_δ with $\delta > 0$; but found very recently a method to show some particular pointwise kernel estimates for G_λ, that are almost as good as those of Agmon [2] for the differential operator case, so that they can be used to prove (1.11) for all $\delta < 1/2$ by a technique of Agmon [2]. This is explained with full proofs in Section 4.2. (The main results were announced in [20].)

We also obtain versions of (1.11) for much more general realizations, including selfadjoint realizations that are not lower bounded, see Section 4.

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2. A motivation: The spectral theory for Douglis–Nirenberg elliptic differential systems

2.1. A *Douglis–Nirenberg elliptic system of differential operators on* Ω *of symmetric type* is a matrix $\boldsymbol{A} = (A_{st})_{s,t=1,\ldots,q}$ where the A_{st} are of order $m_s + m_t$ for a given set of integers $m_1, \ldots, m_q$, and the matrix of principal symbols

$$a^0(x, \xi) = \big(\sigma_{m_s+m_t}(A_{st})(x, \xi)\big)_{s,t\leqslant q}$$

is invertible for $(x, \xi) \in T^*(\Sigma)\setminus 0$. A is said to be *strongly D.–N. elliptic* when $a^0(x, \xi)+a^0(x, \xi)^*$ is positive definite for $\xi \neq 0$; then one can assume that all $m_t \geqslant 0$. By rearranging (in an admissible way for spectral questions) and letting the A_{st} be matrix valued — or act from bundles E_t to E_s, respectively — we can assume that

$$m_1 > m_2 > \ldots > m_q \geqslant 0.$$

Denote $\bigoplus_{t\leqslant q} E_t = E$. Let Σ be a neighborhood of $\bar{\Omega}$, and assume that bundles and operators are extended to Σ.

A famous example is the linearized Navier–Stokes operator

$$A^{(1)} = \begin{pmatrix} -\Delta & -\mathrm{grad} \\ \mathrm{div} & 0 \end{pmatrix} \quad \text{on } \Omega \subset \boldsymbol{R}^3, \tag{2.1}$$

here $E_1 = \Omega \times C^3$ and $E_2 = \Omega \times C$; $m_1 = 1$ and $m_2 = 0$. $A^{(1)}$ is D.–N. elliptic, and $A^{(1)}+cI$ is strongly D.–N. elliptic for $c > 1$. Other examples are

$$A^{(2)} = \begin{pmatrix} \Delta^2 & A_{12}^{(2)} \\ A_{21}^{(2)} & -\Delta \end{pmatrix}, \quad \text{and} \quad A^{(3)} = \begin{pmatrix} \Delta^2 & A_{12}^{(3)} & A_{13}^{(3)} \\ A_{21}^{(3)} & -\Delta & A_{23}^{(3)} \\ A_{31}^{(3)} & A_{32}^{(3)} & I \end{pmatrix} \tag{2.2}$$

with the $A_{st}^{(i)}$ being suitable operators of order m_s+m_t, for $m_1 = 2$, $m_2 = 1$ and $m_3 = 0$.

From the formal expression A one defines realizations by imposing boundary conditions. For the spectral theory, it is convenient to have realizations in $L^2(E)$, but these are somewhat problematic since A acts more naturally between products of Sobolev spaces of different orders. A solution of this problem was given in [12], [14], by the introduction of the so-called *reduced Cauchy data* χu that have a sense on the maximal L^2-domain of A; we refer to [12], [14] for the complete discussion, and mention also that the boundary conditions $B\chi u$ that define selfadjoint lower bounded L^2-realizations A_B are characterized in [17] ([15] treats the case $m_q > 0$). An example is the Dirichlet realization A_γ.

Let A_B be a selfadjoint lower bounded realization, we may assume that A_B is positive. Then

$$A_B^{-1} = C = (C_{st})_{s,t\leqslant q} = \begin{bmatrix} 0 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & C_{qq} \end{bmatrix} + S \tag{2.3}$$

where C_{qq} is of order $-2m_q$ and S is of order $\leqslant -m_q-m_{q-1} \leqslant -2m_q-1$. Since S is of lower order than C_{qq}, one can expect that the spectral properties of A_B^{-1} are close to those of C_{qq}, which has the advantage to be of

a single order, but on the other hand belong to the calculus for pseudo-differential (rather than differential) boundary problems. In fact, if $\tilde{A} = (\tilde{A}_{st})_{s,t\leqslant q}$ is a parametrix of A on Σ, we have that

$$C_{qq} = (\tilde{A}_{qq})_\Omega + G_{qq}, \tag{2.4}$$

where $\tilde{A}_{qq}$ is a parametrix of the ps.d.o. P of order $2m_q$

$$P = A_{qq} - (A_{st})_{\substack{s=q\\ t\leqslant q-1}} (A_{st})^{-1}_{\substack{s\leqslant q-1\\ t\leqslant q-1}} (A_{st})_{\substack{s\leqslant q-1\\ t=q}}, \tag{2.5}$$

and G_{qq} is a singular Green operator of order $-2m_q$ and *class* 0 [10]. (Singular Green operators are part of the Boutet de Monvel calculus [5], more details are in Section 3 and Proposition 2.2 below.) There is now a great difference between the cases $m_q > 0$ and $m_q = 0$.

2.2. The case $m_q > 0$. Here A_B^{-1} and C_{qq} are compact operators, so the spectrum is discrete. We shall use a perturbation theorem (inspired from a theorem of Ky Fan [9]), that will now be stated in a general form:

PROPOSITION 2.1. *Let R_1 and R_2 be compact selfadjoint operators in a Hilbert space H, and let $S = R_2 - R_1$. Let σ, θ, ϱ and c be positive constants with $\sigma > \theta$ and $\sigma > \varrho$. Let $\theta' = \max\{\theta, \varrho(1+\sigma)/(1+\varrho)\}$, and let $\beta = (1+\sigma-\theta)/\sigma$, $\beta' = (1+\sigma-\theta')/\sigma$.*

1° If R_1 and S satisfy, respectively,

$$\lambda_j^+(R_1) = c^\sigma j^{-1/\sigma} + O(j^{-\beta}), \quad \lambda_j^-(R_1) = O(j^{-1/\varrho}), \tag{2.6}$$

and

$$\lambda_j(|S|) = O(j^{-1/\varrho}), \quad \textit{for } j\to\infty, \tag{2.7}$$

then R_2 satisfies

$$\lambda_j^+(R_2) = c^\sigma j^{-1/\sigma} + O(j^{-\beta'}), \quad \lambda_j^-(R_2) = O(j^{-1/\varrho}), \quad \textit{for } j\to\infty. \tag{2.8}$$

2° If R_1 has the inverse A_1, then (2.6) *is equivalent to*

$$N^+(t; A_1) = ct^\sigma + O(t^\theta), \quad N^-(t; A_1) = O(t^\varrho), \quad \textit{for } t\to\infty. \tag{2.9}$$

The proof is a straightforward modification of the proofs of [19], Prop. 6.1 and Lemma 6.2, applying the minimum-maximum principle to the ordered sequences of positive resp. negative eigenvalues λ_j^+ resp. λ_j^-. We denote $(S^*S)^{1/2} = |S|$.

Recall the theorem of Agmon ([1], Theorem 13.6), that when S is selfadjoint $\geqslant 0$ in $L^2(\Omega)$, and continuous from $L^2(\Omega)$ to $H^r(\Omega)$ with $r > n/2$, then $\lambda_j(S) \leqslant Cj^{-r/n}$, where C is a constant depending on the norm of S and on Ω. We shall need the following refinement (based on the calculus of Boutet de Monvel [5]):

PROPOSITION 2.2. *When G is a singular Green operator on $\bar{\Omega}$ of order $-r$ (r integer > 0) and class 0, then there is a constant C so that*

$$\lambda_j(|G|) \leqslant Cj^{-r/(n-1)} \quad \textit{for all } j. \tag{2.10}$$

Proof. The statement was proved in [15], Proposition 3.5, for the case where G is a finite sum $G = \sum_{j\leqslant J} K_jT_j$, where the K_j are Poisson operators of order $-r+\frac{1}{2}$, say, and the T_j are trace operators of order $-\frac{1}{2}$ and class 0. Letting $\mathscr{K} = \{K_j\}_{j\leqslant J}$ and $\mathscr{T} = {}^t\{T_j\}_{j\leqslant J}$, one then has that

$$|G|^2 = \mathscr{T}^*\mathscr{K}^*\mathscr{K}\mathscr{T},$$

where $\mathscr{K}^*\mathscr{K}$ is a pseudo-differential operator of order $-2r$ in Γ; to this (or an iterate) Agmon's theorem applies with $n-1$ instead of n. In general, $G = \sum_{j=1}^{\infty} K_jT_j+S$, where the K_j and T_j have rapidly decreasing symbols, and S is of order $-\infty$; then the result follows for $\sum_{j=1}^{\infty} K_jT_j$ by a passage to the limit, and hence for G by a simple perturbation argument. (An estimate of C can in principle be deduced from the exact form of G.) ∎

Denote $2m_q = l$, and $m_q+m_{q-1} = l'$ (note that $l' > l \geqslant 2$). By the theorem of Agmon, S in (2.3) satisfies $\lambda_j(|S|) = O(j^{-l'/n})$ whereas C_{qq} is of order l, so we shall have occasion to apply Proposition 2.1 with $\sigma = n/l$ and $\varrho = n/l'$. Now C_{qq} is the inverse of a fairly complicated "realization" of P (cf. (2.5), details in [15]), but we can show, using (2.4), that the *Dirichlet* realization P_γ of P satisfies

$$C_{qq}-P_\gamma^{-1} = G,$$

where G is another singular Green operator of order $-l$ and class 0. By Proposition 2.2, $\lambda_j(|G|)$ is $O(j^{-l/(n-1)})$, so we shall have occasion to apply Proposition 2.1 with $\sigma = n/l$ and $\varrho = (n-1)/l$. It remains to determine the spectral asymptotics of P_γ.

This is the main subject of Section 4, where it is shown that for any $\varepsilon > 0$,

$$N(t;\, P_\gamma) = c_P t^{n/l}+O(t^{(n-1/2+\varepsilon)/l}) \quad \text{for } t\to\infty.$$

($N = N^+$, since N^- is zero). Applying Proposition 2.1 twice, as indicated above, we finally deduce:

THEOREM 2.3. *For any* $\varepsilon > 0$,

$$N(t;\, A_B) = c_P t^{n/l}+O(t^{(n-\delta)/l}) \quad \textit{for } t\to\infty,$$

where $\delta = \min\left\{\frac{1}{2}-\varepsilon,\, n\frac{l'-l}{n+l'},\, \frac{l}{n-1+l}\right\}$.

The theorem is a sharpening of the result of [15]. For the example $A^{(2)}$ in (2.2), $l = 2$ and $l' = 3$; so that if $n = 3$, $\delta = \frac{1}{2}-\varepsilon$.

2.3. The case $m_q = 0$. (The results here were obtained jointly with G. Geymonat.) Here C_{qq} is not compact in $L^2(E_q)$, so A_B has a non-trivial essential spectrum. (We define the essential spectrum of an operator F, $\operatorname{ess\,sp} F$, as the complement in $\boldsymbol{C}$ of the set of $\lambda\in\boldsymbol{C}$ for which $F-\lambda I$ has finite dimensional nullspace and closed range with finite codimension.) Since S in (2.3) is of negative order, hence compact, a theorem of H. Weyl can be used to show that

$$\operatorname{ess\,sp} A_B = \{\lambda\neq 0 \mid \lambda^{-1}\in \operatorname{ess\,sp} C_{qq}\}.$$

This set was determined in [10] by the construction of *singular sequences*, that permit *localization* of certain properties of C_{qq} to the properties of its interior and boundary symbol (more explanations of the terminology is given in Section 3). For the interior points (in $T^*(\Omega)\setminus 0$), the sequences are like those used in Hörmander ([21], Sect. 3); but for the boundary points (in $T^*(\Gamma)\setminus 0$), the sequences are a new tool. It is shown that $\lambda^{-1}\in\operatorname{ess\,sp} C_{qq}$ exactly when $C_{qq}-\lambda^{-1}I$ is not elliptic in the sense of Boutet de Monvel; and hence $\lambda\in\operatorname{ess\,sp} A_B$ exactly when *either* $A-\lambda I$ is not elliptic, *or* $A-\lambda I$ is elliptic but $B\chi$ does not satisfy the complementing condition w.r.t. $A-\lambda I$. (This was to be hoped, but is non-trivial in view of the difficulties with the domain of the realization.) $\operatorname{Ess\,sp} A_B$ is determined in [10] also for general elliptic (not strongly elliptic or selfadjoint) A_B. For the Stokes operator $A^{(1)}$ (2.1) with the Dirichlet condition, $\operatorname{ess\,sp} A_B = \{-1,\, -\frac{1}{2}\}$.

When A_B is lower bounded, $\operatorname{ess\,sp} A_B$ turns out to be bounded, so there is in the selfadjoint case an eigenvalue sequence $\lambda_j(A_B)$ going to $+\infty$. It was shown in [11] that *this sequence behaves* (in the first approximation) *like the sequence of eigenvalues of* $\tilde{A}_B$, defined in $L^2(\bigoplus_{i\leqslant q-1} E_i)$ by

$$\tilde{A} = (A_{st})_{s,t\leqslant q-1}, \tag{2.11}$$

$$\text{with the boundary condition } B\chi\{u_1, \ldots, u_{q-1}, 0\} = 0.$$

For the example $A^{(1)}$, the eigenvalues thus behave like the eigenvalues of $-\Delta$ (applied to 3-vectors). For example $A^{(3)}$, the eigenvalues behave like those of an operator like $A^{(2)}$, that was treated above in the case $m_q > 0$.

Altogether, we see that $\operatorname{ess\,sp} A_B$ is determined from the essential spectrum of the block of highest order ($=0$) in A_B^{-1}, and the eigenvalue asymptotics at $+\infty$ is determined from the eigenvalue asymptotics of the block of highest order ($=-2m_{q-1}$) in $(\tilde{A}_B)^{-1}$. In each case, the problem is reduced to a study of pseudo-differential boundary problems.

3. A resolvent construction for boundary problems

Since there is not room here to give a deeper explanation than the presentation in [18], we shall just make some comments on our method. We assume that P is of order $l = 2m$ (m integer > 0) and has the so-called transmission property (cf. [5]). The purpose of the construction is to write

$R_\lambda = (P_\gamma - \lambda I)^{-1}$ as a series

$$R_\lambda = R_\lambda^0 + R_\lambda^1 + \ldots + R_\lambda^N + S_{\lambda,N}, \tag{3.1}$$

where the terms R_λ^k have known homogeneous symbols, and $S_{\lambda,N}$ satisfies good estimates uniformly for $\lambda \in V_0$ (cf. (1.17)). Each term is a sum $R_\lambda^k = Q_{\lambda,\Omega}^k + G_\lambda^k$, where Q_λ^k is the corresponding term in the resolvent $Q_\lambda = (P-\lambda I)^{-1}$ of P on Σ, and G_λ^k is a singular Green operator. Since the domain of P_γ is not the full Sobolev space, it is more convenient to obtain (3.1) as a part of the resolvent construction for the "inhomogeneous boundary problem"

$$\begin{pmatrix} P_\Omega - \lambda I \\ \gamma \end{pmatrix}^{-1} = (R_\lambda \ \ K_\lambda), \tag{3.2}$$

where $\gamma = \{\gamma_0, \ldots, \gamma_{m-1}\}$, and K_λ is a Poisson operator depending on λ.

Let us recall some ingredients of the Boutet de Monvel calculus. It is concerned with operators

$$A = \begin{pmatrix} P_\Omega + G & K \\ T & Q \end{pmatrix}, \tag{3.3}$$

where P is a ps.d.o. with the transmission property at Γ; K is a *Poisson operator* (going from Γ to Ω; a typical example is the solution operator $K\colon \varphi \mapsto u$ of the problem: $\Delta u = 0$ in Ω, $\gamma u = \varphi$ on Γ); T is a *trace operator* (going from Ω to Γ; the definition includes the standard trace operators γ_k, and adjoints of Poisson operators — the latter are of class 0); Q is a ps.d.o. on Γ; and G is a so-called *singular Green operator* (going from Ω to Ω; it is essentially of the form $\sum_j K_j T_j$ where the K_j and T_j are Poisson resp. trace operators). A is called a *Green operator* (cf. [5]).

For such systems A one works on three levels:

(1) the *operator* A that acts in n *dimensions*;

(2) the *interior symbol* $\sigma_\Omega(A)(x, \xi) =$ the symbol of P, which is a (locally matrix valued) *function* on $T^*(\Omega)$;

(3) the *boundary symbol* $\sigma_\Gamma(A)(x', \xi')$, which is a family of special integral operators in *one dimension*, parametrized by $(x', \xi') \in T^*(\Gamma)$.

For each (x', ξ'), $\sigma_\Gamma(P)(x', \xi')$ is, more precisely, a matrix of operators on $H^+ = \mathscr{F}[e^+\mathscr{S}(\overline{\boldsymbol{R}}_+)]$. We prefer to work *with* $\mathscr{S}(\overline{\boldsymbol{R}}_+)$ *itself* (before the Fourier transform $\mathscr{F}$), where $\sigma_\Gamma(P)(x', \xi')$ corresponds to the operator $p(x', \xi', D_{x_n})_\Omega$ [the restriction to $\boldsymbol{R}_+$ of the ps.d.o. on $\boldsymbol{R}$ with symbol $p(x', \xi', \xi_n)$]. The whole $\sigma_\Gamma(A)(x', \xi')$ corresponds to a Green operator $a(x', \xi', D_{x_n})$ on $\boldsymbol{R}_+$; the *boundary symbol operator*. (There is an intricate symbolic calculus for the operators $a(x', \xi', D_{x_n})$ or $\sigma_\Gamma(A)(x', \xi')$ explained in [5], Section 1. Detailed presentations are furthermore given in lecture notes by B. W. Schultze and his colleagues, Akad. Wiss. Berlin. The Appendix of [10] gives a survey.)

When the symbols are series of homogeneous functions, the top order term (principal part) will be denoted by an upper index 0. A is *elliptic*, when $p^0(x, \xi)$ is invertible at each (x, ξ) $(\xi \neq 0)$ and $a^0(x', \xi', D_{x_n})$ is invertible at each (x', ξ') $(\xi' \neq 0)$. The parametrix $\tilde{A}$ of A is then built up of these inverses by a kind of ps.d.o. calculus. The simplest case is when $p^0(x, \xi)$ is *independent of* x_n *near* Γ, then the principal part of $\tilde{A}$ is entirely defined by $a^0(x', \xi', D_{x_n})$ near Γ, and by $p^0(x, \xi)$ further away from Γ.

Consider our operator P, assumed to be strongly elliptic in Σ. The Dirichlet realization P_γ defined by applying the Lax–Milgram lemma to the form (Pu, v) for $u, v \in C_0^\infty(\Omega, E)$ (the Friedrichs extension of $P_\Omega|_{C_0^\infty(\Omega,E)}$ in the case when P is selfadjoint) does indeed coincide with the realization of P_Ω with domain $D(P_\gamma) = H^l(E) \cap H_0^m(E)$, thank to the transmission property; and we have that (3.2) is valid for all $\lambda \in \boldsymbol{C} \setminus \boldsymbol{R}_+$, with $R_\lambda = (P_\gamma - \lambda I)^{-1}$, within the calculus of Boutet de Monvel.

In the analysis of R_λ, we take λI into the principal symbol of P, so that we work with the principal parts

$$p^0(x, \xi) - \lambda I, \quad \text{and} \quad \begin{pmatrix} p^0(x', \xi', D_{x_n})_\Omega - \lambda I \\ \gamma \end{pmatrix} = a_\lambda^0(x', \xi', D_{x_n})$$

(λ can be viewed as homogeneous of degree l in $\mu = |\lambda|^{1/l}$). Here the positivity of p^0 implies nice properties of a_λ^0 and its inverse $(r_\lambda^0 \ k_\lambda^0)$. In fact, when $p^0(x, \xi)$ is independent of x_n near Γ, we get the formula (3.1) with $N = 0$ without having to involve the deeper symbolic calculus of Boutet de Monvel [5]; with a fairly good description of R_λ^0 (or rather G_λ^0, since $Q_{\lambda,\Omega}^0$ is obvious, cf. (4.5)), and satisfactory estimates for S_λ^0 in V_0, cf. [15]. This suffices to obtain (1.15) in the selfadjoint case, by evaluating the kernels and applying a Tauberian argument.

The development in more terms ($N \geqslant 1$), and in particular the inclusion of operators where $p^0(x, \xi)$ is not constant in x_n near Γ, require more intricate calculations. (3.1) is only obtained with up to $l-1$ terms. (In the present method, this comes from the fact that we operate with truly homogeneous symbols, because the operators G_λ^k seem most clear and applicable in this case. However, a similar limitation comes up in a rather different approach to G_λ in Theorem 4.7 below.)

The construction generalizes to other boundary problems with a suitable "ray of minimal growth" property.

4. Remainder estimates

4.1. The case without boundary. Let us first give some indications on the results of [19] for the case without boundary. We assume as usual that P is a selfadjoint, positive elliptic pseudo-differential operator in

$\tilde{E}$, and we now want to analyze the decomposition of $Q_\lambda = (P-\lambda I)^{-1}$ in a sum

$$Q_\lambda = Q_\lambda^0 + Q_\lambda^1 + \ldots + Q_\lambda^N + S_{\lambda,N}, \tag{4.1}$$

for $\lambda \in V_\delta$ (cf. (1.17)) with $\delta > 0$. The Q_λ^k are ps.d.o.s. with symbols $q_\lambda^k(x, \xi)$ determined in local coordinates by successive application of the formula

$$\sum_{|\alpha|\leqslant N} \frac{1}{\alpha!}\partial_\xi^\alpha\Big(\sum_{j=0}^N p^{l-j}(x, \xi) - \lambda I\Big) D_x^\alpha\Big(\sum_{k=0}^N q_\lambda^k(x, \xi)\Big) = I + s_{-N-1} \tag{4.2}$$

for $N = 0, 1, 2, \ldots$; here s_{-N-1} stands for terms of homogeneity degree $\leqslant -N-1$. This essentially fixes the Q_λ^k, and the problem is to estimate the remainder $S_{\lambda,N}$. The method here is to show that

$$(P-\lambda I)(Q_\lambda^0 + \ldots + Q_\lambda^N) = I - S'_{\lambda,N}, \tag{4.3}$$

where $S'_{\lambda,N}$ has a decreasing norm for $|\lambda| \to \infty$ ($\lambda \in V_\delta$), such that $I - S'_{\lambda,N}$ is invertible for large λ with inverse $\sum_{r=0}^{\infty} (S'_{\lambda,N})^r$, and (4.1) holds with

$$S_{\lambda,N} = (Q_\lambda^0 + \ldots + Q_\lambda^N)\sum_{r=1}^{\infty} (S'_{\lambda,N})^r. \tag{4.4}$$

The study of (4.3) requires estimates for compositions of P with the Q_λ^k. Clearly,

$$q_\lambda^0(x, \xi) = \big(p^0(x, \xi) - \lambda I\big)^{-1}. \tag{4.5}$$

When l is integer, one finds in local coordinates, setting

$$\lambda = -e^{i\theta l}\mu^l, \text{ with } \mu = |\lambda|^{1/l},\ \theta = \frac{1}{l}\operatorname{Arg}\lambda \in \,]-\pi/l, \pi/l[, \tag{4.6}$$

the following estimates for all $|\alpha| \leqslant l$, all β and all j:

$$|D_x^\beta D_\xi^\alpha D_\mu^j q_\lambda^0(x, \xi)| \leqslant c_{\alpha,\beta,j}(\mu + |\xi|)^{-l+\delta-(|\alpha|+j)(1-\delta)+|\beta|\delta}, \tag{4.7}$$

uniformly for $\lambda \in V_\delta$. Hence $q_\lambda^0(x, \xi)$ is in some sense in the class $S_{\varrho,\delta}^{-l+\delta}$ (with $\varrho = 1-\delta$) of Hörmander [21], if considered as a function on $T^*(\boldsymbol{R}^{n+1})$. We use this to obtain Sobolev estimates in $\boldsymbol{R}^{n+1}$ for the operator with symbol $\tilde{q}^0(x, t, \xi, \tau) = q^0_{-e^{i\theta l}\tau^l}(x, \xi)$, and this implies uniform estimates for Q_λ^0 in the spaces $H^{s,\mu}(\boldsymbol{R}^n)$, where $H^{s,\mu}$ denotes the space H^s with the norm $\|u\|_{s,\mu} = (\|u\|_s^2 + \mu^{2s}\|u\|_0^2)^{1/2}$.

There are several problems in this approach. For one thing, it turns out that for the next terms Q_λ^k,

$$q_\lambda^k(x, \xi) \in S_{\varrho,\delta}^{-l+\delta-k(1-2\delta)}(\boldsymbol{R}^{n+1}) \quad \text{(in a certain sense)}, \tag{4.8}$$

so the "order" $-l+\delta-k(1-2\delta)$ only improves with increasing k when $\delta < \frac{1}{2}$. In the following, we set

$$\delta = \tfrac{1}{2} - \varepsilon \quad \text{for some } \varepsilon > 0. \tag{4.9}$$

This phenomenon is encountered also for differential operators. [When P is scalar (or the symbol satisfies certain conditions), one may possibly get around it by use of an adapted phase function like in Robert [26], permitting $\delta < 1$.] Another problem is that the symbols $q_\lambda^k(x, \xi)$ are estimated for $\lambda \in V_\delta$, i.e. [cf. (4.6)] for μ on *half-rays* $\{\mu \geqslant \mu_\theta\}$, where $\mu_\theta \to \infty$ when θ approaches $\pm\pi/l$. To use the calculus, we have to extend the symbol to all $\mu \in \boldsymbol{R}$ (for each θ) without ruining the uniform estimates; this is solved in [19], Lemma 3.8. A third problem is that when P is a ps.d.o., the estimates of derivatives in ξ are on lygood up to a certain number (which is lower, the higher the number k in q_λ^k is). Then the calculus of [21] does not apply immediately; some composition formulas have to be proved anew in a weaker form, and it turns out that in order to have (4.1) for a given N, with $S_{N,\lambda}$ of a given "order" (uniformly for $\lambda \in V_\delta$), l must be large, inverse proportionally to ε. (Our method applies to differential operators without that phenomenon.) Altogether, we find the following theorem (where $[a]$ denotes the largest integer $\leqslant a$; $(-\lambda)^a$ is defined to be positive for $\lambda \in \boldsymbol{R}_-$; and $K(S)(x, y)$ denotes the integral operator kernel of an operator S, defined at least locally).

THEOREM 4.1. *Let* $\varepsilon \in \,]0, \frac{1}{4}]$ *and let* $\delta = \frac{1}{2} - \varepsilon$.

(i) *When P is of integer order $l \geqslant \varepsilon^{-1}(n+5)$, and $N = [l/2 - n/4 - 2]$, the resolvent is of the form* (4.1) *with $S_{\lambda,N}$ continuous from $H^{s,\mu}(\tilde{E})$ to $H^{s+l+1,\mu}(\tilde{E})$, uniformly for $\lambda \in V_\delta$* (cf. (4.6)). *In particular, the kernels of the operators exist as continuous functions and satisfy* (*for certain C^∞ functions $c_k(x)$*)

$$K(Q_\lambda^k)(x, x) = c_k(x)(-\lambda)^{-1+(n-k)/l} \quad \textit{for } k = 0, \ldots, N, \tag{4.10}$$

$$|K(S_{\lambda,N})(x, x)| \leqslant c|\lambda|^{-1+(n-1)/l} \quad \textit{for } \lambda \in V_\delta. \tag{4.11}$$

(ii) *Let N' be an integer $\geqslant 0$. Then if $l \geqslant \varepsilon^{-1}(n+N'+5)$, $S_{\lambda,N'}$ satisfies*

$$|K(S_{\lambda,N'})(x, x)| \leqslant c|\lambda|^{-1+(n-N'-1)/l} \quad \textit{for } \lambda \in V_\delta. \tag{4.12}$$

Part (i) is proved in detail in [19], and part (ii) follows by a further application of [19], showing that when $l \geqslant \varepsilon^{-1}(n+N'+5)$, then (4.1) holds with $K(S_{\lambda,N})(x, x)$ being $O(|\lambda|^{-1+(n-N'-1)/l})$ for $N = [l/2 - n/4 - 2]$. Since $N > N'$, we have that $S_{\lambda,N'} = \sum_{N'<k\leqslant N} Q_\lambda^k + S_{\lambda,N}$, where the kernel of each term is $O(|\lambda|^{-1+(n-N'-1)/l})$ for $x = y$.

Part (i) suffices to obtain (1.6) with $\delta = \frac{1}{2} - \varepsilon$ and hence (1.4), for ps.d.o.s. P of any order $l \in \boldsymbol{R}_+$, by use of a Tauberian theorem of Malliavin and Pleijel. Here, (i) is in fact applied to a suitably high, possibly non-integer power of P, defined by the calculus of Seeley [27]. The latter permits us to deduce also a *two-sided* estimate for selfadjoint, not lower bounded, elliptic P. Define

$$c_P^{\pm}(x) = \int_{|\xi|=1} \sum_{\lambda_j(p^0)\in \boldsymbol{R}_\pm} \big|\lambda_j(p^0(x,\xi))\big|^{-n/l}\, d\omega; \qquad \tilde{c}_P^{\pm} = \int_\Sigma c_P^{\pm}(x)\,dx, \tag{4.13}$$

and set $|P| = (P^2)^{1/2}$, and

$$P^+ = \tfrac{1}{2}(P+|P|), \qquad P^- = \tfrac{1}{2}(P-|P|). \tag{4.14}$$

THEOREM 4.2. *If P is selfadjoint elliptic of order $l > 0$ in $\tilde{E}$, not necessarily lower bounded, the spectral functions $e^{\pm}(t;\, x, y)$ of P^+, resp. $-P^-$, satisfy, for any $\varepsilon > 0$,*

$$\operatorname{tr} e^{\pm}(t;\, x, x) = c_P^{\pm}(x)\, t^{n/l} + O\big(t^{(n-1/2+\varepsilon)/l}\big) \quad \text{for } t\to\infty, \tag{4.15}$$

and hence the numbers $N^{\pm}(t;\, P)$ of positive, resp. negative, eigenvalues of P in $[-t, t]$ satisfy

$$N^{\pm}(t;\, P) = \tilde{c}_P^{\pm}\, t^{n/l} + O\big(t^{(n-1/2+\varepsilon)/l}\big) \quad \text{for } t\to\infty. \tag{4.16}$$

4.2. The case with boundary. Now let P be selfadjoint positive of order $l = 2m$ on Σ, having the transmission property at Γ; and define P_γ as in Section 3. As noted earlier, the resolvent $R_\lambda = (P_\gamma - \lambda I)^{-1}$ is of the form, for $\lambda \in \boldsymbol{C}\setminus \boldsymbol{R}_+$,

$$R_\lambda = Q_{\lambda,\Omega} + G_\lambda, \tag{4.17}$$

where $Q_{\lambda,\Omega}$ is the restriction to Ω of Q_λ; and the problem is to get control over the singular Green operator G_λ for $\lambda \in V_\delta$.

For the case of differential operators, Agmon showed certain estimates for the kernel of G_λ, making use of the localness of P [$(P_\Omega - \lambda I)Q_{\lambda,\Omega} = I$, when P is a differential operator]. In our case,

$$\begin{aligned} I-(P_\Omega-\lambda I)Q_{\lambda,\Omega} &= [(P-\lambda I)Q_\lambda]_\Omega - (P-\lambda I)_\Omega Q_{\lambda,\Omega} \\ &= L(P-\lambda I,\, Q_\lambda) \end{aligned} \tag{4.18}$$

is a non-trivial singular Green operator (cf. (1.8)–(1.9)). However, we can obtain some slightly weaker estimates of $K(G_\lambda)$ by a method partly inspired from Agmon [2] and Beals [4].

Denote by e^- and r^- the extension, resp. restriction, operators for $\Sigma\setminus\overline{\Omega}$ (analogous to e^+ and r^+ for Ω), and note that when u is a *function* on Σ, then

$$u - e^+r^+u = e^-r^-u.$$

When S and S' are ps.d.o.s. on Σ, we thus have that if $S'e^+u$ is a function on Σ,

$$L(S, S')u = r^+SS'e^+u - r^+Se^+r^+S'e^+u = r^+Se^-r^-S'e^+u. \tag{4.19}$$

LEMMA 4.3. *Let φ and $\psi \in C^\infty(\Sigma)$, with φ and $1-\psi \in C_0^\infty(\Omega)$ and $\varphi\psi = 0$. Then for all $\lambda \in \boldsymbol{C}\setminus\boldsymbol{R}_+$,*

$$\varphi G_\lambda(\varphi u) = \varphi R_\lambda [(P-\lambda I)\psi Q_\lambda \varphi]_\Omega u \quad \text{for } u \in L^2(E). \tag{4.20}$$

Proof. By (4.17)–(4.19) and (3.2),

$$\begin{aligned} \varphi G_\lambda(\varphi u) &= \varphi(R_\lambda - Q_{\lambda,\Omega})\varphi u \\ &= \varphi\big(R_\lambda[(P_\Omega-\lambda I)Q_{\lambda,\Omega} + L(P-\lambda I,\, Q_\lambda)] - Q_{\lambda,\Omega}\big)\varphi u \\ &= \varphi\big([I - K_\lambda\gamma]Q_{\lambda,\Omega} + R_\lambda L(P-\lambda I,\, Q_\lambda) - Q_{\lambda,\Omega}\big)\varphi u \\ &= -\varphi K_\lambda \gamma Q_{\lambda,\Omega}\varphi u + \varphi R_\lambda L(P-\lambda I,\, Q_\lambda)\varphi u. \end{aligned}$$

Using that $\gamma v = \gamma\psi v$ for $v \in H^m(E)$, and

$$L(P-\lambda I,\, Q_\lambda)\varphi u = L(P-\lambda I,\, \psi Q_\lambda)\varphi u$$

by (4.19) (Q_λ is of order $-2m$), we then find

$$\begin{aligned} \varphi G_\lambda \varphi u &= -\varphi K_\lambda \gamma\psi Q_{\lambda,\Omega}\varphi u + \varphi R_\lambda L(P-\lambda I,\, \psi Q_\lambda)\varphi u \\ &= \varphi[R_\lambda(P_\Omega-\lambda I) - I]\psi Q_{\lambda,\Omega}\varphi u + \varphi R_\lambda[(P-\lambda I)\psi Q_\lambda]_\Omega \varphi u - \\ &\qquad -\varphi R_\lambda(P_\Omega - \lambda I)\psi Q_{\lambda,\Omega}\varphi u \\ &= \varphi R_\lambda [(P-\lambda I)\psi Q_\lambda]_\Omega \varphi u, \text{ since } \varphi\psi = 0. \ \blacksquare \end{aligned}$$

We note that the operator $\varphi G_\lambda \varphi$ is selfadjoint in $L^2(E)$ when $\lambda \in \boldsymbol{R}_-$ (since R_λ and $Q_{\lambda,\Omega}$ are selfadjoint then), and $(\varphi G_\lambda \varphi)^* = \varphi G_{\bar\lambda}\varphi$ in general.

When S is a continuous operator from $H^s(E)$ to $H^t(E)$ [or from $H^s(\tilde{E})$ to $H^t(\tilde{E})$], we denote the operator norm in question by $\|S\|_{s,t}$. The following lemma applies to classical (or $S_{1,0}$) ps.d.o.s.

LEMMA 4.4. *Let $S = \operatorname{Op}\big(s(x,\xi)\big)$ [cf. (1.1)] be of positive integer order k, with $s(x,\xi)$ vanishing for x outside a compact set in $\boldsymbol{R}^n$. Let $\varphi(x_n) \in C_0^\infty(\boldsymbol{R})$, and set $\varphi_r(x_n) = \varphi(x_n/r)$ for $r \in\,]0, 1]$. Let $\sigma > 0$. Then for $u \in \mathscr{S}(\boldsymbol{R}^n)$,*

$$S(\varphi_r u) = \sum_{j=0}^{k-1} \frac{1}{j!}\big(D_{x_n}^j \varphi_r(x_n)\big)\operatorname{Op}\big(\partial_{\xi_n}^j s(x,\xi)\big)u + S_r u, \tag{4.21}$$

where $\|S_r\|_{0,0} \leqslant \operatorname{const} r^{-k-\sigma}$ for $r \in\,]0, 1]$.

Proof. We first note that for integer $a \geqslant 0$, the Fourier transform $\hat{\varphi}_r$ of φ_r satisfies

$$|\tau^a \hat{\varphi}_r(\tau)| = r^{-a+1}|\mathscr{F}(D^a\varphi)(r\tau)| \leqslant c_a r^{-a+1},$$

from which it follows (using interpolation) that for all $a \in \overline{\boldsymbol{R}}_+$

$$|\hat{\varphi}_r(\tau)| \leqslant c'_a r^{-a+1}(1+|\tau|)^{-a} \quad \text{for} \quad r \in\,]0, 1] \text{ and } \tau \in \boldsymbol{R}.$$

In particular we have, taking $a = 1+k+\sigma$,

$$|\tau|^k |\hat{\varphi}_r(\tau)| \leqslant c''_\sigma r^{-k-\sigma}(1+|\tau|)^{-1-\sigma}. \tag{4.22}$$

Now

$$\mathscr{F}[S(\varphi_r u)](\eta) = (2\pi)^{-n}\int e^{i\langle x,\xi-\eta\rangle} s(x,\xi)\widehat{\varphi_r u}(\xi)\,d\xi dx$$
$$= (2\pi)^{-n-1}\int \hat{s}(\eta-\xi,\xi)\hat{\varphi}_r(\xi_n-\theta_n)\hat{u}(\xi',\theta_n)\,d\xi d\eta d\theta_n.$$

Insertion of

$$\hat{s}(\eta-\xi,\xi) = \sum_{j=0}^{k-1}\frac{1}{j!}\,\partial_{2n}^j\hat{s}(\eta-\xi,\xi',\theta_n)(\xi_n-\theta_n)^j +$$
$$+(\xi_n-\theta_n)^k\int_0^1\frac{(1-h)^{k-1}}{(k-1)!}\,\partial_{2n}^k\hat{s}\big(\eta-\xi,\xi',\theta_n+h(\xi_n-\theta_n)\big)\,dh$$

gives the two contributions from (4.21), where the second term satisfies, for all $v\in\mathscr{S}(\boldsymbol{R}^n)$,

$$(\widehat{S_r u},\hat{v}) = c\int(\xi_n-\theta_n)^k\hat{\varphi}_r(\xi_n-\theta_n)\,\hat{u}(\xi',\theta_n)\bar{\hat{v}}(\eta)\int_0^1(1-h)^{k-1}\times$$
$$\times\,\partial_{2n}^k\hat{s}\big(\eta-\xi,\xi',\theta_n+h(\xi_n-\theta_n)\big)\,dh\,d\xi d\eta d\theta_n.$$

Since $\partial_{2n}^k s(x,\xi)$ is of order 0, we have that for any $N\geqslant 0$, there is a constant c_N so that for all ξ, η and θ_n,

$$|\partial_{2n}^k\hat{s}(\eta-\xi,\xi',\theta_n)|\leqslant c_N(1+|\eta'-\xi'|)^{-N}(1+|\xi_n-\eta_n|)^{-N}.$$

Using this and (4.22), we find

$$|(\widehat{S_r u},\hat{v})|\leqslant c_N' r^{-k-\sigma}\int(1+|\xi'-\eta'|)^{-N}(1+|\eta_n-\xi_n|)^{-N}\times$$
$$\times(1+|\xi_n-\theta_n|)^{-1-\sigma}|\hat{u}(\xi',\theta_n)\hat{v}(\eta)|\,d\xi d\eta d\theta_n,$$

which is $\leqslant\text{const}\,\|u\|_0\|v\|_0$ when $N\geqslant n$, by an application of the Cauchy–Schwarz inequality. ∎

Note in particular that the commutator $[\varphi_r,S]=\varphi_r S-S\varphi_r$ satisfies

$$[\varphi_r,S]=\sum_{j=1}^{k-1}\frac{1}{j!}(D^j\varphi_r)\operatorname{Op}\big(\partial_{\xi_n}^j s(x,\xi)\big)+S_r;\tag{4.23}$$

and

$$\psi S(\varphi_r u)=\psi S_r u,\quad\text{when }\psi\varphi_r=0\tag{4.24}$$

(then $\psi=0$ on $\operatorname{supp}\varphi_r$). We remark also that when S is a *differential* operator, $S_r u$ is a function times $(D^k\varphi_r)u$, so $\|S_r\|_{0,0}$ is $O(r^{-k})$; here $\psi S(\varphi_r u)=0$ when $\psi\varphi_r=0$. The formulas generalize easily to operators on Σ, when $\varphi\in C_0^\infty(\Sigma')$ and moreover is a function of x_n, constant in x'.

For $\lambda\in\boldsymbol{C}$ we denote by $d(\lambda)$ the distance from λ to $\boldsymbol{R}_+$, and set $V=\{\lambda|\ d(\lambda)>0,\ |\lambda|\geqslant 1\}$. For the exact resolvents Q_λ and R_λ it is not hard to show, using the ellipticity and positivity of P resp. P_γ:

$$\|Q_\lambda\|_{0,0}\leqslant d(\lambda)^{-1},\quad \|Q_\lambda\|_{0,l}\leqslant c|\lambda|d(\lambda)^{-1}\quad\text{for }\lambda\in V,\tag{4.25}$$

and hence by interpolation

$$\|Q_\lambda\|_{0,s}\leqslant c_s|\lambda|^{s/l}d(\lambda)^{-1}\quad\text{for }\lambda\in V,\tag{4.26}$$

for each $s\in[0,l]$; similarly

$$\|R_\lambda\|_{0,s}\leqslant c_s|\lambda|^{s,l}d(\lambda)^{-1}\quad\text{for }\lambda\in V.\tag{4.27}$$

It then follows moreover from a theorem of Agmon (cf. e.g. [2], Lemma 4.1) that when $l>n$, the kernels of Q_λ and R_λ exist as continuous matrix-valued functions (locally) and are $O\big(|\lambda|^{n/l}d(\lambda)^{-1}\big)$ uniformly in $\lambda\in V$. Then also

$$|K(G_\lambda)(x,x)|\leqslant c|\lambda|^{n/l}d(\lambda)^{-1}\quad\text{for }x\in\bar{\Omega},\ \lambda\in V;\tag{4.28}$$

$K(G_\lambda)(x,x)$ is a section in $\operatorname{Hom}(E,E)$.

Let M be an integer >1. Choose a system of C^∞ functions on $\boldsymbol{R}$: $\tilde{\varphi}_1,\tilde{\varphi}_2,\ldots,\tilde{\varphi}_M$ with support in $]0,1[$ and satisfying

$$\tilde{\varphi}_{j+1}\tilde{\varphi}_j=\tilde{\varphi}_j\quad\text{for }j=1,\ldots,M-1;\quad\text{and}\quad\varphi_1(\tfrac{1}{2})=1.$$

For each j, let $\tilde{\varphi}_{j,r}(t)=\tilde{\varphi}_j(t/r)$, and let $\varphi_{j,r}(x)$ denote the C^∞ function on Σ that is equal to $\tilde{\varphi}_{j,r}(x_n)$, when $x=(x',x_n)\in\Sigma'$, and zero elsewhere.

Proposition 4.5. *Assume that $l\geqslant M$. For any $\sigma>0$, there is a constant c_σ such that for all $\lambda\in\boldsymbol{C}\setminus\boldsymbol{R}_+$ with $|\lambda|\geqslant 1$, and all $r\in\,]|\lambda|^{-1/l},1]$,*

$$\|(P-\lambda I)(1-\varphi_{M,r})Q_\lambda\varphi_{1,r}\|_{0,0}\leqslant c_\sigma\big(r^{-1-\sigma}|\lambda|^{1-1/l}d(\lambda)^{-1}\big)^M.\tag{4.29}$$

Proof. The commutator of each $\varphi_{j,r}$ with Q_λ satisfies

$$[Q_\lambda,\varphi_{j,r}]u=Q_\lambda\varphi_{j,r}u-\varphi_{j,r}Q_\lambda u$$
$$=Q_\lambda\varphi_{j,r}(P-\lambda I)Q_\lambda u-Q_\lambda(P-\lambda I)\varphi_{j,r}Q_\lambda u$$
$$=Q_\lambda[\varphi_{j,r},P]Q_\lambda u,\quad\text{for }u\in L^2(E).$$

Hence we have for each $j<M$

$$[\varphi_{j+1,r},P]Q_\lambda\varphi_{j,r}u=[\varphi_{j+1,r},P][Q_\lambda,\varphi_{j,r}]u+[\varphi_{j+1,r},P]\varphi_{j,r}Q_\lambda u$$
$$=[\varphi_{j+1,r},P]Q_\lambda[\varphi_{j,r},P]Q_\lambda u+(\varphi_{j+1,r}-1)P\varphi_{j,r}Q_\lambda u,$$

since $(\varphi_{j+1,r}-1)\varphi_{j,r}=0$. Denoting $[\varphi_{j,r},P]Q_\lambda=C_{j,r}$ and $(\varphi_{j+1,r}-1)P\varphi_{j,r}Q_\lambda=F_{j+1,r}$, we have shown the operator identity

$$C_{j+1,r}\varphi_{j,r}=C_{j+1,r}C_{j,r}+F_{j+1,r}.\tag{4.30}$$

($F_{j+1,r}$ is 0 when P is a differential operator.) This is applied successively in the following calculation:

$$\begin{aligned}(4.31)\qquad (P-\lambda I)(1-\varphi_{M,r})Q_\lambda\varphi_{1,r} &= (P-\lambda I)(1-\varphi_{M,r})Q_\lambda\varphi_{M-1,r}\cdots\varphi_{1,r}\\ &= [\varphi_{M,r},P]Q_\lambda\varphi_{M-1,r}\cdots\varphi_{1,r}\\ &= C_{M,r}\varphi_{M-1,r}\cdots\varphi_{1,r}\\ &= C_{M,r}C_{M-1,r}\varphi_{M-2,r}\cdots\varphi_{1,r}+F_{M,r}\varphi_{M-2,r}\cdots\varphi_{1,r}=\cdots\\ &= C_{M,r}C_{M-1,r}\ldots C_{1,r}+F_{M,r}\varphi_{1,r}+\ldots+C_{M,r}\ldots C_{2,r}F_{1,r}.\end{aligned}$$

We shall now estimate the factors. In view of Lemma 4.4 [see (4.24)] and (4.25), we have for each j

$$(4.32)\qquad \|F_{j,r}\|_{0,0}\leqslant\|(\varphi_{j+1,r}-1)P\varphi_{j,r}\|_{0,0}\|Q_\lambda\|_{0,0}\leqslant c_\sigma r^{-l-\sigma}d(\lambda)^{-1},$$

for any $\sigma>0$. By (4.23), $C_{j,r}$ is locally of the form

$$C_{j,r}=[\varphi_{j,r},P]Q_\lambda=\sum_{k=1}^{l-1}\frac{1}{k!}(D^k\varphi_{j,r})\operatorname{Op}(\partial_{\xi_n}^k p)Q_\lambda+S_rQ_\lambda,$$

where $\|S_r\|_{0,0}$ is $O(r^{-l-\sigma})$; here (4.26) implies

$$\begin{aligned}\|(D^k\varphi_{j,r})\operatorname{Op}(\partial_{\xi_n}^k p)Q_\lambda\|_{0,0}&\leqslant cr^{-k}\|\operatorname{Op}(\partial_{\xi_n}^k p)\|_{l-k,0}\|Q_\lambda\|_{0,l-k}\\ &\leqslant c_1r^{-k}|\lambda|^{(l-k)/l}d(\lambda)^{-1}\quad\text{ for } k=1,\ldots,l-1;\end{aligned}$$

and we have as above

$$\|S_rQ_\lambda\|_{0,0}\leqslant c'_\sigma r^{-l-\sigma}d(\lambda)^{-1}.$$

Since $r^{-1}\leqslant|\lambda|^{1/l}$, we find altogether that

$$(4.33)\qquad \|C_{j,r}\|_{0,0}\leqslant c''_\sigma r^{-1-\sigma}|\lambda|^{1-1/l}d(\lambda)^{-1}.$$

Then the first term in (4.31) is estimated by $\operatorname{const}\big(r^{-1-\sigma}|\lambda|^{1-1/l}d(\lambda)^{-1}\big)^M$. For the other terms, we note that

$$\begin{aligned}r^{-l-\sigma}d(\lambda)^{-1}&\leqslant r^{-1-\sigma}|\lambda|^{1-1/l}d(\lambda)^{-1},\quad\text{ since } r^{-1}\leqslant|\lambda|^{1/l};\\ r^{-l-\sigma}d(\lambda)^{-1}&\leqslant\big(r^{-1-\sigma'}|\lambda|^{1-1/l}d(\lambda)^{-1}\big)^l,\quad\text{ since } d(\lambda)\leqslant|\lambda|,\end{aligned}$$

with $\sigma'=\sigma/l$, and hence by interpolation, (4.32) gives

$$(4.34)\qquad \|F_{j,r}\|_{0,0}\leqslant c_\sigma\big(r^{-1-\sigma}|\lambda|^{1-1/l}d(\lambda)^{-1}\big)^s\quad\text{ for any }\sigma>0,$$

for each $s\in[1,l]$. Since $M\leqslant l$, we find (4.29) by applying (4.34) for various choices of s, together with (4.33), to the remaining terms in (4.31). ■

Remark 4.6. It might have seemed more natural to try to prove (4.29) by estimating $(1-\varphi_{M,r})Q_\lambda\varphi_{1,r}$ directly using the calculus of pseudo-differential operators (cf. [19]), but this seems to give weaker results under heavier assumptions (e.g. high order l). The point is that the continuity properties of Q_λ, in their dependence on $d(\lambda)$, do not fully reflect the fact that Q_λ is the inverse of $P-\lambda I$, a property that is used to advantage in the above proof, where commutations are carried over to P, which is independent of λ.

For $x\in\Sigma$, we set $d(x)=x_n$ when $x\in\Sigma^0$, and $d(x)=1$ elsewhere. Then we can conclude for the kernel $K(G_\lambda)(x,y)$ of G_λ:

THEOREM 4.7. *Assume that $l>n$. Let $M\in[0,l]$ and let $\sigma>0$. There is a constant C so that for all $x\in\Omega$, and all $\lambda\in\mathbf{C}\setminus\mathbf{R}_+$ with $|\lambda|\geqslant1$,*

$$(4.35)\qquad |K(G_\lambda)(x,x)|\leqslant C|\lambda|^{n/l}d(\lambda)^{-1}\big(d(x)^{-1-\sigma}|\lambda|^{1-1/l}d(\lambda)^{-1}\big)^M.$$

Proof. The inequality was proved for $M=0$ in (4.28). This also shows that the inequality is trivial when $d(x)\leqslant|\lambda|^{-1/l}$, for then $d(x)^{-\sigma}d(x)^{-1}|\lambda|^{1-1/l}d(\lambda)^{-1}\geqslant1$, since $d(\lambda)\leqslant|\lambda|$ and $d(x)\leqslant1$. It remain to consider $x\in\Omega$ with $d(x)\geqslant|\lambda|^{-1/l}$. If $x=(x',x_n)\in\Sigma'$ with $x_n\leqslant\frac12$ we apply Proposition 4.5 with $r=2x_n$ and $M=l$. By Lemma 4.3, (4.27) and (4.29),

$$\begin{aligned}\|\varphi_{1,r}G_\lambda\varphi_{1,r}\|_{0,0}&=\|\varphi_{1,r}R_\lambda r^+(P-\lambda I)(1-\varphi_{l,r})Q_\lambda e^+\varphi_{1,r}\|_{0,0}\\ &\leqslant c\|R_\lambda\|_{0,0}\|(P-\lambda I)(1-\varphi_{l,r})Q_\lambda\varphi_{1,r}\|_{0,0}\\ &\leqslant c_1|\lambda|d(\lambda)^{-1}\big(x_n^{-1-\sigma}|\lambda|^{1-1/l}d(\lambda)^{-1}\big)^l,\end{aligned}$$

and similarly (since multiplication by $\varphi_{1,r}$ is continuous in $H^{l,\mu}$, uniformly when $\mu\geqslant r^{-1}$)

$$\|\varphi_{1,r}G_\lambda\varphi_{1,r}\|_{0,l}\leqslant c_2d(\lambda)^{-1}\big(x_n^{-1-\sigma}|\lambda|^{1-1/l}d(\lambda)^{-1}\big)^l.$$

Since $(\varphi_{1,r}G_\lambda\varphi_{1,r})^*=\varphi_{1,r}G_{\bar\lambda}\varphi_{1,r}$, with similar properties, a theorem of Agmon (e.g. [2], Lemma 4.13) can be applied to give

$$\begin{aligned}(4.36)\qquad |K(G_\lambda)(x,x)|&=|K(\varphi_{1,r}G_\lambda\varphi_{1,r})(x,x)|\\ &\leqslant c_3|\lambda|^{n/l}d(\lambda)^{-1}\big(d(x)^{-1-\sigma}|\lambda|^{1-1/l}d(\lambda)^{-1}\big)^l,\end{aligned}$$

where c_3 is independent of $d(x)$. For the points $x\in\Omega$ with $d(x)>\frac12$ one chooses a fixed system of functions $\varphi_1',\ldots,\varphi_l'$ in $C_0^\infty(\Omega)$ with $\varphi_1'(x)=1$ for $d(x)\geqslant\frac12$ and $\varphi_{j+1}'\varphi_j'=\varphi_j'$, $j<l$. Then (4.36) is shown by a simpler version of the above deductions. Finally, the validity of (4.35) for $M\in\,]0,l[$ is obtained by interpolation. ■

This result together with Theorem 4.1 (ii) permit an application of the method of proof of Agmon [2], Theorem 3.1. Let $\varepsilon\in\,]0,\frac14]$ (another restriction is added below), and let $\delta=\frac12-\varepsilon$. We use Theorem 4.1 (ii) with $N'=n$, and in this case, l has to be $\geqslant\varepsilon^{-1}(2n+5)$. We use Theorem 4.7 with $M=\delta(\frac12-\delta)^{-1}=1/2\varepsilon-1$, and we assume that $\delta\in\,]n/2(n+1),1/2[$, which means that we restrict to $\varepsilon\in\,]0,1/2(n+1)[$. Here $M\leqslant l$ if

$l \geqslant 1/2\varepsilon - 1$, this is already satisfied when $l \geqslant \varepsilon^{-1}(2n+5)$. Now Agmon's proof applies, when the function called $\delta(x)$ there is replaced by $d(x)^{1+\sigma}$. This gives (since the estimates for larger ε follow from the estimates for small ε):

THEOREM 4.8. *Let $\varepsilon \in]0, \frac{1}{2}]$, let $\sigma > 0$, and assume that $l > \max\{\varepsilon^{-1} \times \times (2n+5),\ 2(n+1)(2n+5)\}$. Then there is a constant c such that the spectral function $e(t;\ x, y)$ of P_γ satisfies* (cf. (1.3))

$$|\operatorname{tr} e(t;\ x, x) - c_P(x) t^{n/l}| \leqslant c t^{(n-1/2+\varepsilon)/l} d(x)^{-1/2+\varepsilon-\sigma}, \tag{4.37}$$

for $t \geqslant 1$ and $x \in \Omega$. [(4.37) holds with $\sigma = 0$ when $\varepsilon = \frac{1}{2}$ and $l > n$.]

Finally, the asymptotic estimate of the eigenvalues of P_γ is obtained as a corollary. Here, it is easy to remove the restrictions on l, and to obtain the same estimate for other realizations, by use of Proposition 2.1 and 2.2. In fact, if G is a singular Green operator of order l, and T is a trace operator, such that the system $\begin{pmatrix} P_\Omega + G \\ T \end{pmatrix}$ is elliptic and invertible (in the framework of [5]) with inverse $(R_T\ K_T)$, and such that R_T is *selfadjoint* and *continuous from $L^2(E)$ to $H^l(E)$*, then we define the realization $(P+G)_T$ as the operator acting like $P_\Omega + G$ and with the domain $R_T(L^2(E))$ [so $(P+G)_T$ is the inverse of R_T: $L^2(E) \to R_T(L^2(E))$]; P_γ is of this kind with $G = 0$, $T = \gamma$. [Theorem 4.8 in itself can be proved for positive selfadjoint P_T without the G and with a standard trace operator T.]

The above hypotheses mean in particular, that $(P+G)_T^{-1} = (P^{-1})_\Omega + G'$, where G' is a singular Green operator of order $-l$ and class 0; and it then follows from the calculus of Boutet de Monvel that for any integer $k > 0$,

$$(P+G)_T^{-k} = (P^{-k})_\Omega + G'', \tag{4.38}$$

where G'' is a singular Green operator of order $-lk$ and class 0. We can finally obtain

THEOREM 4.9. *Let l be even > 0, let P be a selfadjoint positive elliptic operator in $\tilde{E}$, having the transmission property at Γ, and let $(P+G)_T$ be a selfadjoint (not necessarily lower bounded) realization as described above. For any $\varepsilon > 0$, the numbers $N^{\pm}(t;\ (P+G)_T)$ of positive, resp. negative, eigenvalues in the interval $[-t, t]$ satisfy* (cf. (1.12))

$$N^+(t;\ (P+G)_T) = c_P t^{n/l} + O(t^{(n-1/2+\varepsilon)/l}) \quad \textit{for } t \to \infty, \tag{4.39}$$

$$N^-(t;\ (P+G)_T) = O(t^{(n-1)/l}) \quad \textit{for } t \to \infty. \tag{4.40}$$

Proof. Let $\varepsilon > 0$, and let r be an integer, for which $rl > \max\{\varepsilon^{-1}(2n+5),\ 2(n+1)(2n+5)\}$. Then (4.37) holds for the Dirichlet realization of the iterate P^r, called $(P^r)_\gamma$, so we obtain by an integration over Ω:

$$N(t;\ (P^r)_\gamma) = c_{P^r} t^{n/rl} + O(t^{(n-1/2+\varepsilon)/rl}) \quad \text{for } t \to \infty. \tag{4.41}$$

In view of (4.38) (applied with $k = 1$ to $(P^r)_\gamma$),

$$(P+G)_T^{-r} - (P^r)_\gamma^{-1} = \tilde{G},$$

where $\tilde{G}$ is a singular Green operator of order $-rl$ and class 0, for which Proposition 2.2 gives

$$\lambda_j(|\tilde{G}|) = O(j^{-rl/(n-1)}) \quad \text{for } j \to \infty. \tag{4.42}$$

Now Proposition 2.1 applies with $R_1 = (P^r)_\gamma^{-1}$ and $R_2 = (P+G)_T^{-r}$, $\sigma = n/rl$, $\theta = (n - \frac{1}{2} + \varepsilon)/rl$ and $\varrho = (n-1)/rl$. Since $rl \geqslant n-1$, θ' equals $(n - \frac{1}{2} + \varepsilon)/rl$. Since $N^{\pm}(t;\ (P+G)_T) = N^{\pm}(t^r;\ (P+G)_T^r)$, we find (4.39)–(4.40). ∎

The main results in this section were announced in [20].

Let us conclude with some remarks on improvements of (4.40). For one thing, another application of our Proposition 2.2 shows that if P is a scalar differential operator, the estimate (4.39) can be improved to be

$$N^+(t;\ (P+G)_T) = c_P t^{n/l} + O(t^{(n-1+\varepsilon)/l}) \quad \text{for } t \to \infty, \tag{4.43}$$

for any $\varepsilon > 0$, by use of (1.13); this is new for the non lower bounded realizations (and for realizations including a G). Secondly, let us remark that the estimate in Theorem 4.7 is useful for resolvent studies in V_δ, for *any* $\delta \geqslant 0$, so a better estimate of $S_{N,\lambda}$ in (4.1) (possibly obtained by a generalization of the proof in Robert [26]) could imply (4.43) for general scalar ps.d.o.s. P (and some systems).

Added in proof (June 1983). Since the above results were presented, various improvementes of remainder estimates have been achieved, however, mainly, for pseudo-differential systems in the case of manifolds without boundary, and for realizations of differential operators in the case of manifolds with boundary. (Cf. e.g. G. V. Rozenblyum: Zap. Nauchn. Sem. Leningr. 96 (1980); V. Ja. Ivrii: Dokl. Akad. Nauk SSSR 250.6 (1980), 258.5 (1980), 263.3 (1982); and G. Métivier: Journées E. D. P. St. Jean de Monts 1982, exposé 1.)

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SLOWLY DECREASING ENTIRE FUNCTIONS AND CONVOLUTION EQUATIONS

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Introduction

In this paper we present a method for constructing entire functions F: $\mathbf{C} \to \mathbf{C}$ having properties of the following three kinds: 1. F satisfies an estimate from above of Paley–Wiener type; 2. along the real axis F does not decrease very fast in the mean; 3. F has a sequence of real zeros of orders as high as possible. Entire functions with such properties arise naturally in the Fourier analysis of convolution equations in various spaces of distributions. The main ideas for the construction are due to Ehrenpreis and Malliavin [3], a slightly weaker version of the one to be found below appears in [5].

Let us discuss the meaning of the above three types of conditions one by one.

The first one is used to characterize those entire functions which are the Fourier transforms of the convolution operators acting on a given space of distributions. For example, by the celebrated Paley–Wiener–Schwartz theorem an entire function F: $\mathbf{C}^n \to \mathbf{C}$ is the Fourier transform $\hat{f}$ of some $f \in \mathscr{E}'(\mathbf{R}^n)$ ($=$ the space of Schwartz distributions on $\mathbf{R}^n$ with compact support) if and only if there are constants $\tilde{N} \in \mathbf{R}$ and $A > 0$ such that with $\omega = \log(1+|\cdot|)$

$$\text{(PW)} \qquad |F(x+iy)| \leqslant \mathrm{const}\exp\big(\tilde{N}\omega(x)+A|y|\big), \qquad x, y \in \mathbf{R}^n.$$

Recall that $\mathscr{E}'(\mathbf{R}^n)$ is the space of convolution operators on the space $\mathscr{D}'(\mathbf{R}^n)$ of Schwartz distributions on $\mathbf{R}^n$ and also on the space $\mathscr{D}'_F(\mathbf{R}^n)$ of distributions of finite order.

The second type of the above conditions serves for the characterization of the convolutors which are surjective on a fixed space. For example, let m: $[0, +\infty) \to \mathbf{R}$ be a strictly monotonically increasing convex function, and denote by $\mathscr{K}_m$ the Fréchet space of C^∞ functions φ: $\mathbf{R}^n \to \mathbf{C}$ such that $|D^\alpha \varphi| \leqslant \mathrm{const}\exp\big(-m(c|\cdot|)\big)$ for arbitrary $\alpha \in \mathbf{Z}^n_+$ and $c > 0$;