PARTIAL DIFFERENTIAL EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 10
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1983

PARAMETRICES FOR A CLASS OF P. D. OPERATORS AND APPLICATIONS

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We consider a class of anisotropic P.D. operators defined in a region $\Omega \subset \boldsymbol{R}^n$, and degenerating on a symplectic submanifold of $T^*\Omega\setminus 0$, which is a generalization of certain well-known operator classes (see [6]). The results obtained are used in the construction of parametrices for degenerate operators.

1. Notation

Write $\boldsymbol{R}_z^n = \boldsymbol{R}_x^\nu \times \boldsymbol{R}_y^{n-\nu}$ $(1 \leqslant \nu < n)$ and let $(z, \zeta) = (x, y;\ \xi, \eta)$ be the points in $T^*\boldsymbol{R}^n$.

Let $M = (M_1, M_2)$ be a pair of positive integers and denote by $\mathrm{OPS}_M^m(\Omega)$ $(m \in \boldsymbol{R}, \Omega \subset \boldsymbol{R}^n)$ the class of all P. D. operators of the form

$$Pf(z) = (2\pi)^{-n} \int e^{i\langle z, \zeta\rangle} p(z, \zeta)\hat{f}(\zeta)\, d\zeta \tag{1}$$

such that:

(i) the symbol $p(z, \zeta)$ belongs to $C^\infty(\Omega \times \boldsymbol{R}^n)$;

(ii) there exists a sequence $(p_{m-j})_{j=0}^\infty$ of C^∞-functions on $\Omega \times (\boldsymbol{R}^n \setminus \{0\})$ with the following properties:

(a) $p_{m-j}(x, y; \lambda^{M_1}\xi, \lambda^{M_2}\eta) = \lambda^{m-j} p_{m-j}(x, y; \xi, \eta)$, $\forall j$, $\forall \lambda > 0$,

(b) $p \sim \sum_{j \geqslant 0} p_{m-j}$, i.e.

$$\Big|\partial_z^\alpha \partial_\xi^\beta \partial_\eta^\gamma \Big(p - \sum_{j<N} p_{m-j}\Big)\Big| = O\big((|\xi|^{1/M_1} + |\eta|^{1/M_1})^{m-N-|\beta|M_1-|\gamma|M_2}\big)$$

as $|\xi| + |\eta| \to \infty$, for all N, α, β, γ, locally uniformly in Ω.

The function p_m is called the *principal symbol* of P, and we set

(2) $$\mathrm{Char}(P) = \{(z,\zeta)\in T^*\Omega\setminus 0 \mid p_m(z,\zeta) = 0\}.$$

Operators in $\mathrm{OPS}^m_M(\Omega)$ extend by continuity to operators from $\mathscr{E}'(\Omega)$ to $D'(\Omega)$.

For every $f\in\mathscr{E}'(\Omega)$ we define:

(3) $$\mathrm{WF}_M(f) = \bigcap_{P\in\mathrm{OPS}^0_M(\Omega),\, Pf\in C^\infty(\Omega)} \mathrm{Char}(P).$$

$\mathrm{WF}_M(f)$ is a closed subset of $T^*\Omega\setminus 0$, stable under the dilations $(x,y;\ \xi,\eta)\to(x,y;\ \lambda^{M_1}\xi,\lambda^{M_2}\eta)$, $\lambda>0$, and projects onto sing supp(f) under the natural mapping $\pi\colon T^*\Omega\to\Omega$. It can be proved that for every $P\in\mathrm{OPS}^m_M(\Omega)$

(4) $$\mathrm{WF}_M(Pf)\subset\mathrm{WF}_M(f)\subset\mathrm{WF}_M(Pf)\cup\mathrm{Char}(P),\qquad \forall f\in\mathscr{E}'(\Omega).$$

For a linear continuous operator $A\colon C_0^\infty(\Omega)\to D'(\Omega)$ with distribution kernel $K_A\in D'(\Omega\times\Omega)$ we can define the wave front set $\mathrm{WF}_{(M,M)}(K_A)\subset T^*(\Omega\times\Omega)\setminus 0$ in the obvious way, and so we see that the first inclusion in (4) is a trivial consequence of the fact that for any $P\in\mathrm{OPS}^m_M(\Omega)$ we have

(5) $$\mathrm{WF}_{(M,M)}(K_P)\subset T^*_\Delta(\Omega\times\Omega)\setminus 0\qquad (\Delta = \text{diagonal of } \Omega\times\Omega).$$

For any $t\in\boldsymbol{R}$ we define $H^t_{M,\mathrm{loc}}(\omega)$ $(\omega\subset\Omega)$ as the set of all distributions $u\in D'(\omega)$ such that:

(6) $$\iint (1+|\xi|^{2/M_1}+|\eta|^{2/M_2})^t\,|\widehat{\varphi u}(\xi,\eta)|^2\,d\xi d\eta<\infty,$$

for all $\varphi\in C_0^\infty(\omega)$.

We say that $P\in\mathrm{OPS}^m_M(\Omega)$ is *hypoelliptic in Ω with loss of r anisotropic derivatives*, $r\geqslant 0$, iff for any $t\in\boldsymbol{R}$ and for all $\omega\subset\Omega$ the following implication holds:

(7) $$f\in\mathscr{E}'(\Omega),\qquad Pf\in H^t_{M,\mathrm{loc}}(\omega)\Rightarrow f\in H^{t+m-r}_{M,\mathrm{loc}}(\omega).$$

Operators with loss of 0-derivatives are exactly those operators P for which $\mathrm{Char}(P)$ is empty. For most of the concepts introduced above refer, e.g. to R. Lascar [3].

2. A class of P. D. operators

Let

$$\Sigma = \{(x,y;\ \xi,\eta)\in T^*\Omega\setminus 0 \mid x=\xi=0\}$$

and let k, l be positive rational numbers. By $N^{m,k}_{M,l}(\Omega;\Sigma)$ we denote the class of all operators $P\in\mathrm{OPS}^m_M(\Omega)$ such that:

(i) $\partial_x^\alpha\partial_\xi^\beta p_m|_\Sigma = 0$ if $|\alpha|M_1/l+|\beta|M_1<k$

and

$$\sum_{|\alpha|M_1/l+|\beta|M_1=k}\frac{1}{\alpha!\beta!}\,\partial_x^\alpha\partial_\xi^\beta p_m\Big|_\Sigma t^\alpha\tau^\beta\neq 0\qquad\text{if}\qquad (t,\tau)\neq(0,0);$$

(ii) For $j\leqslant kl/(1+l)$ one has:

$$\partial_x^\alpha\partial_\xi^\beta p_{m-j}|_\Sigma = 0\qquad\text{if}\qquad |\alpha|M_1/l+|\beta|M_1<k-\frac{l+1}{l}j.$$

Remark. When $M_1=M_2=l=1$, $k\in\boldsymbol{Z}_+$, we obtain the classes considered by Sjöstrand [6] (written in a particular system of coordinates).

Various other classes, e.g. those considered by Menikoff [5], are included in $N^{m,k}_{M,l}$.

Let us observe that $\bigcup_{m,k} N^{m,k}_{M,l}$ is an algebra with respect to composition and that it is closed under the involution $P\to P^*$.

Examples (Models of operators), $n=2$:

1) $P = aD_x^2+bx^{2h}D_y^2+cx^{h-1}D_y$
2) $P = aD_x^2+bx^hD_y$
3) $P = ax^lD_y^2+bD_x+cx^{(l-1)/2}D_y$

a, b, c being suitable functions in $C^\infty(\boldsymbol{R}^2)$.

With any operator $P\in N^{m,k}_{M,l}(\Omega;\Sigma)$ we associate a family of differential operators with polynomial coefficients in $\boldsymbol{R}^\nu$, depending on the parameter $\varrho\in\Sigma$. More precisely, for any $\varrho\in\Sigma$ we put:

(8) $$P_\Sigma(\varrho;\ t,D_t) = \sum_{j=0}^{kl/(1+l)}\ \sum_{\frac{|\alpha|M_1}{l}+|\beta|M_1=k-\frac{l+1}{l}j}\frac{1}{\alpha!\beta!}(\partial_x^\alpha\partial_\xi^\beta p_{m-j})(\varrho)\,t^\alpha D_t^\beta.$$

It can be shown that $P_\Sigma(\varrho;\ t,D_t)\colon\mathscr{S}'(\boldsymbol{R}^\nu)\to\mathscr{S}'(\boldsymbol{R}^\nu)$ has a finite index $(\ker P_\Sigma(\varrho)\subset\mathscr{S}(\boldsymbol{R}^\nu),\ldots)$.

Main Theorem. *Let $P\in N^{m,k}_{M,l}(\Omega;\Sigma)$; then:*

I. *If P is hypoelliptic with loss of r derivatives in Ω, then $r\geqslant kl/(1+l)$. If P is hypoelliptic with loss of $kl/(1+l)$ derivatives, then*

$(*)$ $$\ker P_\Sigma(\varrho;\ t,D_t) = \{0\},\qquad\forall\varrho\in\Sigma.$$

II. *Let $(*)$ be satisfied (resp. let $P_\Sigma(\varrho)$ be, for any ϱ, a surjective map from $\mathscr{S}(\boldsymbol{R}^\nu)$ onto $\mathscr{S}(\boldsymbol{R}^\nu)$); then there exists a linear operator*

$$E\colon H^t_{M,\mathrm{loc}}(\Omega)\to H^{t+m-\frac{kl}{1+l}}_{M,\mathrm{loc}}(\Omega),$$

continuous for all t, such that:

(i) $\mathrm{WF}_{(M,M)}(K_E) \subset T^*_\Delta(\Omega\times\Omega)\setminus 0$,

(ii) *$EP-I$ is regularizing (resp. $PE-I$ is regularizing).*

III. *Let* $(*)$ *be satisfied and suppose that* $M_1 < (1+l)M_2$; *then the left parametrix E of* II *is microlocal in the usual sense, i.e.*

$$\mathrm{WF}(K_E) \subset T^*_\Delta(\Omega\times\Omega)\setminus 0 .$$

IV. *If, moreover, P is also a classical isotropic P.D. operator of order $m \geqslant 0$, if $(*)$ is satisfied and if $M_1 < M_2$, then P is hypoelliptic in Ω with loss of $m - m/M_2 + r/M_2$ isotropic derivatives, i.e.*

$$f\in\mathscr{E}'(\Omega),\qquad Pf\in H^t_{\mathrm{loc}}(\omega) \Rightarrow f\in H^{t+m/M_2-r/M_2}_{\mathrm{loc}}(\omega). \tag{9}$$

The proofs of I and II run along more or less well-known lines (in particular, for the construction of the parametrix E we follow closely Sjöstrand [6]).

As regards assertion III, there are strong reasons to think that condition $M_1 < (1+l)M_2$ is almost necessary for the microhypoellipticity of P.

Consider the following example. Let $m_1, m_2 \in \boldsymbol{N}$, $m_1 < m_2$, and write $m_1/m_2 = \lambda_1/\lambda_2$ with λ_1 and λ_2 relatively prime ($m_1 = h\lambda_1$, $m_2 = h\lambda_2$). Consider the operator P in $\boldsymbol{R}^1_x\times\boldsymbol{R}^{n-1}_y$,

$$P = \sum_{j=0}^{h} A_j(y, D_y)\, x^{m_1-\lambda_1 j} D_x^{j\lambda_1},$$

where A_j is a differential operator of order $m_2 - \lambda_2 j$ in $\boldsymbol{R}^{n-1}$. Suppose that the following conditions hold:

(i) $$\sum_{j=0}^{h} \sigma(A_j)(y,\eta)\, x^{m_1-\lambda_1 j}\xi^{j\lambda_1} = 0 \quad \text{iff} \quad x = \xi = 0$$

($\sigma(A_j)$ denoting the principal symbol of A_j);

(ii) $\forall (y,\eta)\in T^*\boldsymbol{R}^{n-1}\setminus 0$, the ordinary equation:

$$\sum_{j=0}^{h} \sigma(A_j)(y,\eta)\, x^{m_1-\lambda_1 j} D_x^{j\lambda_1}\varphi(x) = 0, \qquad \varphi\in\mathscr{S}(\boldsymbol{R}^1_x),$$

has only the trivial solution $\varphi = 0$.

Then we can apply the theorem by choosing $m = k = h\lambda_1\lambda_2$, $M_1 = \lambda_2$, $M_2 = \lambda_1$, $l = 1$. It follows, in particular, that

$$\mathrm{WF}_M(Pf) = \mathrm{WF}_M(f), \qquad \forall f\in\mathscr{E}'. \tag{10}$$

Nevertheless, if $M_1 \geqslant 2M_2$, P is not microlocally hypoelliptic. Indeed, consider the open cone

$$\Gamma = \{(x, y;\ \xi, \eta)\in T^*\boldsymbol{R}^n\setminus 0 \mid\ \eta\neq 0\}.$$

The isotropic principal symbol of P on Γ is $x^{m_1}\sigma(A_0)(y,\eta)$, which vanishes exactly of order m_1 on the surface $x = 0$ and whose hamiltonian vector field is not collinear with the radial vector field. Let Λ be a classical P.D. operator with principal symbol $x(|\xi|^2+|\eta|^2)^{1/2}$; since $\lambda_2 \geqslant 2\lambda_1$; it is easy to see that we can find classical P.D. operators A_α of order $m_2 - m_1$ such that $P \equiv \sum_{0\leqslant\alpha\leqslant m_1} A_\alpha \Lambda^\alpha$ (Levi's condition). We can then apply a result of Chazarain [1] to conclude that not only the equality $\mathrm{WF}(f) = \mathrm{WF}(Pf)$ fails to hold (though $\operatorname{sing\,supp}(f) = \operatorname{sing\,supp}(Pf)$!), but, moreover, that there is a propagation of singularities in the fibers $T^*_{0,y}(\boldsymbol{R}^n)\cap\Gamma$. A typical example is given by the Kannai operator $iD_x - xD_y^2$.

3. Applications

We sketch only some of them.

A. *Operators with symplectic characteristic manifold.* Let M be a manifold and let $P\in\mathrm{OPS}^m(M)$ be a classical P.D. operator in M with symbol $p\sim p_m + p_{m-1} + \ldots$ Let Σ_1, Σ_2 be two conic sub-manifolds of $T^*M\setminus 0$, of codimension ν, and such that

(i) Σ_1 is regular involutive (regular = radial vector field $\notin T(\Sigma_1)^\perp$);

(ii) Σ_2 is involutive;

(iii) $\Sigma = \Sigma_1\cap\Sigma_2 \neq \emptyset$ with transversal intersection, and such that

$$\operatorname{rank}\sigma|_{T_\varrho(\Sigma)^\perp} = 2\nu, \qquad \forall\varrho\in\Sigma \quad \left(\sigma = \sum d\xi_j\wedge dx_j\right).$$

We impose on P the following hypothesis:

$$p_m(z,\zeta) \approx |\zeta|^m\left[d_{\Sigma_1}(z, \zeta/|\zeta|)^k + d_{\Sigma_2}(z,\zeta/|\zeta|)^r\right], \tag{11}$$

for some $k, r\in\boldsymbol{N}$ (d_{Σ_j} being the distance to Σ_j). In the cases $k = 1$ or $k > 1$ and $r = 1$ which imply $\nu = 1$, we obtain a parametrix (left or right) for a class of sub-elliptic operators. These results are well known, only the proof seems to be simple.

EXAMPLE. $D_x + ix^r|D_y|$, $D_x + ix^r D_y$ (r even), $D_x^2 + ixD_y^2$.

When k and r are greater than 1, some hypotheses on the lower order terms in the symbol of P are needed to ensure e.g. hypoellipticity.

Suppose $k = 2$ and $r \geqslant 2$; and assume the hypothesis:

$$|p_{m-1}(z,\zeta)| \lesssim |\zeta|^{m-1}\left[d_{\Sigma_1}(z,\zeta/|\zeta|) + d_{\Sigma_2}(z,\zeta/|\zeta|)^{r/2}\right]^{\left(2-\frac{r+2}{r}\right)_+} \tag{12}$$

where $t_+ = \max(0, t)$. Note that conditions (11) and (12) are invariant under general homogeneous canonical transformations. Then for every $\varrho\in\Sigma$ we construct in $N_\varrho = T_\varrho(T^*M)/T_\varrho(\Sigma)$ a differential operator P_ϱ such that P is microlocally hypoelliptic with loss of $2r/(r+2)$ derivatives iff P_ϱ is injective for every ϱ.

EXAMPLE. $D_x^2+x^{2l}D_y^2+\lambda x^{l-1}D_y$.

The proofs of all the above results consist in reducing by a suitable canonical transform, to the case of

$$\Sigma = \{(x, y;\ \xi, \eta) \in T^*(R_x^\nu \times R_y^{n-\nu})\setminus 0 \mid x = \xi = 0\}$$

so that, microlocally, P belongs to some $N_{1,l}^{m,k}(\boldsymbol{R}^n;\ \Sigma)$, and recognizing that the hypotheses imposed on P allow to apply the main theorem.

B. *Operators with involutive characteristic manifold.* We make the same assumptions on M, Σ_1, Σ_2 as before. Suppose that $P \in \mathrm{OPS}^m(M)$ and suppose that:

$$p_m(z, \zeta) \approx |\zeta|^m d_{\Sigma_1}(z, \zeta/|\zeta|)^k, \qquad k \in \boldsymbol{N}. \tag{13}$$

For well-known reasons we consider only the case $k \geqslant 2$ and define:

$$p'(z, \zeta) = p_{m-1}(z, \zeta) + \frac{i}{2}\sum_{j=1}^{n}\frac{\partial^2 p_m}{\partial z_j \partial \zeta_j}(z, \zeta), \tag{14}$$

which is called the *sub-principal* symbol of P.

For every $\varrho \in \Sigma_1$, let X be a smooth vector field on T^*M which is transversal to Σ_1 at ϱ and define:

$$I_p(\varrho;\ X) = \frac{1}{k!}(X^k p_m)(\varrho) + p'(\varrho). \tag{15}$$

Then if $I_p(\varrho; X) \neq 0$ for every ϱ and X, we get that P is microlocally hypoelliptic with loss of 1 derivative.

EXAMPLE. $D_x^k + iD_y^{k-1}$.

Condition $I_p(p; X) \neq 0$ implies that $p'(\varrho) \neq 0$. In the case where $p'|_{\Sigma_1}$ is identically zero (and $k = 2$), various results are known concerning propagation of singularities for the solutions of $Pu = f$ (see [7]).

We consider here the case of $k = 2$ and suppose that $p'|_{\Sigma_1}$ vanishes exactly of order r on $\Sigma_1 \cap \Sigma_2$. We make the following hypotheses:

(i) p_m takes values in a closed convex cone $\Lambda \subset \boldsymbol{C}$ (with opening $< \pi$);

(ii) $p'|_{\Sigma_1}$ takes values in a closed convex cone $\Lambda' \subset \boldsymbol{C}$;

(iii) $\Lambda \cap (-\Lambda') = \{0\}$.

Under the above hypotheses P is microlocally hypoelliptic with loss of $(2r+2)/(r+2)$ derivatives (a two-sided parametrix can be constructed).

EXAMPLE. $D_x^2 + ix^r D_y$.

Lascar [4] has obtained similar results by a different technique. We can also consider the case of $k \geqslant 2$, $r = 1$ (under some hypotheses on the range of p_m and $p'|_{\Sigma}$) and obtain a two-sided microlocal parametrix.

EXAMPLE. $D_x^k + \lambda x D_y^{k-1}$.

If $r > 1$, then one is usually forced to assume certain hypotheses on the lower order terms of the symbol of P. As before, the idea of the proof is to reduce to the case $\Sigma = \{(x, y;\ \xi, \eta) \mid x = \xi = 0\}$ and to recognize that, at least microlocally, P belongs to some $N_{M,l}^{m,k}(\boldsymbol{R}^n;\ \Sigma)$; but this time M_1, M_2 are not equal to 1 and, anyway, $M_1 < M_2$.

C. *Operators with non-regular characteristic manifold.* We consider here only a well-known example. Let $\Omega \subset \boldsymbol{R}^n$, $\varphi \in C^\infty(\Omega)$ be real function with $X = \varphi^{-1}(0) \neq \emptyset$ and $\operatorname{grad}\varphi \neq 0$ in Ω. Consider the operator

$$P = \varphi(x)^h \sum_{j,k=1}^{n}{}' a_{jk}(x)\frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{n} b_j(x)\frac{\partial}{\partial x_j} + c(x), \tag{16}$$

with smooth coefficients in Ω and $h \in \boldsymbol{N}$. Suppose that

(i) The matrix $A(x) = (a_{jk}(x))$ is real symmetric, $A(x) \geqslant 0$ and $\ker A(x)$ is generated by $\operatorname{grad}_x\varphi(x)$;

(ii) $\langle \operatorname{Re} b(x), \operatorname{grad}_x\varphi(x)\rangle + \varphi(x)^h \operatorname{Tr}(A(x)\operatorname{Hess}\varphi(x)) > 0$, $x \in \Omega$. Then P is hypoelliptic in Ω, has a left parametrix, and is microlocally hypoelliptic iff $h > 1$.

By a suitable change of variables near X, we can reduce (16) to the operator:

$$\tilde{P} = y^h \sum_{1}^{n-1} a_{ij}(y, z)\frac{\partial^2}{\partial z_j \partial z_i} + \beta(y, z)\frac{\partial}{\partial y} + \ldots \tag{17}$$

The hypotheses (i), (ii) allow us to conclude that, locally,

$$\tilde{P} \in N_{M,h}^{2,2}(\boldsymbol{R}^n;\ \Sigma) \quad \text{with} \quad M = (2, 1) \text{ and } \Sigma = \{\ldots \mid y = \eta = 0\}.$$

The condition $(*)$ of the Main Theorem is satisfied. When $h = 1$, P is not microlocally hypoelliptic, as follows from a well-known theorem of Duistermät–Hörmander [2].

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Presented to the Semester
Partial Differential Equations
September 11–December 16, 1978