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SLOWLY DECREASING ENTIRE FUNCTIONS AND CONVOLUTION EQUATIONS

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Introduction

In this paper we present a method for constructing entire functions F: $\mathbf{C}\to\mathbf{C}$ having properties of the following three kinds: 1. F satisfies an estimate from above of Paley–Wiener type; 2. along the real axis F does not decrease very fast in the mean; 3. F has a sequence of real zeros of orders as high as possible. Entire functions with such properties arise naturally in the Fourier analysis of convolution equations in various spaces of distributions. The main ideas for the construction are due to Ehrenpreis and Malliavin [3], a slightly weaker version of the one to be found below appears in [5].

Let us discuss the meaning of the above three types of conditions one by one.

The first one is used to characterize those entire functions which are the Fourier transforms of the convolution operators acting on a given space of distributions. For example, by the celebrated Paley–Wiener–Schwartz theorem an entire function F: $\mathbf{C}^n\to\mathbf{C}$ is the Fourier transform $\hat{f}$ of some $f\in\mathscr{E}'(\mathbf{R}^n)$ (= the space of Schwartz distributions on $\mathbf{R}^n$ with compact support) if and only if there are constants $\tilde{N}\in\mathbf{R}$ and $A>0$ such that with $\omega=\log(1+|\cdot|)$

$$\text{(PW)}\qquad |F(x+iy)|\leqslant \operatorname{const}\exp\big(\tilde{N}\omega(x)+A|y|\big),\qquad x,y\in\mathbf{R}^n.$$

Recall that $\mathscr{E}'(\mathbf{R}^n)$ is the space of convolution operators on the space $\mathscr{D}'(\mathbf{R}^n)$ of Schwartz distributions on $\mathbf{R}^n$ and also on the space $\mathscr{D}'_F(\mathbf{R}^n)$ of distributions of finite order.

The second type of the above conditions serves for the characterization of the convolutors which are surjective on a fixed space. For example, let m: $[0,+\infty)\to\mathbf{R}$ be a strictly monotonically increasing convex function, and denote by $\mathscr{K}_m$ the Fréchet space of C^∞ functions φ: $\mathbf{R}^n\to\mathbf{C}$ such that $|D^\alpha\varphi|\leqslant\operatorname{const}\exp\big(-m(c|\cdot|)\big)$ for arbitrary $\alpha\in\mathbf{Z}^n_+$ and $c>0$;

then $f*\mathscr{K}'_m \subset \mathscr{K}'_m$ for every $f \in \mathscr{E}'(\boldsymbol{R}^n)$, i.e. each $f \in \mathscr{E}'(\boldsymbol{R}^n)$ is a convolutor on $\mathscr{K}'_m$. From the theory of convolution equations we cite without proof (see [2], [6], [8], [9], [4], [10]):

THEOREM 1. *Let $f \in \mathscr{E}'(\boldsymbol{R}^n)$, and let X be one of the spaces $\mathscr{D}'$, $\mathscr{D}'_F$, $\mathscr{K}'_m$. Then the following conditions are equivalent:*

(i) $f*X = X$.

(ii) *f has a fundamental solution in X.*

(iii) *The Fourier transform F of f satisfies an estimate of the form*

$$\text{(SD)} \qquad M_F\big(x, \varrho(\omega(x))\big) \geqslant \operatorname{const}\exp\big(-N\omega(x)\big) \quad \text{for } x \in \boldsymbol{R}^n,\ |x| \text{ large},$$

(notation: $M_F(x, r) := \sup\{|F(w)|;\ w \in \boldsymbol{C}^n,\ |w| \leqslant r\}$), *where* $\omega = \log(1+|\cdot|)$, *where N is a real constant, and where $\varrho\colon [0, +\infty) \to (0, +\infty)$ is a function such that*

$$\varrho(t) = \begin{cases} O(t),\ t \to +\infty & \text{if } X = \mathscr{D}', \\ o(t),\ t \to +\infty & \text{if } X = \mathscr{D}'_F, \\ O\left(\dfrac{t}{m^{-1}(t)}\right),\ t \to +\infty & \text{if } X = \mathscr{K}'_m. \end{cases}$$

Recall that F is called *slowly* (resp. *very slowly*; resp. *extremely slowly*) *decreasing* if (SD) is satisfied with $\varrho(t) = O(t)$ (resp. $o(t)$; resp. $\equiv$ const) and $\omega = \log(1+|\cdot|)$.

The third of the above-mentioned conditions comes into the picture if one wants to distinguish between solvability and non-solvability of a convolution equation in different distribution spaces X. A trivial consequence of Theorem 1 is the observation that every convolutor $f \in \mathscr{E}'(\boldsymbol{R}^n)$ which operates surjectively on some of the spaces X does so on every larger one as well. The more difficult question is to the other direction: if f acts surjectively on some X, does it so on any smaller one, i.e. does its Fourier transform satisfy a stronger condition of (SD)-type? In some cases the answer is in the affirmative (see § 2 below), in general, however, it is in the negative. To show this, one employs the following method: from the estimates (PW) and (SD) one derives corresponding estimates for the orders of the real zeros of F (see § 3); then one constructs examples F showing that these estimates are sharp (see § 5), i.e. these F cannot satisfy stronger conditions of (SD)-type, and consequently the corresponding convolutors do not operate surjectively on the smaller space in question.

It was this line of arguments Ehrenpreis and Malliavin used in [3], § 2, in order to prove the existence of distributions $f \in \mathscr{E}'(\boldsymbol{R})$ such that $f*\mathscr{D}'(\boldsymbol{R}) = \mathscr{D}'(\boldsymbol{R})$ but $f*\mathscr{D}'_F(\boldsymbol{R}) \neq \mathscr{D}'_F(\boldsymbol{R})$.

For further results of this kind see [5]. In this article we apply the improved version (see § 4 below) of the construction lemma in [5] to obtain examples in $\mathscr{E}'(\boldsymbol{R}^n)$ which in combination with Theorem 1 lead to the following theorem (for the proof see § 6):

THEOREM 2. *Let m, m_1 be strictly monotonically increasing convex functions on $[0, +\infty)$. Then the following conditions are equivalent:*

(i) *For every $f \in \mathscr{E}'(\boldsymbol{R}^n)$ such that $f*\mathscr{K}'_m = \mathscr{K}'_m$ one has $f*\mathscr{K}'_{m_1} = \mathscr{K}'_{m_1}$.*

(ii) *There is a constant $\varepsilon > 0$ such that $m_1(t) \geqslant m(t^\varepsilon)$ for sufficiently large t.*

Similar results hold for arbitrary convolution operators on $\mathscr{K}'_m$; they will be treated elsewhere (see [10]).

Finally, in § 7, we deal with estimates which hold for arbitrary $f \in \mathscr{E}'(\boldsymbol{R}^n)$. The following result — due to Ehrenpreis ([2], Prop. 4.5) — asserts that the Fourier transform of every $f \in \mathscr{E}'(\boldsymbol{R}^n)$ satisfies an estimate of (SD)-type.

THEOREM 3 (Ehrenpreis). *Let $f \in \mathscr{E}'(\boldsymbol{R}^n)$. Then for every $\varepsilon > 0$ there is a constant $B > 0$ such that for arbitrary $x \in \boldsymbol{R}^n$, $|x| \geqslant B$, one has $M_{\hat f}(x, \varepsilon|x|) \geqslant \exp(-\varepsilon|x|)$.*

We show that the corresponding estimate for the real zeros of $\hat f$ is sharp in general. Here the method of § 4 seems to fail, we make use of the much more refined Beurling–Malliavin Theorem from [1].

2. Sharpening the estimate (SD)

In this section we show to what extent the estimate (SD) can be sharpened for functions satisfying (PW).

From now on let $\omega\colon \boldsymbol{R}^n \to [0, +\infty)$ be a continuous function such that

$$(\alpha) \qquad \omega(x+y) \leqslant \omega(x)+\omega(y) \quad \text{for arbitrary } x, y \in \boldsymbol{R}^n;$$

$$(\beta) \qquad \omega \leqslant d|\cdot|+D, \quad \text{where } d, D \text{ are constants} > 0;$$

$$(\gamma) \qquad \lim_{|x|\to\infty} \omega(x) = +\infty.$$

THEOREM 4. *Let $F\colon \boldsymbol{C}^n \to \boldsymbol{C}$ be an entire function satisfying* (PW) *for some constants $\tilde N$ and A, and let $\varrho_j\colon [0, +\infty) \to (0, +\infty)$, $j = 1, 2$, be continuous functions such that*

$$(1) \qquad \varrho_2(t) < \varrho_1(t) < t \quad \text{for sufficiently large } t.$$

If F satisfies (SD) *for some N with $\varrho = \varrho_1$ and if*

$$(2) \qquad \limsup_{t\to\infty} \frac{\log \dfrac{t}{\varrho_2(t)}}{\log \dfrac{t}{\varrho_1(t)}} < +\infty$$

then F also satisfies (SD) *for some N with $\varrho = \varrho_2$.*

Proof. By (2) we may choose $\eta, t_0 > 0$ such that

$$\log\frac{t}{\varrho_1(t)} \geqslant \frac{1}{1+\eta}\log\frac{t}{\varrho_2(t)} \quad \text{for every } t \geqslant t_0. \tag{3}$$

We fix $x \in \boldsymbol{R}^n$ and set $\tau := \varrho_2(\omega(x))$, $r := \varrho_1(\omega(x))$ and $R := r(r/\tau)^{1/\eta}$. Since in view of (γ) $\omega(x) \geqslant t_0$ if $|x|$ is sufficiently large, it follows from (3) that $\omega(x) \geqslant R$ if $|x|$ is large. Using (α) and (β) and the last estimate we obtain from (PW):

$$M_F(x, R) \leqslant \operatorname{const}\exp\big((\tilde{N} + |\tilde{N}|d + A)\,\omega(x)\big), \qquad x \in \boldsymbol{R}^n.$$

Since in view of (1) and (γ) we have $\tau < r < R$ if $|x|$ is large enough, Hadamard's Three-Circles-Theorem yields

$$M_F(x, \tau) \geqslant M_F(x, r)^{1+\eta} M_F(x, R)^{-\eta}$$

(compare [4], Satz 9). The assertion follows. ∎

3. Estimates for the orders of the zeros of slowly decreasing entire functions

In this section we give the estimates for the orders of the real zeros of entire functions satisfying estimates of the form (PW) and (SD). By $\operatorname{ord}(x, F)$ we denote the order of x as a zero of F (which is by definition equal to 0 if $F(x) \neq 0$).

LEMMA 1. *Let $F\colon \boldsymbol{C}^n \to \boldsymbol{C}$ be an entire function. Let $\tilde{N} \in \boldsymbol{R}$ and $A > 0$ be constants such that* (PW) *holds. And let $\varrho\colon [0, +\infty) \to (0, +\infty)$ be a function and $N \in \boldsymbol{R}$ be a constant such that* (SD) *is fulfilled. Then*

$$\limsup_{|x|\to\infty} \frac{\operatorname{ord}(x, F)}{\omega(x)} \log\left(c\frac{\omega(x)}{\varrho(\omega(x))}\right) \leqslant N + \tilde{N} + (A + d|\tilde{N}|)c$$

for every $c > \limsup\limits_{t\to\infty} \dfrac{\varrho(t)}{t}$.

Proof. Fix $x \in \boldsymbol{R}^n$. For $r > 0$ choose $w \in \boldsymbol{C}^n$ such that $|w| = r$ and $M_F(x, r) = |F(x+w)|$. Application of the maximum principle to the entire function $\boldsymbol{C} \ni \lambda \mapsto \lambda^{-\operatorname{ord}(0,G)} G(\lambda)$ where $G(\lambda) := F(x + \lambda w)$ yields the inequalities

$$\operatorname{ord}(x, F) \leqslant \frac{1}{\theta}\big(\log M_F(x, re^\theta) - \log M_F(x, r)\big) \quad \text{for every } \theta > 0 \tag{4}$$

(see [5], Lemma 1). Setting

$$\theta := \log\left(c\frac{\omega(x)}{\varrho(\omega(x))}\right)$$

we have $\varrho(\omega(x))e^\theta = c\omega(x)$. Since by (α) and (β)

$$\tilde{N}\omega(y) \leqslant \tilde{N}\omega(x) + |\tilde{N}|d\varrho(\omega(x))e^\theta + |\tilde{N}|D$$

for every $y \in K(x, \varrho(\omega(x))e^\theta)$, it follows from (PW) that

$$M_F(x, \varrho(\omega(x))e^\theta) \leqslant \operatorname{const}\exp\big((\tilde{N} + (|\tilde{N}|d + A)c)\,\omega(x)\big). \tag{5}$$

If $c > \limsup\limits_{t\to\infty}(\varrho(t)/t)$ and if $|x|$ is sufficiently large then — in view of (γ) — $\theta > 0$, and the assertion follows by combining (4), (5) and (SD) and taking (γ) into account. ∎

The following results are immediate consequences of Lemma 1. They contain parts of those of Ehrenpreis ([2], Prop. 6.1) and Grudzinski ([5], § 4).

THEOREM 5. *Let $F\colon \boldsymbol{C}^n \to \boldsymbol{C}$ be an entire function satisfying* (PW), *and let $\varrho\colon [0, +\infty) \to (0, +\infty)$ be a function such that* (SD) *holds.*

(i) *If F is slowly decreasing with respect to ω, i.e. $\varrho(t) = O(t)$ as $t \to \infty$, then $\operatorname{ord}(x, F) = O(\omega(x))$ as $|x| \to \infty$.*

(ii) *If F is very slowly decreasing with respect to ω, i.e. $\varrho(t) = o(t)$ as $t \to \infty$, then $\operatorname{ord}(x, F) = o(\omega(x))$ as $|x| \to \infty$; more precisely*

$$\limsup_{|x|\to\infty} \frac{\operatorname{ord}(x, F)}{z(\omega(x))} \leqslant N + \tilde{N},$$

where N, $\tilde{N}$ are the constants in (SD), (PW) *respectively and where (for large t)*

$$z(t) = \frac{t}{\log\dfrac{t}{\varrho(t)}}. \tag{6}$$

COROLLARY 1. *If F is extremely slowly decreasing with respect to ω, i.e.* (SD) *holds with $\varrho \equiv$ const, then*

$$\operatorname{ord}(x, F) = O\left(\frac{\omega(x)}{\log\omega(x)}\right) \quad \text{as} \quad |x| \to \infty.$$

Combining the case $\omega = |\cdot|$ of assertion (ii) of Theorem 5 with Theorem 3 one obtains

COROLLARY 2. *If $f \in \mathscr{E}'(\boldsymbol{R}^n)$ then $\operatorname{ord}(x, f) = o(|x|)$ as $|x| \to \infty$, $x \in \boldsymbol{R}^n$.*

4. The main lemma for the construction of slowly decreasing functions with high order zeros

The lemma of this section is a slight improvement upon Lemma 2 in [5]. The latter is essentially due to Ehrenpreis and Malliavin ([3], § 4).

Let $(t_k)_{k\in\boldsymbol{N}}$ be a sequence of real numbers, and let (m_k) and (l_k) be sequences of positive numbers. Our main hypothesis for the construction of slowly decreasing entire functions postulates the existence of a sequence of numbers $\nu_k \geqslant 1$ with

(7) $$\nu := \sum_{k\in\boldsymbol{N}} m_k \exp\Big(-\frac{\pi}{2}\nu_k\Big) < +\infty$$

such that the points t_k lie so far apart from each other that

(8) the intervals $J_k := [t_k-\tau_k, t_k+\tau_k]$ (where $\tau_k := \nu_k l_k$) are pairwise disjoint having distance greater than 1.

Moreover, we assume that the sequence (l_k) converges to $+\infty$ so that we can define a continuous function $h\colon \boldsymbol{R}\to[0, +\infty)$ by

(9) $$h(y) := \frac{\pi}{2}\sum_{k\in\boldsymbol{N}} \frac{m_k}{l_k}\max\{|y|-l_k, 0\}, \qquad y\in\boldsymbol{R}.$$

LEMMA 2. *Under the preceding assumptions there exists an entire function $F\colon \boldsymbol{C}\to\boldsymbol{C}$ having the following properties*:

(10) $$F\cdot(1+|\cdot|)^{-4}\exp\big(-h(\operatorname{Im}(\cdot))-2\pi|\operatorname{Im}(\cdot)|\big)\in L_2(\boldsymbol{R}^2);$$

(11) $$\operatorname{ord}(t_k, F) > m_k-1$$

and

(12) $$\sup\{|F(x+u)|;\ u\in\boldsymbol{R},\ |u|\leqslant 1\} \geqslant \begin{cases} 1 & \text{if } x\in\boldsymbol{R}\setminus\bigcup_{k\in\boldsymbol{N}} J_k,\\ \Big(\dfrac{1}{e}\min\{1, c_k(x)\}\Big)^{m_k} & \text{if } x\in J_k,\end{cases}$$

where

$$c_k(x) := \frac{1}{l_k}\min\{|y-t_k|;\ y\in x+[-1,1],\ |y-t_k|\geqslant\tfrac{1}{2}\}.$$

Proof. The idea of the proof is as follows: First a suitable subharmonic function is constructed which reflects the desired properties of F (this step is essentially [3], Lemma 4); then F is found by means of the so-called Oka principle: first a C^∞ function having the desired properties is constructed and is then made holomorphic by an application of the solvability theory of the inhomogeneous Cauchy–Riemann equations as developed in Hörmander [7]. The second step is suggested by a result of Bombieri (see for example Hörmander [7], Theorem 4.4.4).

Step 1. Define $g\colon \boldsymbol{C}\to[-\infty, 0]$ by

$$g(z) := \begin{cases} 0 & \text{if } |\operatorname{Im} z|\geqslant 1,\\ \log\left|\dfrac{\exp(\frac{1}{2}\pi z)-1}{\exp(\frac{1}{2}\pi z)+1}\right| & \text{if } |\operatorname{Im} z|\leqslant 1\end{cases}$$

and set

$$v(z) := \sum_{k\in\boldsymbol{N}} m_k g\Big(\frac{z-t_k}{l_k}\Big), \qquad z\in\boldsymbol{C}.$$

Since — as is easily calculated —

(13) $$g(x+iy)\geqslant -4\exp(-\tfrac{1}{2}\pi|x|) \quad \text{for every } x, y\in\boldsymbol{R} \text{ with } |x|\geqslant 1/\pi,$$

it follows from (7) and (8) that

(14) $$v(z)\geqslant -4\nu+\begin{cases} 0 & \text{if } \operatorname{Re} z\notin\bigcup_{k\in\boldsymbol{N}} J_k,\\ m_k g\Big(\dfrac{z-t_k}{l_k}\Big) & \text{if } \operatorname{Re} z\in J_k.\end{cases}$$

Hence $v\colon \boldsymbol{C}\to[-\infty, 0]$ is a well-defined upper semicontinuous function belonging to $L^1_{\mathrm{loc}}(\boldsymbol{R}^2)$. It is not difficult to compute

$$\Delta v = \sum_{k\in\boldsymbol{N}} 2\pi m_k\delta_{t_k} - \sum_{k\in\boldsymbol{N}}\frac{m_k}{l_k}\psi_k\otimes(\delta_{l_k}+\delta_{-l_k})$$

where $\delta_{t_k}\in\mathscr{E}'(\boldsymbol{R}^2)$ and $\delta_{l_k}, \delta_{-l_k}\in\mathscr{E}'(\boldsymbol{R})$ are the Dirac distributions at t_k and l_k, $-l_k$ respectively and

$$\psi_k(x) := \frac{\pi}{2}\Big(\cosh\Big(\frac{\pi}{2}\,\frac{x-t_k}{l_k}\Big)\Big)^{-1}, \qquad x\in\boldsymbol{R},$$

(for details see [5], proof of Lemma 2, step 1). Since $\psi_k\leqslant\pi/2$ and since

$$\frac{d^2h}{dy^2} = \frac{\pi}{2}\sum_{k\in\boldsymbol{N}}\frac{m_k}{l_k}(\delta_{l_k}+\delta_{-l_k})$$

it follows that the function $w\colon \boldsymbol{C}\to[-\infty, +\infty)$, $z\mapsto v(z)+h(\operatorname{Im} z)$, is subharmonic.

Step 2. Choose $0<\varepsilon<1/8$ such that

(15) $$\log\frac{1}{1-4\varepsilon}<\frac{1}{5}.$$

For $x\in\boldsymbol{R}$ we denote by $S(x,\varepsilon)$ the open square $\{t+iy;\ |t-x|<\varepsilon,\ |y|<\varepsilon\}$. Let us choose a bump function $\chi\in C_0^\infty(\boldsymbol{R}^2)$ such that

(16) (a) $\operatorname{supp}\chi\subset S(0, 2\varepsilon)$, (b) $\chi|_{S(0,\varepsilon)}\equiv 1$, (c) $\chi\geqslant 0$.

By $(x_n)_{n\in\boldsymbol{N}}$ we denote the sequence of numbers $x\in\frac{1}{2}\boldsymbol{Z}$ such that the distance of x to every t_k is not smaller than $\frac{1}{2}$. Define $G\colon \boldsymbol{C}\to[0,+\infty)$ by

$$G:=\begin{cases}0 & \text{on } \boldsymbol{R}^2\setminus\bigcup_{n\in\boldsymbol{N}}S(x_n,2\varepsilon),\\ c_n\tau_{x_n}\chi & \text{on } S(x_n,2\varepsilon)\end{cases}\tag{17}$$

where

$$c_n:=\begin{cases}1 & \text{if } S(x_n,2\varepsilon)\cap\bigcup_{k\in\boldsymbol{N}}J_k=\emptyset,\\ \exp\left(m_k\left(-1+\log\min\left\{1,\dfrac{|x_n-t_k|}{l_k}\right\}\right)\right) & \text{if } S(x_n,2\varepsilon)\cap J_k\neq\emptyset.\end{cases}$$

Note that the c_n and G are well-defined by (16.a) and (8). Since G is constant on every square $S(x,\varepsilon)$, $x\in\frac{1}{2}\boldsymbol{Z}$, $\partial G/\partial\bar z$ vanishes there, and we can define a C^∞ function $H\colon\boldsymbol{C}\to\boldsymbol{C}$ by

$$H(z):=\begin{cases}\dfrac{\partial G}{\partial\bar z}(z)/\sin 2\pi z & \text{if } z\in\operatorname{supp}\dfrac{\partial G}{\partial\bar z},\\ 0 & \text{otherwise.}\end{cases}\tag{18}$$

Since $g-\log|\cdot|$ is harmonic in the strip $\{z\in\boldsymbol{C};\ |\mathrm{Im}\,z|<1\}$ we find — making use of the maximum principle —:

$$g(z)-\log|z|\geqslant g(1)-\log\sqrt{2}\geqslant-\tfrac{4}{5}\quad\text{for every } z\in S(0,1).$$

Now let $z\in S(x_n,2\varepsilon)$ such that $\mathrm{Re}\,z\in J_k$. If $|\mathrm{Re}\,z-t_k|\leqslant l_k$ we then conclude that

$$\exp\left(m_kg\left(\frac{z-t_k}{l_k}\right)\right)\geqslant\exp\left(-m_k\left(\frac{4}{5}+\log\frac{l_k}{|x_n-t_k|-2\varepsilon}\right)\right)$$
$$\geqslant c_n\exp\left(m_k\left(\frac{1}{5}-\log\frac{|x_n-t_k|}{|x_n-t_k|-2\varepsilon}\right)\right)\geqslant c_n,$$

here the last inequality is valid since because of $|x_n-t_k|\geqslant\frac{1}{2}$ and (15) the last exponent is nonnegative. If on the other hand $|\mathrm{Re}\,z-t_k|\geqslant l_k$, then by (13) we have $g((z-t_k)/l_k)\geqslant-1$, and the inequality

$$\exp\left(m_kg\left(\frac{z-t_k}{l_k}\right)\right)\geqslant c_n\quad\text{for every } z\in S(x_n,2\varepsilon)\text{ such that }\mathrm{Re}\,z\in J_k\tag{19}$$

is proved in this case as well. Hence we deduce from (14) and (17) that

$$\left|\frac{\partial G}{\partial\bar z}\right|\leqslant\mathrm{const}\,e^v.$$

Since $|\sin(x+iy)|^2=|\sin x|^2+|\sinh y|^2$, the same inequality (with a new constant) holds for H. Consequently the function $He^{-w}(1+|\cdot|^2)^{-1}$ belongs to $L_2(\boldsymbol{R}^2)$. By Hörmander ([7], Theorem 4.4.2) we obtain a solution $u\in L^2_{\mathrm{loc}}(\boldsymbol{R}^2)$ of the equation

$$\frac{\partial u}{\partial\bar z}=-H\tag{20}$$

such that

$$ue^{-w}(1+|\cdot|^2)^{-2}\in L_2(\boldsymbol{R}^2).\tag{21}$$

We set

$$F(z):=G(z)+u(z)\sin 2\pi z,\qquad z\in\boldsymbol{C}.$$

By (20) and (18) we have $\partial F/\partial\bar z=0$, i.e. F is holomorphic. Since by (19) and (17) $|G|$, too, is majorized by $\mathrm{const}\,e^v$, (21) implies

$$F\cdot(1+|\cdot|)^{-4}\exp\left(-w-2\pi|\mathrm{Im}(\cdot)|\right)\in L_2(\boldsymbol{R}^2).\tag{22}$$

Since $v\leqslant 0$, (10) is a consequence of (22). Now, (22) and the harmonicity of $g-\log|\cdot|$ near 0 imply that the function $z\mapsto|z-t_k|^{n_k}$ is locally square-integrable near t_k where $n_k:=\mathrm{ord}(t_k,F)-m_k$; this can only be true if $n_k>-1$, and (11) is proved. Finally, since $d(x,\{x_n;\ n\in\boldsymbol{N}\})\leqslant 1$ for every $x\in\boldsymbol{R}$ and since $F(x_n)=G(x_n)=c_n$, (12) follows from the definition of c_n. ∎

5. Applying the construction lemma

Let us suppose in this section that $\omega\colon\boldsymbol{R}\to[0,+\infty)$ is a continuous function satisfying (α) and (γ) and instead of (β) the stronger condition

$$\omega(x)=O\left(\frac{|x|}{\log|x|}\right)\quad\text{as } |x|\to\infty.\tag{β$'$}$$

So we may fix constants $d,s>0$ such that

$$\sup\{\omega(x);\ |x|\leqslant t\}\leqslant d\frac{t}{\log t}\quad\text{for every } t\geqslant s.\tag{23}$$

With many applications the data of Lemma 2 are sequences (t_k), (m_k) and (l_k) such that

$$\sum_{k\in\boldsymbol{N}}\frac{m_k}{\omega(t_k)}<+\infty\tag{24}$$

and

$$l_k<\frac{\pi}{2d}\,\omega(t_k)\quad\text{for every } k\in\boldsymbol{N}.\tag{25}$$

In the following completely elementary technical lemma we specify additional assumptions under which the hypotheses of Lemma 2 are fulfilled if we set

$$\nu_k := \frac{2}{\pi} \log \omega(t_k). \tag{26}$$

LEMMA 3. *Under the preceding assumptions the following assertions are true.*

(i) (7) *is valid.*

(ii) (8) *holds if* $|t_1| \geqslant \max\{e^d, s\}$ *and if for every* $k \in N$

$$\left|\frac{t_{k+1}}{t_k}\right| > \frac{1+|t_k|^{-1}+L_k}{1-L_{k+1}} \quad \textit{where } L_k := \frac{2d}{\pi}\frac{l_k}{\omega(t_k)}. \tag{27}$$

(iii) *If* $\nu_k l_k \geqslant s$ *we have for every* $x \in J_k$

$$(1-d\varepsilon_k)\,\omega(t_k) \leqslant \omega(x) \leqslant (1+d\varepsilon_k)\,\omega(t_k)$$

where $\varepsilon_k := \max\left\{\frac{2}{\pi}\frac{l_k}{\omega(t_k)},\ \frac{1}{\log \nu_k l_k}\right\}$.

Proof. (i). (7) is (24) in view of (26).

(ii). (26) and (23) imply $\nu_k l_k \leqslant L_k |t_k|$. Since (27) means

$$(1-L_{k+1})\,|t_{k+1}| > (1+L_k)\,|t_k|+1$$

and since by (25) $L_k < 1$, (8) follows.

(iii). By (α) and by (23) we have

$$|\omega(t_k)-\omega(x)| \leqslant d\frac{\tau_k}{\log \tau_k}.$$

From this inequality the assertion follows since

$$\frac{\tau_k}{\log \tau_k} \leqslant \begin{cases} \dfrac{1}{\log \tau_k}\,\omega(t_k) & \text{if } \tau_k \leqslant \omega(t_k), \\[2mm] \dfrac{2}{\pi}\dfrac{l_k}{\omega(t_k)}\,\omega(t_k) & \text{if } \tau_k \geqslant \omega(t_k). \end{cases} \blacksquare$$

For the rest of this section we fix a function $\sigma\colon \boldsymbol{R} \to (0, +\infty)$ and an unbounded subset Σ of $\boldsymbol{R}$ such that

$$\limsup_{|x|\to\infty,\, x\in\Sigma} \sigma(x) = +\infty. \tag{28}$$

The following theorem shows how sharp the estimate for the orders of the zeros in assertion (ii) of Theorem 5 is.

THEOREM 6. *Let* $z\colon [0, +\infty) \to [1, +\infty)$ *be a function such that* $z(t) = o(t)$ *as* $t \to \infty$. *Define* $\varrho\colon [0, +\infty) \to [0, +\infty)$ *by* (6), *i.e.*

$$\varrho(t) = t\exp\left(-\frac{t}{z(t)}\right). \tag{29}$$

Suppose that $\varrho(t) \geqslant 1$ *for sufficiently large* t *and that there is a constant* $\eta > 0$ *such that*

$$\limsup_{t\to\infty} \max\left\{\frac{\varrho(t)}{\varrho(\tau)};\ \tau \in [(1-\eta)t, (1+\eta)t]\right\} < +\infty. \tag{30}$$

Then for every $c > 0$ *there exist an entire function* $F\colon \boldsymbol{C} \to \boldsymbol{C}$ *and a sequence of numbers* $t_k \in \Sigma$ *such that*

$$|F(x+iy)| \leqslant \mathrm{const}(1+|x|)^4 \exp(7|y|), \qquad x, y \in \boldsymbol{R}, \tag{31}$$

such that (SD) *is valid for any* $N > c$, *and such that*

$$\operatorname{ord}(t_k, F) \geqslant cz\big(\omega(t_k)\big) \quad \textit{for every } k \in \boldsymbol{N} \tag{32}$$

and

$$\lim_{k\to\infty} \sigma(t_k) = +\infty. \tag{33}$$

COROLLARY 3 ([5], Theorem 4′). *Let* $z\colon [0, +\infty) \to (0, +\infty)$ *be a function such that* $z(t) = o\big(\omega(t)\big)$ *as* $t \to \infty$. *Then there are* $f \in \mathscr{E}'(\boldsymbol{R})$ *and* $(t_k) \subset (0, +\infty)$ *such that* $\hat{f}$ *is very slowly decreasing with respect to* ω *and* $\operatorname{ord}(t_k, \hat{f}) \geqslant z(t_k)$ *for every* $k \in \boldsymbol{N}$.

The next corollary improves ([5], Theorem 8) and shows that the estimate in Corollary 1 of Theorem 5 is sharp.

COROLLARY 4. *For every* $c > 0$ *there are* $f \in \mathscr{E}'(\boldsymbol{R})$ *and* $(t_k) \subset (0, +\infty)$ *such that* $\hat{f}$ *is extremely slowly decreasing with respect to* ω *and*

$$\operatorname{ord}(t_k, \hat{f}) \geqslant c\frac{\omega(t_k)}{\log \omega(t_k)} \quad \textit{for every } k \in \boldsymbol{N}.$$

The following more general corollary will be needed for the proof of Theorem 2.

COROLLARY 5. *Let* $m\colon [0, +\infty) \to \boldsymbol{R}$ *be a strictly increasing convex function. Then there exist* $f \in \mathscr{E}'(\boldsymbol{R})$ *and* $(t_k) \subset \boldsymbol{R}$ *such that* (SD) *holds for some* N *with* $\varrho(t) = t/m^{-1}(t)$, *such that* (33) *holds and*

$$\operatorname{ord}(t_k, \hat{f}) \geqslant \frac{\omega(t_k)}{\log m^{-1}\big(\omega(t_k)\big)} \quad \textit{for every } k \in \boldsymbol{N}.$$

Proof of Theorem 6. Let $(\eta_k) \subset (0, \pi/2d)$ be a sequence decreasing to 0 as $k \to \infty$. Choose a sequence $(t_k) \subset \Sigma$ satisfying (33) such that $|t_1| \geqslant \max\{e^d, s\}$, such that

$$\left|\frac{t_{k+1}}{t_k}\right| > \frac{2\pi + 2d\eta_k}{\pi - 2d\eta_{k+1}}$$

and

$$\sum_{k\in N} \frac{z(\omega(t_k))}{\eta_k \omega(t_k)} < \frac{7-2\pi}{\pi c}. \tag{34}$$

Set $m_k := cz(\omega(t_k))+1$ and $l_k := \eta_k \omega(t_k)$. Then the assumptions of the assertions (i) and (ii) of Lemma 3 are satisfied. Hence Lemma 2 applies and yields an entire function $F\colon \boldsymbol{C} \to \boldsymbol{C}$ such that (32) holds. Since by (34)

$$\frac{\pi}{2} \sum_{k\in N} \frac{m_k}{l_k} < 7 - 2\pi,$$

(10) implies (31). To deduce (SD) from (12) we fix $\varepsilon > 0$ and choose, using (30), constants $\delta, T > 0$ such that

$$\delta\varrho(t) \leqslant \varrho(\tau) \quad \text{for arbitrary } \tau \in [(1-\eta)t, (1+\eta)t] \text{ and } t \geqslant T. \tag{35}$$

Now fix $k \in \boldsymbol{N}$. If $\varrho(\omega(t_k)) \leqslant 4/\delta$ we have in view of (γ), provided k is sufficiently large:

$$m_k = cz(\omega(t_k))+1 \leqslant c\frac{\omega(t_k)}{\log\frac{\delta\omega(t_k)}{4}} + 1 \leqslant (c+\varepsilon)\frac{\omega(t_k)}{\log\omega(t_k)}$$

and, using the notation of (12),

$$c_k(x)^{m_k} \geqslant \left(\frac{d}{\pi}\right)^{m_k} \exp\left(-(c+\varepsilon)\omega(t_k)\right) \quad \text{for every } x \in \boldsymbol{R}. \tag{36}$$

If $\varrho(\omega(t_k)) \geqslant 4/\delta$ we have for every $y \in \boldsymbol{R}$ such that $|y - t_k| \geqslant \delta\varrho(\omega(t_k)) - 1$: $l_k c_k(y) \geqslant \frac{1}{2}\varrho(\omega(t_k))$ or

$$c_k(y) \geqslant \frac{\delta}{2\eta_k} \exp\left(-\frac{\omega(t_k)}{z(\omega(t_k))}\right);$$

since because of (γ) and $\lim\limits_{t\to\infty} z(t) = +\infty$

$$\frac{m_k\omega(t_k)}{z(\omega(t_k))} \leqslant (c+\varepsilon)\omega(t_k) \quad \text{for } k \text{ large},$$

we conclude that for sufficiently large k

$$c_k(y)^{m_k} \geqslant \left(\frac{d\delta}{\pi}\right)^{m_k} \exp\left(-(c+\varepsilon)\omega(t_k)\right). \tag{37}$$

Since $z(t) = o(t)$, there is for every $b > 0$ a constant $k_b \in \boldsymbol{N}$ such that

$$b^{m_k} \geqslant \exp\left(-\varepsilon\omega(t_k)\right) \quad \text{for every } k \geqslant k_b. \tag{38}$$

Inserting (36) resp. (37) into (12) and applying (38) we see that for sufficiently large k

$$M_F\left(x, \max\{\delta\varrho(\omega(t_k)), 1\}\right) \geqslant \exp\left(-(c+2\varepsilon)\omega(t_k)\right), \quad x \in J_k. \tag{39}$$

Since $\lim\limits_{k\to\infty} \nu_k l_k = +\infty$, the assumption of assertion (iii) of Lemma 3 is fulfilled for large k, and the sequence of numbers ε_k defined there converges to 0. Hence Lemma 3, (iii), has two consequences: $(c+2\varepsilon)\omega(t_k) \leqslant (c+3\varepsilon)\omega(x)$ and, in combination with (γ) and (35), $\delta\varrho(\omega(t_k)) \leqslant \varrho(\omega(x))$ for arbitrary $x \in J_k$ and sufficiently large k. Hence (39) implies (SD) with $N = c+3\varepsilon$. ∎

6. Proof of Theorem 2

Let $\omega = \log(1+|\cdot|)$. Setting

$$\varrho_1(t) := \frac{t}{m^{-1}(t)} \quad \text{and} \quad \varrho_2(t) := \frac{t}{m_1^{-1}(t)}$$

we see that condition (ii) of Theorem 2 is equivalent to (2). Hence the implication (ii) $\Rightarrow$ (i) is a special case of Theorem 4. If, on the other hand, condition (ii) is false this means that (28) holds for

$$\sigma(t) := \frac{\log m_1^{-1}(\omega(t))}{\log m^{-1}(\omega(t))} \quad \text{and} \quad \Sigma = \boldsymbol{R}.$$

Let $f \in \mathscr{E}'(\boldsymbol{R})$ and (t_k) have the properties of the assertion of Corollary 5. Set $g := f\otimes \ldots \otimes f \in \mathscr{E}'(\boldsymbol{R}^n)$. It is then evident that $\hat{g}$ satisfies (SD) with $\varrho = \sqrt{n}\,\varrho_1$ so that by Theorem 1: $g * \mathscr{K}'_m = \mathscr{K}'_m$. Since

$$\begin{aligned}\operatorname{ord}\left((t_k, 0, \ldots, 0), \hat{g}\right) &\geqslant \operatorname{ord}(t_k, \hat{f}) \\ &\geqslant \frac{\omega(t_k)}{\log m^{-1}(\omega(t_k))} = \sigma(t_k)\frac{\omega(t_k)}{\log m_1^{-1}(\omega(t_k))},\end{aligned}$$

Theorem 5, (ii), says in view of (33): there is no constant $c > 0$ such that (SD) holds for some N with $\varrho = c\varrho_2$. Hence by Theorem 1: $g * \mathscr{K}'_{m_1} \neq \mathscr{K}'_{m_1}$. This proves (i) $\Rightarrow$ (ii). ∎

7. A consequence of the Beurling–Malliavin Theorem

The construction in § 5 does not answer the question whether or not the estimates in assertion (i) of Theorem 5 and in Corollary 2 are sharp. In both cases the method of Lemma 2 seems to fail: as for the general. In both cases the method of Lemma 2 seems to fail: as for the first case, setting $m_k := l_k := \omega(t_k)$ entails that h is not majorized by const$|\cdot|$; as for the second case, the same thing happens even if we set $m_k := t_k/\log t_k$ and try to satisfy (7) and (8). In fact, in both cases the estimates are sharp in general. For the first one this was proved in Ehrenpreis and Malliavin ([3], § 2) by a different method which relies heavily on potential theory. For the second one this readily follows from the Beurling–Malliavin Theorem in [1] as we are going to show now

THEOREM 7. *Let z: $[0, +\infty) \to (0, +\infty)$ be a function such that $z(t) = o(t)$ as $t \to \infty$. Then there are a C^∞ function f: $\boldsymbol{R} \to \boldsymbol{C}$ with support contained in $(-1, 1)$ and a sequence $(t_k) \subset (0, +\infty)$ such that* $\operatorname{ord}(t_k, \hat{f}) \geqslant z(t_k)$ *for every $k \in \boldsymbol{N}$.*

Proof. Choose (t_k) such that $t_{k+1} \geqslant 2t_k$ and such that $\sum_{k\in \boldsymbol{N}} z(t_k)/t_k$ converges. Let Λ be the increasing sequence which contains t_k precisely $z(t_k)$ times (we may suppose that $z(t_k) \in \boldsymbol{N}$) for every $k \in \boldsymbol{N}$. By dN_Λ we denote the measure $\sum_{k\in\boldsymbol{N}} z(t_k)\delta_{t_k}$. Let $\varepsilon > 0$. Choose j_ε and $k_\varepsilon > j_\varepsilon$ such that

$$\sum_{j=j_\varepsilon+1}^{\infty} \frac{z(t_j)}{t_j} < \frac{\varepsilon}{4} \tag{40}$$

and

$$\sum_{j=1}^{\infty} \frac{z(t_j)}{t_j} \frac{t_{j_\varepsilon}}{t_{k_\varepsilon} - t_{j_\varepsilon}} < \frac{\varepsilon}{2}. \tag{41}$$

Assertion 1. Every real interval I which has the property

$$\frac{1}{|I|} \int_I dN_\Lambda \geqslant \varepsilon \tag{42}$$

is contained in $\bigcup_{k=0}^{\infty} I_k$ where $I_0 := \bigl(-\sum_{j=1}^{k_\varepsilon-1} \mu_j, t_{k_\varepsilon}\bigr)$ and $I_k := [t_k - \mu_k, t_k + \mu_k]$ for $k \geqslant 1$ with $\mu_j := (1/\varepsilon) z(t_j)$.

Proof of Assertion 1. Let I be an interval satisfying (42), and let $t_l, \ldots, t_k$ (where $l \leqslant k$) be the points of the sequence (t_j) which lie in I. If $l = k$ then by (42) $z(t_k) = \int_I dN_\Lambda \geqslant \varepsilon|I|$, and $I \subset t_k + [-|I|, |I|] \subset I_k$. If $l < k$ then $|I| \geqslant t_k - t_l$, and by (42):

$$\frac{1}{t_k - t_l} \sum_{j=l}^{k} z(t_j) \geqslant \varepsilon; \tag{43}$$

if k were not smaller than k_ε we would have by (40) and (41)

$$\sum_{j=l}^{k} \frac{z(t_j)}{t_k - t_l} \leqslant \sum_{j=l}^{j_\varepsilon} \frac{z(t_j)}{t_j} \frac{t_{j_\varepsilon}}{t_{k_\varepsilon} - t_{j_\varepsilon}} + \sum_{j=j_\varepsilon+1}^{k} \frac{z(t_j)}{t_j} \frac{t_k}{t_k - t_{k-1}} < \varepsilon$$

in contradiction to (43). Hence $k < k_\varepsilon$, and $I \subset (-\infty, t_{k_\varepsilon})$. Since $|I| \leqslant \sum_{j=l}^{k} \mu_j$ by (42), it follows that $I \subset I_0$.

Assertion 2. Let D_k be the isoscele with base I_k and of height $|I_k|$, $k \geqslant 0$, and let $D := \bigcup_{k=0}^{\infty} D_k$. Then

$$\int_D \frac{dxdy}{1+x^2+y^2} < +\infty.$$

Proof of Assertion 2. If k is so large that $\mu_k < \frac{1}{2}t_k$ we have

$$\int_{D_k} \frac{dxdy}{1+x^2+y^2} \leqslant |I_k| \int_{I_k} \frac{dx}{x^2} = \frac{4\mu_k^2}{t_k^2 - \mu_k^2} \leqslant 4\frac{\mu_k}{t_k} = \frac{4}{\varepsilon}\frac{z(t_k)}{t_k}.$$

In view of the choice of (t_k) Assertion 2 follows.

End of the proof of Theorem 7. The Assertions 1 and 2 imply that, in the terminology of [1], the set $\Omega_\varepsilon := \{I; I \text{ satisfies } (42)\}$ is negligible for every $\varepsilon > 0$, and hence that dN_Λ is regular and of density 0. By combining Theorems II and A of Beurling and Malliavin [1] one obtains an entire function F: $\boldsymbol{C} \to \boldsymbol{C}$ of exponential type <1 such that $|F| \leqslant \exp(-|\cdot|^{1/2})$ on $\boldsymbol{R}$ and such that $\operatorname{ord}(t_k, F) \geqslant z(t_k)$ for every $k \in \boldsymbol{N}$. By the Paley–Wiener Theorem there is a C^∞ function f: $\boldsymbol{R} \to \boldsymbol{C}$ with support contained in $(-1, 1)$ such that $F = \hat{f}$. The proof is complete. ■

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Added in proof:

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ON THE "FUNDAMENTAL PRINCIPLE" OF L. EHRENPREIS

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1. Introduction and statement of the "Fundamental Principle"

It is very well known that every solution u to a homogeneous linear ordinary differential equation with constant coefficients

$$P\left(\frac{d}{dx}\right)u = 0,$$

P a non-trivial polynomial, is an exponential-polynomial of the form

$$u(x) = \sum_{P(\lambda)=0,\lambda\in C} q_\lambda(x)e^{\lambda x}, \quad x \in \boldsymbol{R},$$

where the polynomials q_λ have degree strictly less than the multiplicity of the root λ. Around 1960 L. Ehrenpreis [3] stated a "Fundamental Principle" which enabled him to give a genuine extension of this result to homogeneous linear partial differential equations with constant coefficients. In this paper we shall give a detailed proof of the "Fundamental Principle" for a single equation. The proof will be self-contained except for some standard facts from functional analysis and for solvability results on the Cauchy–Riemann equations.

Let $n \in \boldsymbol{N}$ and let P be a polynomial in n variables. We have

$$P(\partial/\partial x)\exp\langle z, x\rangle = P(z)\exp\langle z, x\rangle \quad \text{for } x \in \boldsymbol{R}^n,\ z \in \boldsymbol{C}^n.$$

Hence the equation

$$P(\partial/\partial x)u = 0 \tag{1.1}$$

holds if $u = \exp\langle z, \cdot\rangle$ for some $z \in \boldsymbol{C}^n$ with $P(z) = 0$. Roughly speaking the "Fundamental Principle" states that any solution u of (1.1) is a superposition of such special solutions. Since the zero-variety of P

$$V = \{z \in \boldsymbol{C}^n;\ P(z) = 0\}$$