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# ON THE HYPOELLIPTICITY OF A PSEUDO-DIFFERENTIAL SYSTEMS WITH A DIAGONAL PRINCIPAL SYMBOL

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## 1

The aim of this article is to give some conditions sufficient for the hypoellipticity of a pseudo-differential system with a diagonal principal symbol. All results which will be formulated below are a logical generalization of the relevant theorems of the scalar case and can be deduced from them. The basic facts of the scalar case used here have been already published [3], [5], [6], [8]. It should also be noted that the study of a pseudo-differential system with an arbitrary principal symbol faces considerable difficulties even in the case of the so-called *simple characteristics*. An examination of systems which are, in a sense, close to the scalar equation is necessiated by this circumstance.

## 2

In a domain $\Omega \subset \boldsymbol{R}^n$ we shall investigate some systems of pseudo-differential equations with the following symbol:

$$p(x, \xi) \sim p_m^0 I_N + p_{m-1} + p_{m-2} + \dots, \tag{1}$$

where $p_m^0(x, \xi)$ is a scalar function, positively homogeneous of order $m$ with respect to $\xi$, and $I_N$ is the identity operator in $\boldsymbol{C}^N$. The remaining matrix-functions $p_{m-j}(x, \xi)$ are positively homogeneous with respect to $\xi$ of order $(m-j)$ $(\mathrm{ord}_\xi p_{m-j} = m-j)$. We remind that the subprincipal symbol $p_{m-1}$ is defined in the following way:

$$p'_{m-1}(x, \xi) = p_{m-1}(x, \xi) + \frac{i}{2}\sum_{j=1}^{n} \frac{\partial^2 p_m^0}{\partial x_j \partial \xi_j}(x, \xi) I_N.$$

It is known that the subprincipal symbol $p'_{m-1}$ of the operator (1) is invariant under homogeneous canonical transformations at those points $(x^0, \xi^0)$ for which $p_m^0(x^0, \xi^0) = \mathrm{grad}_{x,\xi} p_m^0(x^0, \xi^0) = 0$.

Suppose that $E$ is a real finite-dimensional symplectic space in which there acts the standard symplectic form $\sigma = \sum_{j=1}^{n} d\xi_j \wedge dx_j$. Then each symmetric quadratic form $Q\colon E \to \boldsymbol{C}^1$ possesses a polarized form $Q\colon \tilde{E}\times\tilde{E} \to \boldsymbol{C}^1$, which is symmetric over the complexification $\tilde{E}$ of the initial space $E$. Owing to the non-degeneracy of $\sigma$ we conclude that there exists a linear map $F_Q\colon \tilde{E} \to \tilde{E}$ skew-symmetric with respect to $\sigma$ such that

$$(2) \qquad Q(X, Y) = \sigma(X, F_Q Y), \quad \forall (X, Y) \in \tilde{E}\times\tilde{E}.$$

DEFINITION 1. The map $F_Q$ defined by (2) will be called a *fundamental (Hamilton) map of* $Q$.

Denote by $\Gamma$ a closed angle in $\boldsymbol{C}^1$ whose opening is strictly less than $\pi$. Assume that $Q(X) \in \Gamma$, $\forall X \in E$. Then it is quite clear that the eigenvalues of $\frac{1}{i} F_Q$ belong to $\Gamma$ or $(-\Gamma)$. In the sequel we shall denote those eigenvalues which lie in $\Gamma$ by $\mu_j$. In the special case of $Q\colon E \to \boldsymbol{R}^1$, $Q \geqslant 0$ it is easy to understand that $\mathrm{spec}(F_Q) \subset i\boldsymbol{R}^1$ is symmetrically situated with respect to the origin.

Here $N_0$ stands for the linear subspace of $\boldsymbol{C}^N$ which consists of all generalized eigenvectors belonging to the eigenvalue 0 (i.e., $\tilde{X} \in N_0 \Leftrightarrow \exists k \in \boldsymbol{Z}_+\colon F^k \not\equiv 0,\ F^k\tilde{X} = 0$). Suppose that the space $\boldsymbol{C}^N$ is supplied with the standard Hermitian scalar product.

Let us note that if $p_m^0(x^0, \xi^0) = \mathrm{grad}_{x,\xi} p_m^0(x^0, \xi^0) = 0$ then the Taylor expansion of $p_m^0(x, \xi)$ in a neighbourhood of $(x^0, \xi^0)$ begins with a symmetric quadratic part which may be denoted by $Q(X)$, $X \in T\ (T^*_{(x^0,\xi^0)}(\Omega))$. All the results of linear algebra mentioned above are valid for $Q$ if we define $F_Q = F_{p_m^0(x^0,\xi^0)}$.

After these preliminaries of linear algebra we pass to the formulation of the basic results.

THEOREM 1. *Let $P(x, D)$ be an operator of type* (1) *with a real principal symbol $p_m^0$. Suppose that for each $\varepsilon > 0$ and each compact set $K \Subset \Omega$ there exists a constant $C(K, \varepsilon)$ such that*

$$(3) \qquad \mathrm{Re}(Pu, u) + \varepsilon \|u\|^2_{\frac{m-1}{2}} \geqslant C(K, \varepsilon)\|u\|^2_{\frac{m-2}{2}}, \qquad \forall u \in C_0^\infty(K, \boldsymbol{C}^N).$$

*Then the estimate* (3) *is equivalent to the requirement that the following algebraic conditions be fulfilled:*

(i) $p_m^0(x, \xi) \geqslant 0$,

(ii) *at each point* $(x^0, \xi^0) \in T^*(\Omega)\setminus\{0\}$ *for which* $p_m^0(x^0, \xi^0) = 0$ *we have the inequalities*:

$$\gamma_k(x^0, \xi^0) + \sum_{j=1}^{n} \mu_j(x^0, \xi^0) \geqslant 0, \qquad k = 1, \ldots, N,$$

*where $\gamma_k(x^0, \xi^0)$ are the eigenvalues of the Hermitian part of the matrix* $p_{m-1}(x^0, \xi^0)$ (i.e. $\gamma_k(x^0, \xi^0) \in \mathrm{spec}\,\frac{p_{m-1}+p^*_{m-1}}{2}(x^0, \xi^0)$).

Let $\Sigma$ denote the characteristic set of the symbol $p_m^0$:

$$\Sigma = \{(x, \xi) \in T^*(\Omega)\setminus\{0\}\colon p_m^0(x, \xi) = 0\}.$$

Then we have two corollaries to Theorem 1:

COROLLARY 1. *If the condition* (i) *is satisfied for the system* (1) *and if at each point $(x, \xi) \in \Sigma \cap K$ the inequalities*

$$\gamma_k(x, \xi) + \sum_{j=1}^{n} \mu_j(x, \xi) > 0, \qquad k = 1, 2, \ldots, N$$

*are fulfilled, then there exist constants $C(K) > 0$, $C_1(K)$ such that:*

$$\mathrm{Re}(Pu, u) \geqslant C(K)\|u\|^2_{\frac{m-1}{2}} + C_1(K)\|u\|^2_{\frac{m-2}{2}}, \qquad \forall u \in C_0^\infty(K, \boldsymbol{C}^N).$$

COROLLARY 2. *Suppose that the principal symbol of the system* (1) *is non-negative and that the matrix $p_{m-1}$ is skew-symmetric* (i.e. $p^*_{m-1} = -p_{m-1}$). *Further, suppose that for each point $(x^0, \xi^0) \in \Sigma$ either the fundamental matrix $F_{p_m^0(x^0,\xi^0)}$ has a non-zero spectrum or the matrix $p'_{m-1}(x^0, \xi^0)$ is non-degenerate. Then the system* (1) *is hypoelliptic with loss of one derivative with respect to elliptic operators: the conditions $Pu \in H^s_{\mathrm{loc}}$, $u \in \mathscr{E}'(\Omega, \boldsymbol{C}^N)$ imply $u \in H^{s+m-1}_{\mathrm{loc}}$, $p_{m(j)} I_N u \in H^{s-1/2}_{\mathrm{loc}}$, $p_m^{(j)} I_N u \in H^{s+1/2}_{\mathrm{loc}}$, $j = 1, \ldots, n$; $\forall s \in \boldsymbol{R}^1$.*

And now we present a theorem on the hypoellipticity of a system of type (1) whose principal symbol $p_m^0$ describes a closed angle $\Gamma$ with opening strictly less than $\pi$:

$$(4) \qquad p_m^0(x, \xi) \in \Gamma, \quad \forall (x, \xi) \in T^*(\Omega)\setminus\{0\} \quad (\Gamma \subset \boldsymbol{C}^1).$$

THEOREM 2. *Let the principal symbol $p_m^0$ of a system* (1) *satisfy condition* (4) *and let it vanish precisely to the second order on the smooth manifold* $\Sigma$.

*Suppose that for each compact set $K \Subset \Omega$ and for each real number $s$ one can find a constant $C(K, s)$ such that*

$$(5) \qquad \|u\|_{s+m-1} \leqslant C(K, s)(\|Pu\|_s + \|u\|_{s+m-2}), \qquad \forall u \in C_0^\infty(K, \boldsymbol{C}^N).$$

*Then the estimate* (5) *is equivalent to the accomplishment of the algebraic condition* (iii) *at each point* $(x^0, \xi^0) \in \Sigma$:

(iii) $\gamma_k(x^0, \xi^0) + Q(\bar{v}, v) + \sum (2a_j+1)\mu_j(x^0, \xi^0) \neq 0$, $k = 1, \ldots, N$; $\forall v \in N_0$, $\forall a_j \in \boldsymbol{Z}_+$, *where* $\gamma_k(x^0, \xi^0)$ *are the eigenvalues of the subprincipal matrix* $p'_{m-1}(x^0, \xi^0)$.

Under the assumption of (iii) the operator $P(x, D)$ is hypoelliptic with loss of one derivative in comparison with elliptic operators.

Obviously there arises the problem of investigating the hypoellipticity of systems for which the range of values of the principal symbol $p_m^0$ is a half-plane or even the whole plane. Before formulating the respective theorems we remind that the principal symbol $C^0_{2m-1}(x, \xi)$ of the commutator $[p_m^{0*}, p_m]$ may be represented as:

$$C^0_{2m-1}(x, \xi) = 2\,\mathrm{Im}\sum_{j=1}^{n} \overline{\frac{\partial p_m^0}{\partial \xi_j}}\,\frac{\partial p_m^0}{\partial x_j}.$$

Further, we impose two conditions on $\mathrm{grad}_{x,\xi} p_m^0(x, \xi)$ and $C^0_{2m-1}(x, \xi)$:

(iv) (a) $(x^0, \xi^0) \in \Sigma$, $\mathrm{grad}_{x,\xi} p_m^0(x^0, \xi^0) \neq 0 \Rightarrow C^0_{2m-1}(x^0, \xi^0) > 0$;

(iv) (b) if $(x^0, \xi^0) \in \Sigma$ and $\mathrm{grad}_{x,\xi} p_m^0(x^0, \xi^0) = 0$ then one can find a conic neighbourhood $\omega \ni (x^0, \xi^0)$ and constant $c(\omega) > 0$ such that

$$C^0_{2m-1}(x, \xi) \geqslant c(\omega)\left(|\mathrm{grad}_x p_m^0|^2 |\xi|^{-1} + |\mathrm{grad}_\xi p_m^0|^2 |\xi|\right), \qquad \forall (x, \xi) \in \omega;$$

(v) the fundamental matrix $F_{C^0_{2m-1}(x^0,\xi^0)}$ has a non-zero spectrum at each point $(x^0, \xi^0) \in \Sigma$ at which the condition (iv) (b) is satisfied.

THEOREM 3. *Suppose that for a system* (1) *the conditions* (iv), (v) *are fulfilled. Then the operator* $P$ *is hypoelliptic* (*of the lower order terms being without importance*) *and it admits loss of one derivative in comparison with elliptic equations.*

COROLLARY 3. *Let us assume that the characteristic manifold* $\Sigma$ *of the function* $p_m^0$ *is symplectic, i.e.* $\sigma|_{T(\Sigma)}$ *is a non-degenerate 2-form. Besides, let* $\mathrm{grad}_{x,\xi} p_m^0|_\Sigma = 0$ *and let* $C^0_{2m-1}$ *vanish precisely to the second order on* $\Sigma$, $C^0_{2m-1}|_{U\setminus\Sigma} > 0$, *where* $U$ *is a certain neighbourhood of* $\Sigma$. *Then the operator* $P$ *is hypoelliptic, the lower-order terms being without importance.*

On getting acquainted with Theorem 2 there arises the problem of investigating the system in the case where condition (iii) is violated. This question is interesting even in the scalar case. That is why we present a result which to a certain degree is opposite to that formulated in Theorem 2.

DEFINITION 2. Call a polynomial $p(t;\ \xi_1, \eta)$, $t \in \boldsymbol{R}^{n-1}$, $x_1 \in \boldsymbol{R}^1$, $(\xi_1, \eta) \in \boldsymbol{R}^1 \times \boldsymbol{R}^{n-1}$ *quasihomogeneous* if there exist real constants $\delta > 0$, $r$ such that:

$$p(t/\lambda;\ \lambda^{1+\delta}\xi_1, \lambda\eta) = \lambda^r p(t;\ \xi_1, \eta), \qquad \forall \lambda > 0.$$



THEOREM 4. *Suppose that a scalar quasihomogeneous operator* $P$ *satisfies the following requirement:*

$$\dim\ker\left(P(t, 1, D_t) \cap S(\boldsymbol{R}^{n-1})\right) \neq 0$$

$$\left(\dim\ker\left(P(t, -1, D_t) \cap S(\boldsymbol{R}^{n-1})\right) \neq 0\right)$$

*and* $P(t, 1, D_t)$ $(P(t, -1, D_t))$ *is elliptic. Then* $P^*$ *is locally nonsolvable in the class of distributions* $D'$ *in each neighbourhood of the origin.*

($S(\boldsymbol{R}^N)$ denotes the space of smooth functions rapidly decreasing at infinity.)

COROLLARY 4. *Let* $P$ *be a quasihomogeneous second order differential operator in* $\boldsymbol{R}^2$ *with non-vanishing coefficient before* $D_t^2$. *Assume that*

$$\dim\ker\left(P(t, 1, D_t) \cap S(\boldsymbol{R}^1)\right) \neq 0$$

$$\left(\dim\ker\left(P(t, -1, D_t) \cap S(\boldsymbol{R}^1)\right) \neq 0\right).$$

*Then* $P^*(t, D_{x_1}, D_t)$ *is locally non-solvable at the origin.*

Evidently, $P$ is a non-hypoelliptic operator if the requirements of Theorem 4 are satisfied.

## 3

The proofs of Theorems 1 and 2 follow with insignificant alterations the proofs of Theorem 3.1 from [5] and Theorem 1.1 from [6], respectively. That is why we shall restrict ourselves only to the formulation of two basic lemmas, matrix analogues of Theorem 2.4 from [5] and Theorem 3.3 from [6]. The remaining arguments are completed by using the method of localization, in much the same way as it has been done in the two articles just referred to.

Let us consider in $\boldsymbol{C}^N$ the following system:

$$\sum_{|\alpha+\beta|=2} \frac{1}{\alpha!\beta!}\, a_{\alpha\beta}\, y^\beta D^\alpha I_N u + Au = f, \qquad u \in C_0^\infty(\boldsymbol{R}^n, \boldsymbol{C}^N).$$

In the above equation, $Q(x, \xi) = \sum_{|\alpha+\beta|=2} \frac{a_{\alpha\beta}}{\alpha!\beta!}\, y^\beta \xi^\alpha$ is a polynomial with real coefficients and $A$ is an arbitrary $(N\times N)$-matrix with constant elements.

LEMMA 1. *The inequality*

$$\mathrm{Re}\left(\left(Q(x, D)I_N + A\right)u, u\right) \geqslant 0, \qquad \forall u \in C_0^\infty(\boldsymbol{R}^n, \boldsymbol{C}^N) \tag{6}$$

*holds if and only if:*

(vi) $Q(x, \xi) \geqslant 0$ $\forall (x, \xi) \in T^*(\boldsymbol{R}^n)$,

(vii) $\gamma_k + \sum_{j=1}^{n} \mu_j \geqslant 0$; $k = 1, \ldots, N$.

*We should remind once again that*

$$\gamma_k \in \operatorname{spec} \frac{A+A^*}{2}, \quad \mu_j \in \operatorname{spec}\left(\frac{1}{i} F_Q\right), \quad \mu_j \geqslant 0; \quad j = 1, 2, \ldots, n.$$

To prove this lemma, let us note that $A = \dfrac{A+A^*}{2} + \dfrac{A-A^*}{2} = A_1 + A_2$. Since $A_1$ is a Hermitian matrix, $(A_1u, u)$ is real, while $(A_2u, u)$ is purely imaginary. We know from linear algebra that there exists a unitary matrix $U$ such that $U^*A_1U = \operatorname{diag}(\gamma_1, \ldots, \gamma_N)$ where $\operatorname{diag}(\gamma_1, \ldots, \gamma_N)$ denotes the diagonal matrix with the real eigenvalues $\gamma_1, \ldots, \gamma_N$ of $A_1$ on its diagonal. Let us set $u = Uv$. Then (6) is equivalent to

$$\operatorname{Re}\big((QI_N + A_1)Uv, Uv\big) \geqslant 0, \quad \forall v \in C_0^\infty(\boldsymbol{R}^n, \boldsymbol{C}^N),$$

i.e.

$$\operatorname{Re}\big((U^*QU + U^*A_1U)v, v\big) \geqslant 0.$$

Since $U^*U = I_N$, then (6) is fulfilled if and only if

$$\text{(6')} \qquad \operatorname{Re}\sum_{l=1}^{N} \big(Q(x, D)v_l + \gamma_k v_l, v_l\big) \geqslant 0, \quad \forall v_l \in C_0^\infty(K).$$

The scalar theorem of Melin, applied to the inequality (6′), gives the desired result.

Let us consider the following differential quadratic form:

$$Q^s(x, D) = \frac{1}{2} \sum_{|\alpha+\beta|=2} \frac{a_{\alpha\beta}}{\alpha!\beta!}(y^\beta D^\alpha + D^\alpha y^\beta) = Q(x, D) + \frac{1}{2i}\sum_{|\alpha|=1} a_{\alpha\alpha}.$$

Suppose that the condition (4) is fulfilled for $Q$, i.e. the range of $Q$ is contained in a closed angle $\Gamma \subset \boldsymbol{C}^1$ of opening strictly less than $\pi$. As in Lemma 1, $A$ will be an arbitrary matrix of the type $N \times N$.

LEMMA 2. *Assume that the symbol $Q$ satisfies* (4), $\mu_j \in \operatorname{spec}\left(\frac{1}{i} F_Q\right)$, $\mu_j \in \Gamma \setminus \{0\}$ *and let $N_0$ be the space of generalized eigenvectors of $F_Q$, belonging to the eigenvalue* 0. *Then the estimate:*

$$\text{(7)} \qquad \|u\|_0 \leqslant C\big\|\big(Q^s(x, D)I_N + A\big)u\big\|_0, \quad \forall u \in C_0^\infty(\boldsymbol{R}^n, \boldsymbol{C}^N)$$

($C = \text{const} > 0$) *is valid if and only if for each* $\gamma_k \in \operatorname{spec}(A)$

$$\gamma_k + Q(\bar{v}, v) + \sum (2a_j + 1)\mu_j \neq 0$$

*for* $v \in N_0$ *and* $0 \leqslant a_j \in \boldsymbol{Z}$; $k = 1, \ldots, N$.

One can easily deduce from the proof of Theorem 3.3 from [6] that the inequality (7) holds only under the following condition:

$$\big(Q^s(x, D)I_N + A\big)v = 0, \quad v \in S(\boldsymbol{R}^n, \boldsymbol{C}^N) \Leftrightarrow v = 0.$$

In fact, $Q^sI_N + A$ is a Fredholm operator whose index is equal to zero. Let a matrix $B$ transform $A$ into the Jordan normal form: $B^{-1}AB = J = \operatorname{diag}(J_1, \ldots, J_{\tilde{s}})$, where $J_r$ is the corresponding Jordan block. Let us consider the equation $\big(Q^s(x, D)I_N + A\big)v = 0$, $v \in S(\boldsymbol{R}^n, \boldsymbol{C}^N)$. After setting $v = Bw$ we get:

$$B^{-1}Q^sBw + B^{-1}ABw = 0 \Rightarrow (Q^sI_N + B^{-1}AB)w = 0,$$

$w \in S(\boldsymbol{R}^n, \boldsymbol{C}^N)$.

For simplicity we write down only the first Jordan block $J_1$:

$$J_1 = \begin{pmatrix} \gamma_1 & 1 & & \\ & \gamma_1 & \ddots & \\ & & \ddots & 1 \\ & & & \gamma_1 \end{pmatrix}_{(r\times r)}.$$

Evidently,

$$\begin{aligned} Q^s(x, D)w_1 + \gamma_1 w_1 + w_2 &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ Q^s(x, D)w_{r-1} + \gamma_1 w_{r-1} + w_r &= 0, \\ Q^s(x, D)w_r + \gamma_1 w_r &= 0. \end{aligned}$$

Then according to the scalar theorem of Hörmander

$w_r = 0 \Leftrightarrow$

$$\text{(8)} \qquad \gamma_1 + Q(\bar{v}, v) + \sum(2a_j+1)\mu_j \neq 0, \quad \forall v \in N_0, \ 0 \leqslant a_j \in \boldsymbol{Z}.$$

The above fact leads to the conclusion that (8) implies: $w_{r-1} = w_{r-2} = \ldots \ldots = w_1 = 0$. Thus everything is proved.

A short proof of Corollary 2 will be given here. To this end it suffices to apply Theorem 1 to the following operator $T$:

$$T_{2m-1}(x, \xi) = |\xi|^{m-1}P + (P^* - P)^* \cdot (P^* - P) + 4[P^*, P].$$

Thus we obtain an a priori estimate:

$$\text{(9)} \qquad \|u\|_{m-1}^2 \leqslant C(K)(\|Pu\|_0^2 + \|u\|_{m-2}^2), \quad \forall u \in C_0^\infty(K, \boldsymbol{C}^N)$$

$(C(K) > 0)$. (In fact $\operatorname{spec}(p'_{m-1}p'^*_{m-1}) \subset \boldsymbol{R}^1_+$.)

The well-known inequality

$$|\mathrm{grad}_x p_m^0|^2\,|\xi|^{-1+2s}+|\mathrm{grad}_\xi p_m^0|^2\,|\xi|^{2s+1}\leqslant \mathrm{const}\,p_m^0\,|\xi|^{2s+m-1}$$

shows that for each compact set $K \Subset \Omega$ and for each real number $s$ one can find a positive constant $C(K, s)$ such that:

$$\sum_{j=1}^{n}\big(\|p_m^{(j)}(x, D)u\|_{s+1/2}^2+\|p_{m,(j)}(x, D)u\|_{s-1/2}^2\big) \leqslant C(K, s)\big(\|p_m(x, D)u\|_s^2+\|u\|_{s+m-1}^2\big), \quad \forall u \in C_0^\infty(K), \tag{10}$$

where

$$p_m^{(j)}(x, \xi) = \frac{\partial p_m}{\partial \xi_j}(x, \xi) \quad \text{and} \quad p_{m,(j)}(x, \xi) = \frac{\partial p_m}{\partial x_j}(x, \xi).$$

The two estimates (9) and (10) give us the desired hypoellipticity [6].

*Remark* 1. Theorem 2 can be proved without using the method of localizations in the special case when the subprincipal matrix has a full system of single-valued continuous eigenvalues $\gamma_1(x, \xi), \ldots, \gamma_N(x, \xi)$. It is sufficient to apply a special pseudo-differential partition of unity and Theorem 4.4 from [11].

## 4

The proof of Theorem 3 is quite different from the proof of the corresponding result in the scalar case since it is not obligatory for the matrix $p_{m-1}$ to be normal. That is why a detailed proof will be given here.

In order to make our exposition clear we now formulate two results regarding the scalar case [9].

LEMMA 3. *Assume that the symbol $p_m^0$ satisfies the conditions* (iv) *and* (v). *Then for every compact set $K \Subset \Omega$ there exist a constant $E > 0$, depending on $K$ and the principal symbol $p_m^0$ but independent of the lower order terms, and a constant $C(K, P) > 0$, such that*

$$\|u\|_{m-1} \leqslant E(K)\,\|Pu\|_0 + C(K, P)\,\|u\|_{m-2}, \quad \forall u \in C_0^\infty(K). \tag{9'}$$

LEMMA 4. *Under the assumptions of Lemma 3 for each real number $s$ and for each compact $K \Subset \Omega$ one can find a constant $C(K, s) > 0$ such that*

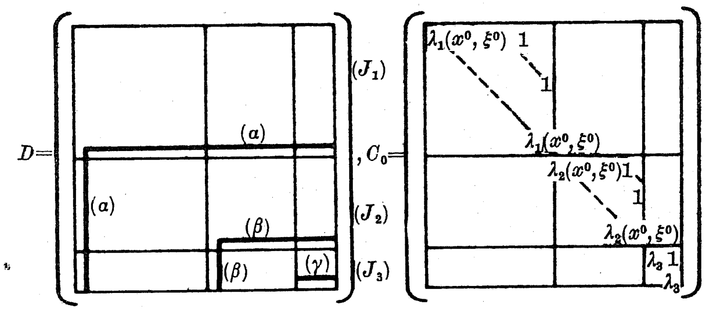


$$\sum_{j=1}^{n}\big(\|p_m^{(j)}(x, D)u\|_{s+1/2}^2+\|p_{m,(j)}(x, D)u\|_{s-1/2}^2\big) \leqslant C(s, K)\big(\|p_m^0(x, D)u\|_s^2+\|u\|_{s+m-1}^2\big), \quad \forall u \in C_0^\infty(K).$$

Since $p_{m-1}(x, \xi) = |\xi|^{m-1}p_{m-1}(x, \xi/|\xi|)$, it is sufficient to examine the matrix $p_{m-1}$ on the unit sphere $S_\xi^{n-1} = \{\xi \in \boldsymbol{R}^n\colon |\xi| = 1\}$ only. Let us fix an arbitrary point $(x^0, \xi^0)$, $\xi^0 \in S^{n-1}$, and suppose that $\lambda_i(x^0, \xi^0) \in \mathrm{spec}\big(p_{m-1}(x^0, \xi^0)\big)$. Then there exists a non-degenerate matrix $B$ with constant elements such that $B^{-1}p_{m-1}(x^0, \xi^0)B = C_0$ is represented in the Jordan normal form: $C_0 = \mathrm{diag}(J_1, \ldots, J_{\tilde{s}})$. According to Theorem 4.4 from [11] one can find in some neighbourhood of $(x^0, \xi^0)$ a non-degenerate matrix $G(x, \xi)$ smoothly depending on the parameter $(x, \xi)$, $\xi \in S^{n-1}$, and such that

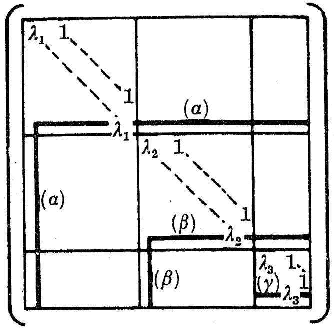
$$G^{-1}(x, \xi)B^{-1}p_{m-1}(x, \xi)BG(x, \xi) = C_0 + \text{Sylvestre matrix } D = L.$$

Let us remind briefly that the Sylvestre matrix $D$ can be written as follows:




The non-zero elements of the matrix $D$ are marked in bold type print. They are smooth functions of $(x, \xi)$ and vanish at the point $(x^0, \xi^0)$.

Therefore the following sketch is justified in a certain neighbourhood of the point $(x^0, \xi^0)$:

(11) $\quad (BG(x, \xi))^{-1} p_{m-1} (BG(x, \xi)) =$  $= L$



This normal form is obtained on the manifold $\Omega \times S^{n-1}$ when $|\xi - \xi^0| < \varepsilon$, $\xi \in S^{n-1}$ and $|x - x^0| < \varepsilon$ for some $\varepsilon > 0$.

Further, we continue the elements of $G(x, \xi)$ to positively homogeneous functions of order zero with respect to $\xi$ in a conic neighbourhood of $(x^0, \xi^0)$. Finally, we continue the elements of $D$ and $C_0$ to positively homogeneous symbols of order $(m-1)$ with respect to $\xi$.

Thus

$$(BG(x, \xi))^{-1} p_{m-1}(x, \xi) (BG(x, \xi)) = |\xi|^{m-1} L(x, \xi/|\xi|),$$

$\operatorname{ord}_\xi (BG(x, \xi)) = 0$ in some conic neighbourhood $\Gamma \ni (x^0, \xi^0)$. The desired hypoellipticity will be deduced from two estimates of the type (9), (10).

We shall also need the following auxiliary facts:

LEMMA 5. *Consider the pseudo-differential system with the symbol*

$$p(x, \xi) \sim p_m^0 I_N + p_{m-1} + p_{m-2} + \ldots, \qquad (x, \xi) \in T^*(\Omega) \setminus \{0\}.$$

*Suppose that the function $p_m^0$ fulfills conditions* (iv), (v) *and the matrix $p_{m-1}$ has representation* (11). *Then there exist constants $C_1 > 0$, $C_2 > 0$ such that if*

$$\max_{x \in \tilde{K} \supset K, \xi \in S^{n-1}} (|\alpha| + |\beta| + |\gamma|) < C_1$$

*then*

(12) $$\|u\|_{m-1} \leqslant C_2(K)(\|Pu\|_0 + \|u\|_{m-2}), \qquad \forall u \in C_0^\infty(K, C^N).$$

Thus the estimate (12) is valid if the uniform norms of the elements $(\alpha)$, $(\beta)$, $(\gamma)$ of the matrix (11) are sufficiently small (when $x \in \tilde{K} \Supset K$, $\bar{\tilde{K}} = \tilde{K}$, $\tilde{K} \Subset \Omega$ and $\xi \in S^{n-1}$, of course).



Without any loss of generality Lemma 5 will be proved in the special case where the representation (11) consists of two Jordan blocks only.

Let us remind that if $\gamma(x, D)$ is a scalar pseudo-differential operator with symbol $\gamma(x, \xi)$, $\operatorname{ord}_\xi \gamma = m-1$, $(x, \xi) \in T^*(\Omega) \setminus \{0\}$, then for each $\varepsilon > 0$ one can find a constant $C(K, \varepsilon)$ such that

$$\|\gamma u\|_0 \leqslant (\max_{x \in \tilde{K}, \xi \in S^{n-1}} |\gamma| + \varepsilon) \|u\|_{m-1} + C(K, \varepsilon) \|u\|_{m-2}, \qquad \forall u \in C_0^\infty(K).$$

($\tilde{K}$ is a compact neighbourhood of $K$.)

Thus

$$\left[ p_m^0 I_N + \left( \begin{array}{ccccc|ccccc}
\lambda & |\xi|^{m-1} & & & & & & & & \\
 & \ddots & \ddots & & & & & & & \\
 & & & & |\xi|^{m-1} & & & & & \\
\alpha_1 & \cdots & \cdots & & \lambda + \alpha_r & \alpha_{r+1} & \cdots & \cdots & & \alpha_{r+s} \\
\hline
\gamma_1 & & & & & \mu & |\xi|^{m-1} & & & \\
\vdots & & & & & & \ddots & \ddots & & \\
\vdots & & & & & & & & & |\xi|^{m-1} \\
\gamma_s & & & & & \beta_1 & \cdots & \cdots & & \mu + \beta_s
\end{array} \right) \right] \left( \begin{array}{c} u_1 \\ \vdots \\ u_r \\ \hline u_{r+1} \\ \vdots \\ u_{s+r} \end{array} \right) = \left( \begin{array}{c} f_1 \\ \vdots \\ f_r \\ \hline f_{r+1} \\ \vdots \\ f_{s+r} \end{array} \right)$$

Therefore,

$$\begin{aligned}
(p_m + \lambda) u_1 &= f_1 - \Lambda^{m-1} u_2, \\
(p_m + \lambda) u_2 &= f_2 - \Lambda^{m-1} u_3, \\
&\ldots\ldots\ldots\ldots\ldots \\
(p_m + \lambda) u_{r-1} &= f_{r-1} - \Lambda^{m-1} u_r, \\
[p_m + (\lambda + \alpha_r)] u_r &= f_r - \sum_{i=1}^{r-1} \alpha_i u_i - \sum_{i=r+1}^{r+s} \alpha_i u_i, \\
(p_m + \mu) u_{r+1} &= f_{r+1} - \gamma_1 u_1 - \Lambda^{m-1} u_{r+2}, \\
&\ldots\ldots\ldots\ldots\ldots \\
[p_m + (\mu + \beta_s)] u_{r+s} &= f_{r+s} - \gamma_s u_1 - \sum_{i=1}^{s-1} \beta_i u_{r+i},
\end{aligned}$$

where $\Lambda$ stands for an operator with the symbol $|\xi|$ and $u_i \in C_0^\infty(K)$. ($\operatorname{ord}_\xi \mu = \operatorname{ord}_\xi \alpha_i = \operatorname{ord}_\xi \beta_j = \operatorname{ord}_\xi \gamma_k = \operatorname{ord}_\xi \lambda = m-1$.)

The fact that the constant $E$ from Lemma 3 does not depend on the lower order terms plays an important role in our proof. In what follows the symbols $\varrho_p$, $\tilde{\varrho}_p$, $\varrho_p'$, $p = 1, \ldots, r+s$ will denote non-negative functions of $\max\limits_{\tilde{K}\times S^{n-1}} (|\alpha|+|\beta|+|\gamma|)$ which tend to zero as the above-mentioned uniform norm tends to zero. Const stands for any constants which are bounded as $\varrho_p \to 0$ $(\tilde{\varrho}_p, \varrho'_p \to 0)$.

So

$$\|u_{r+s}\|_{m-1} \leqslant E(K)\Big\|f_{r+s}-\gamma_s u_1 - \sum_{i=1}^{s-1} \beta_i u_{r+i}\Big\|_0 + O(\|u\|_{m-2}),$$

i.e.,

$$\|u_{r+s}\|_{m-1} \leqslant E(K)\|f_{r+s}\|_0 + E(K)\varrho_{r+s}\Big(\|u_1\|_{m-1} + \sum_{i=1}^{s-1}\|u_{r+i}\|_{m-1}\Big) + O(\|u\|_{m-2}).$$

It follows analogously:

$$\|u_{r+s-1}\|_{m-1} \leqslant E(K)\|f_{r+s-1}\|_0 + E(K)\varrho_{r+s-1}\|u_1\|_{m-1} + \tfrac{3}{2}E(K)\|u_{r+s}\|_{m-1} + O(\|u\|_{m-2}),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\|u_{r+1}\|_{m-1} \leqslant E(K)\|f_{r+1}\|_0 + E(K)\varrho_{r+1}\|u_1\|_{m-1} + \tfrac{3}{2}E(K)\|u_{r+2}\|_{m-1} + O(\|u\|_{m-2}),$$

$$\|u_r\|_{m-1} \leqslant E(K)\|f_r\|_0 + E(K)\varrho_r\Big(\sum_{i=1}^{r-1}\|u_i\|_{m-1} + \sum_{i=1+r}^{s+r}\|u_i\|_{m-1}\Big) + O(\|u\|_{m-2}),$$

$$\|u_{r-1}\|_{m-1} \leqslant E(K)\|f_{r-1}\|_0 + \tfrac{3}{2}E(K)\|u_r\|_{m-1} + O(\|u\|_{m-2}),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\|u_1\|_{m-1} \leqslant E(K)\|f_1\|_0 + \tfrac{3}{2}E(K)\|u_2\|_{m-1} + O(\|u\|_{m-2}).$$

Consequently:

$$\|u_{r+s-1}\|_{m-1} \leqslant E(K)\|f_{r+s-1}\|_0 + E(K)\varrho_{r+s-1}\|u_1\|_{m-1} + \tfrac{3}{2}E^2(K)\|f_{r+s}\|_0 + E(K)\tilde{\varrho}_{r+s-1}\Big(\|u_1\|_{m-1} + \sum_{i=1}^{s-1}\|u_{r+i}\|_{m-1}\Big) + O(\|u\|_{m-2})$$

and we find that for sufficiently small $\tilde{\varrho}_{r+s-1}$:

$$\|u_{r+s-1}\|_{m-1} \leqslant \text{const}\,\|f\|_0 + \varrho'_{r+s-1}\Big(\|u_1\|_{m-1} + \sum_{i=1}^{s-2}\|u_{r+i}\|_{m-1}\Big) + O(\|u\|_{m-2}) \quad (f = (f_1, \dots, f_{r+s})).$$



By induction we conclude that:

$$\|u_{r+s-j}\|_{m-1} \leqslant \text{const}\,\|f\|_0 + \varrho'_{r+s-j}\Big(\|u_1\|_{m-1} + \sum_{i=1}^{s-j-1}\|u_{r+i}\|_{m-1}\Big) + O(\|u\|_{m-2}), \quad j = 0, \dots, s-2,$$

for sufficiently small $\varrho'_{r+s-j}$.

Thus

$$\|u_{r+1}\|_{m-1} \leqslant \text{const}\,\|f\|_0 + \varrho'_{r+1}\|u_1\|_{m-1} + O(\|u\|_{m-2}).$$

Therefore there exists a number $\varepsilon_0 > 0$ such that if $\varrho'_p < \varepsilon_0$, $p = r+1, \dots, r+s$, then

$$\|u_{r+s}\|_{m-1} \leqslant \text{const}\,\|f\|_0 + \tilde{\varrho}\|u_1\|_{m-1} + O(\|u\|_{m-2}).$$

Further,

$$\|u_r\|_{m-1} \leqslant E(K)\|f_r\|_0 + \varrho_r\Big(\sum_{i=1}^{r-1}\|u_i\|_{m-1} + \sum_{i=r+1}^{r+s}\|u_i\|_{m-1}\Big) + O(\|u\|_{m-2}),$$

which implies that

$$\|u_r\|_{m-1} \leqslant \text{const}\,\|f\|_0 + \varrho'_r\sum_{i=1}^{r-1}\|u_i\|_{m-1} + O(\|u\|_{m-2}).$$

Using induction once again, we find that

$$\|u_j\|_{m-1} \leqslant \text{const}\,\|f\|_0 + \varrho'_j\sum_{i=1}^{j-1}\|u_i\|_{m-1} + O(\|u\|_{m-2}), \quad j = 2, \dots, r.$$

The assumption that the norm $\max\limits_{\tilde{K}\times S^{n-1}}(|\alpha|+|\beta|+|\gamma|)$ is small enough implies the following inequalities:

$$\|u_i\|_{m-1} \leqslant \text{const}\,\|f\|_0 + O(\|u\|_{m-2}), \quad i = 1, \dots, r+s, \quad \forall u_i \in C_0^\infty(K),$$

ending the proof.

LEMMA 6. *Suppose that a complex-valued matrix $q_1$ smoothly depending on the parameter $(x, \xi)$ is defined in a conic neighbourhood $\Gamma$ of the point $(x^0, \xi^0)$. Assume that $q_1$ is positively homogeneous of order $k$ with respect to $\xi$ and $\det q_1 \neq 0$ in $\Gamma$. Then one can find a conic neighbourhood $\Gamma_1 \Subset \Gamma$, $(x^0, \xi^0) \in \Gamma_1$ and a smooth elliptic matrix $q(x, \xi)$, $\operatorname{ord}_\xi q = k$, $(x, \xi) \in T^*\Omega\setminus\{0\}$ for which $q_1|_{\Gamma_1} = q|_{\Gamma_1}$.*

*In this way every complex-valued elliptic matrix has a smooth elliptic continuation over the bundle $T^*(\Omega)\setminus\{0\}$.*

*Proof.* Let $\xi^0$ belong to $S^{n-1}$ and $t \in [0,1]$. Evidently $q_1(x^0, \xi^0) = tq_1(x^0, \xi^0)+(1-t)q_1(x^0, \xi^0)$. On account of the homogeneity of $q_1$ we can restrict ourselves to the case of $\xi \in S^{n-1}$. Note that the set of the non-degenerate matrices is open in the set of all matrices $M_N$. Therefore for each $t \in [0,1]$ there exist a pair of open neighbourhoods $V_t \ni t$, $U_t \ni (x^0, \xi^0)$, $U_t \subset \Omega \times S^{n-1}$, such that:

$$\det\big(tq_1(x, \xi)+(1-t)q_1(x^0, \xi^0)\big) \neq 0, \qquad \forall t \in V_t \text{ and } \forall (x, \xi) \in U_t.$$

Let $V_{t_1}, \ldots, V_{t_s}$ form an open covering of the unit interval $[0,1]$ and consider the intersection $U = \bigcap_{i=1}^{s} U_{t_i} \ni (x^0, \xi^0)$. Then the matrix $\tilde{q}(x, \xi) = tq_1(x, \xi)+(1-t)q_1(x^0, \xi^0)$ is non-degenerate for each $t \in [0,1]$ and each $(x, \xi) \in U$. Moreover, $\tilde{q} \in C^\infty(U)$. Consider the function

$$\Phi \in C_0^\infty(\Omega \times S^{n-1}), \qquad 0 \leqslant \Phi \leqslant 1, \qquad \operatorname{supp}\Phi \subset U,$$

which is equal to 1 in some neighbourhood of the point $(x^0, \xi^0)$.

Let us define the following matrix-valued function:

$$q(x, \xi) = \Phi q_1(x, \xi)+(1-\Phi)q_1(x^0, \xi^0).$$

Obviously, $q(x, \xi) \in C^\infty(\Omega \times S_\xi^{n-1})$ and $(x, \xi) \in U$ imply that $\det q(x, \xi) \neq 0$. On the other hand, if $(x, \xi) \notin U$, then $q(x, \xi) = q_1(x^0, \xi^0)$. To make the proof complete, the matrix $q$ has to be continued to a positively homogeneous function of order $k$ with respect to $\xi$.

Our subsequent considerations show that the operator $P$ fulfils the estimate (9). This fact together with Lemma 4 will give us the hypoellipticity sought for. The inequality (9) will be proved by using a special partition of unity and by applying the results of Lemmas 5 and 6.

We suppose that $\tilde{K}$ is a compact set with $\tilde{K} \Supset K$ and we consider the set $K \times S_\xi^{n-1}$. Then in a neighbourhood of each point $(x, \xi) \in \Sigma$ the matrix $p_{m-1}$ can be represented in the Sylvestre normal form. Therefore there exist a finite number of bounded open sets $\{\omega_i\}_{i=1}^M$, $\{\omega_i'\}_{i=1}^M$, $\omega_i \Subset \omega_i' \Subset \Omega \times S_\xi^{n-1}$, satisfying the following conditions:

1. $\bigcup_{i=1}^{M} \omega_i \supset (\tilde{K} \times S^{n-1}) \cap \Sigma$,

2. $p_{m-1}$ has the Sylvestre normal form in $\omega_i'$. In fact, $\Sigma$ is a compact subset of $\tilde{K} \times S_\xi^{n-1}$.

It is easy to see that one can find two open bounded sets $\omega_0 \Subset \omega_0'$, $\mathrm{pr}_x \omega_0' \subset \Omega$ such that $\omega_0 \cup \big(\bigcup_{i=1}^{M} \omega_i\big) \supset K \times S_\xi^{n-1}$ and $p_m|_{\overline{\omega}_0'}$ is an elliptic symbol.

Then we construct a pseudo-differential partition of unity $\{\psi_j(x, \xi)\}$ subordinated to the covering $\{\omega_i\}_{i=0}^M$ and we continue the functions $\psi_j$ to positively homogeneous symbols of order 0 with respect to $\xi$. Thus $\operatorname{supp}\psi_j \subset \omega_j$ and $\sum_{j=0}^{M} \psi_j(x, \xi) \equiv 1$ for each $x$ belonging to some open neighbourhood of $K$ when $\xi \in S^{n-1}$. If $u \in C_0^\infty(K)$ then $u = \sum \psi_j u + (1-\sum\psi_j)u$. According to the pseudolocality of a pseudo-differential operator, $1-\sum\psi_j$ is an operator of order $(-\infty)$ in $C_0^\infty(K, C^N)$. So our considerations should concentrate on the study of $\psi_j(x, D)u$. Obviously, $\psi_j(x, D)u \in C_0^\infty(\mathrm{pr}_x \omega_j', C^N)$. We can assume that each $\omega_j'$, $1 \leqslant j \leqslant M$, is an open ball with centre at $(x^j, \xi^j)$ and sufficiently small radius.



As proved above, there exists a smooth non-degenerate matrix $G$ defined in $\omega_j'$ and such that

$$G^{-1}p_{m-1}G = C_0+D = L \quad \text{in } \omega_j'.$$

The matrix $L$ will be continued to $\tilde{K} \times S_\xi^{n-1}$ without increasing the uniform norms of its elements.

To this end the following cut-off function $h$ is defined:

$$h \in C_0^\infty(\tilde{K} \times S^{n-1}), \qquad h|_{\omega_j} \equiv 1, \qquad h \in C_0^\infty(\omega_j').$$

Denote $\tilde{D} = hD \Rightarrow L-\tilde{L} = (h-1)D$. Thus if the diameter of $\omega_j'$, $1 \leqslant j \leqslant M$, is small enough, then the estimate

$$\|\psi_j u\|_{m-1} \leqslant C\|(p_m^0 I_N+\tilde{L})(\psi_j u)\|_0 + O(\|u\|_{m-2})$$

holds for any $u \in C_0^\infty(K, C^N)$.

(We may assume that the constant $C$ does not depend on $j$, since there are a finite number of neighbourhoods $\omega_j'$.)

According to Lemma 6 (diam $\omega_j'$ being small enough), one can continue the matrix $G$ outside $\omega_j'$ so as to get

$$\mathrm{ord}_\xi G = 0, \qquad \det G \neq 0 \qquad \text{in } \Omega \times S^{n-1}.$$

Denote this continuation by $\tilde{G}$. Then

$$\mathrm{ord}_\xi \tilde{G} = 0, \qquad \tilde{G}|_{\omega_j} = G|_{\omega_j},$$
$$\tilde{G}^{-1}p_{m-1}\tilde{G}|_{\omega_j} = G^{-1}p_{m-1}G|_{\omega_j} = L|_{\omega_j} = \tilde{L}|_{\omega_j}.$$

Now form the difference $(\tilde{G}^{-1}p_{m-1}\tilde{G}-\tilde{L})\circ\psi_j$. In view of the fact that the compact $\operatorname{supp}\psi_j$ is in $\omega_j$, the pseudo-differential operator under consideration has a zero symbol, i.e. its order is equal to $(-\infty)$.

Therefore

$$\|\psi_j u\|_{m-1} \leqslant C\|(p_m^0 I_N+\tilde{G}^{-1}\circ p_{m-1}\circ\tilde{G})(\psi_j u)\|_0 + O(\|u\|_{m-2}), \qquad \forall u \in C_0^\infty(K, C^N).$$

In the above estimate the operator $\tilde{G}^{-1}\circ p_m^0\circ\tilde{G}$ will be written everywhere instead of $p_m^0 I_N$. Simple calculations based on the principal results from the algebra of pseudo-differential operators [2] show that

$$\tilde{G}^{-1}\circ p_m\circ\tilde{G} = p_m I_N + \sum_{l=1}^{n}(p_{m,(l)}T_{l,(-1)} + p_m^{(l)}T_{l,0}) + p_m T_{-1} + T_{-2},$$

where $T_{l,(-1)}$ are matrix-valued operators of order $(-1)$, $T_{l,0}$ of order 0 and $T_{-1}$ $(T_{-2})$ of order $(-1)$ $((-2))$.

So we conclude that for a convenient constant $C_1 > 0$ holds:

$$\|\psi_j u\|_{m-1} \leqslant C\|(\tilde{G}^{-1}\circ p_m\circ\tilde{G} + \tilde{G}^{-1}\circ p_{m-1}\circ\tilde{G})(\psi_j u)\|_0 + $$
$$+ C_1\Big[\sum_{l=1}^{n}(\|p_{m,l}(\psi_j u)\|_{-1} + \|p_m^{(l)}(\psi_j u)\|_0)\Big] + C_1\|p_m^0(\psi_j u)\|_{-1} + O(\|u\|_{m-2}),$$
$$\forall u \in C_0^\infty(K, C^N),$$

i.e.,

$$\|\psi_j u\|_{m-1} \leqslant C\|\tilde{G}^{-1}\circ(p_m^0 I_N + p_{m-1})\circ\tilde{G}(\psi_j u)\|_0 +$$
$$+ C_2\Big[\sum_{l=1}^{n}(\|p_{m,l}u\|_{-1} + \|p_m^{(l)}u\|_0)\Big] + C_2\|p_m u\|_{-1} + O(\|u\|_{m-2}).$$

In the inequality obtained just now, change $\psi_j(x, \xi)$ by $(\psi_j\tilde{G}^{-1})(x, \xi)$. Then $(\psi_j\tilde{G}^{-1})(x, D)u \in C_0^\infty(\mathrm{pr}_x\omega_j, C^N)$. Obviously, $(\psi_j\tilde{G}^{-1})(x, D) = \tilde{G}^{-1}\circ\psi_j I_N + T_{-1}(x, D)$ where $T_{-1}$ is a matrix-valued operator whose order is equal to $(-1)$. Because of the ellipticity of $\tilde{G}^{-1}$:

$$\|(\psi_j\tilde{G}^{-1})u\|_{m-1} = \|\tilde{G}^{-1}(\psi_j u)\|_{m-1} + O(\|u\|_{m-2})$$
$$\geqslant C_0\|\psi_j u\|_{m-2} + O(\|u\|_{m-2}),$$

$C_0 > 0$, $C_0 = \mathrm{const}$ since $\psi_j u \in C_0^\infty(\mathrm{pr}_x\omega_j, C^N)$. Bearing in mind that $G$ is a bounded operator, it is sufficient to consider the operators, as follows:

$$(p_m^0 I_N + p_{m-1})\circ\tilde{G}\circ(\tilde{G}^{-1}\circ\psi_j I_N + T_{-1}) = (p_m^0 I_N + p_{m-1})\circ(\psi_j I_N + \tilde{G}\circ T_{-1})$$
$$= (p_m^0 I_N + p_{m-1})\circ\psi_j + \tilde{G}\circ T_{-1}\circ(p_m^0 I_N + p_{m-1}) +$$
$$+ [p_m^0 I_N + p_{m-1}, \tilde{G}\circ T_{-1}].$$

The commutator $[p_m^0 I_N + p_{m-1}, \tilde{G}\circ T_{-1}]$ is of order $(m-2)$ since $p_m^0 I_N$ is a diagonal operator. Thus we get:

$$\|\tilde{G}\circ T_{-1}\circ(p_m^0 I_N + p_{m-1})u\|_0 \leqslant \mathrm{const}\,\|(p_m^0 I_N + p_{m-1})u\|_{-1}$$
$$\leqslant \mathrm{const}\,\|p_m^0 u\|_{-1} + O(\|u\|_{m-2}),$$



i.e.,

$$\|\psi_j u\|_{m-1} \leqslant \mathrm{const}\Big(\|(p_m^0 I_N + p_{m-1})(\psi_j u)\|_0 + \|p_m^0 u\|_{-1} +$$
$$+ \sum_{l=1}^{n}(\|p_{m,l}u\|_{-1} + \|p_m^{(l)}u\|_0)\Big) + O(\|u\|_{m-2}), \qquad \forall u \in C_0^\infty(K, C^N).$$

Lemma 4, applied for $s = -1/2$ yields the estimate:

$$\|\psi_j u\|_{m-1} \leqslant \mathrm{const}\big(\|(p_m^0 I_N + p_{m-1})(\psi_j u)\|_0 + \|p_m u\|_{-1/2}\big) + O(\|u\|_{m-3/2}).$$

Now we have to consider the commutator

$$(p_m^0 I_N + p_{m-1})\circ\psi_j = \psi_j\circ(p_m^0 I_N + p_{m-1}) + [\psi_j, p_m^0]I_N + [\psi_j I_N, p_{m-1}].$$

Then $\|[\psi_j I_N, p_{m-1}]u\|_0 \leqslant \mathrm{const}\,\|u\|_{m-2}$ ($\psi_j I_N$ is a diagonal operator).

In view of Lemma 4

$$\|[\psi_j, p_m^0]u\|_0 \leqslant \mathrm{const}\,\|(p_m^0 I_N + p_{m-1})u\|_{-1/2} + O(\|u\|_{m-3/2}),$$
$$\forall u \in C_0^\infty(K, C^N).$$

Combining the above results we can conclude that there exists a constant $C_j > 0$ such that

$$\|\psi_j u\|_{m-1} \leqslant C_j\big(\|(p_m^0 I_N + p_{m-1})u\|_0 + \|u\|_{m-3/2}\big),$$
$$\forall u \in C_0^\infty(K, C^N),\ j = 1, 2, \ldots, M. \tag{13}$$

The proof of Lemma 6 can be completed by studying the behaviour of our operator in $\omega_0$. To this end we continue the principal symbol $p_m^0$ to an elliptical symbol $\tilde{p}_m$, $\mathrm{ord}_\xi\tilde{p}_m = m$, defined in the whole space $T^*(\Omega)\setminus\{0\}$ $(\tilde{p}_m|_{\omega_0} = p_m|_{\omega_0})$. Since $\mathrm{supp}\,\psi_0 \subset \omega_0$, then $\psi_0(x, D)u \in C_0^\infty(\mathrm{pr}_x\omega_0, C^N)$. The ellipticity of $\tilde{p}_m^0$ implies that

$$\|\psi_0 u\|_{m-1} \leqslant C\big(\|(\tilde{p}_m^0 I_N + p_{m-1})\psi_0 u\|_{-1} + \|u\|_{m-2}\big), \qquad \forall u \in C_0^\infty(K, C^N).$$

Now replace $\tilde{p}_m$ by $p_m$. According to the pseudo-locality of a pseudo-differential operator, the following estimate is valid:

$$\|\psi_0 u\|_{m-1} \leqslant C\|(p_m^0 I_N + p_{m-1})\psi_0 u\|_{-1} + O(\|u\|_{m-2}).$$

Thus we get immediately

$$\|\psi_0 u\|_{m-1} \leqslant C_0\|(p_m^0 I_N + p_{m-1})u\|_{-1} + O(\|u\|_{m-2}), \qquad \forall u \in C_0^\infty(K, C^N). \tag{14}$$

The two inequalities (13), (14) give

$$\|u\|_{m-1} \leqslant \sum_{j=0}^{M}\|\psi_j u\|_{m-1} + O(\|u\|_{m-2})$$
$$\leqslant \tilde{C}\big(\|(p_m^0 I_N + p_{m-1})u\|_0 + \|u\|_{m-3/2}\big), \qquad \forall u \in C_0^\infty(K, C^N),$$
$$\tilde{C} = \mathrm{const} > 0. \tag{15}$$

A simple conclusion from this estimate and from Lemma 6 is that the operator $P$ is hypoelliptic with loss of one derivative.

## 5

In order to prove Theorem 4, we remind the necessary condition of Hörmander for the local solvability of a linear differential operator [1]. Thus, let an operator $P(x, D)$ be locally solvable in a neighbourhood of the origin in the class of distributions $D'$. Then there exist a neighbourhood of the origin $\omega \ni 0$, an integer $M \geqslant 0$ and a constant $C > 0$ such that

$$\left|\int f(x) v(x)\, dx\right| \leqslant C \sum_{|\alpha| \leqslant M} \sup |D^\alpha f(x)| \sum_{|\alpha| \leqslant M} \sup |D^\alpha (P^* v)| \tag{16}$$

for every pair of functions $f, v \in C_0^\infty(\omega)$. (As usual, $P^*$ is the formally adjoint operator of the operator $P$.)

Theorem 4 will be proved if we show that for every choice of $\omega \ni 0$, $M \in \mathbf{Z}_+$, $C > 0$ the estimate (16) fails to hold for at least one pair $f, v \in C_0^\infty(\omega)$.

Suppose that the function $u(t) \in S(\mathbf{R}^{n-1})$ is a non-trivial solution of the equation $P(t, 1, D_t) u = 0$. In view of the ellipticity of $P(t, 1, D_t)$ we conclude that $u(t)$ is an analytic function, i.e. there exists a multiindex $\alpha_0$ for which $D_t^{\alpha_0} u(0) \neq 0$. Using the quasihomogeneity of $P$ we see that the funct on

$$w_\lambda(x_1, t) = \int \psi(\lambda\varrho) e^{ix_1\lambda^2\varrho} u\big((\varrho\lambda^2)^{1/(1+\delta)} t\big)\, d\varrho, \qquad \lambda = \text{const} > 0$$

satisfies the equation $P(t, D_{x_1}, D_t) w_\lambda = 0$ if $\psi \in C_0^\infty(\mathbf{R}^1)$, $\psi \geqslant 0$, $\operatorname{supp}\psi \subset [1, 2]$, $\int \psi(\varrho)\, d\varrho = 1$.

Thus fix $\omega \ni 0$, $C = \text{const} > 0$ and $M \geqslant 0$ for which (16) is valid. For $f_\lambda$ we take the function

$$f_\lambda(x_1, t) = D_t^{\alpha_0}\big(F(\lambda^2 x_1, \lambda^2 t)\big), \qquad \lambda \geqslant 1, \tag{17}$$

where $F \in C_0^\infty(\omega)$ and $\iint F(x_1, t)\, dx_1\, dt = 1$.

One can easily see that $\operatorname{supp} f_\lambda \subset \omega$ for $\lambda$ large enough. If $\omega' \Subset \omega$ is a neighbourhood of the origin, take a function $\varphi \in C_0^\infty(\omega')$, $\varphi|_{\omega'} = 1$ and consider the function defined as follows:

$$\begin{aligned} v_\lambda(x_1, t) &= \varphi w_\lambda \\ &= \varphi(x_1, t) \int \psi(\lambda\varrho) e^{ix_1\lambda^2\varrho} u\big((\varrho\lambda^2)^{1/(1+\delta)} t\big)\, d\varrho \\ &\qquad\qquad (v_\lambda \in C_0^\infty(\omega)). \end{aligned} \tag{18}$$



We shall prove that the pair of functions (17), (18) violate the inequality (16) when $\lambda \to \infty$. We begin with examining the bilinear form $\int f_\lambda v_\lambda\, dx$:

$$\begin{aligned} &\iiint D_t^{\alpha_0}\big(F(\lambda^2 x_1, \lambda^2 t)\big) \psi(\lambda\varrho) \varphi(x_1, t) e^{ix_1\lambda^2\varrho} u\big((\varrho\lambda^2)^{1/(1+\delta)} t\big)\, dx_1\, dt\, d\varrho \\ &\equiv \iiint F(\lambda^2 x_1, \lambda^2 t) \psi(\lambda\varrho) e^{ix_1\lambda^2\varrho} (\varrho\lambda^2)^{|\alpha_0|/(1+\delta)} (D_t^{\alpha_0} u) \times \\ &\qquad \times \big((\varrho\lambda^2)^{1/(1+\delta)} t\big) \varphi(x_1, t)\, dx_1\, dt\, d\varrho + \\ &\quad + \sum_{|\alpha_0| \geqslant |\gamma| \geqslant 1} C_\gamma \iiint F(\lambda^2 x_1, \lambda^2 t) \psi(\lambda\varrho) e^{ix_1\lambda^2\varrho} (\varrho\lambda^2)^{|\alpha_0 - \gamma|/(1+\delta)} \times \\ &\qquad \times \varphi_\gamma(x_1, t) (D_t^{\alpha_0 - \gamma} u) \big((\varrho\lambda^2)^{1/(1+\delta)} t\big)\, dx_1\, dt\, d\varrho. \end{aligned}$$

In the above identity $\varphi_\gamma \equiv D_t^\gamma \varphi$, $\varphi_\gamma \equiv 0$ in $\omega'$ and $C_\gamma$ are constants.

Applying the change of variables $\lambda^2 x_1 \to x_1$, $\lambda^2 t \to t$, $\lambda\varrho \to \varrho$, we find that:

$$\iint f_\lambda v_\lambda\, dx_1\, dt = \lambda^{-2n-1}[D_t^{\alpha_0} u(0) + o(1)], \qquad \lambda \to \infty.$$

On the other hand,

$$\sup_{|\alpha| \leqslant M} |D^\alpha f_\lambda| \leqslant C\lambda^{2M + 2|\alpha_0|}, \qquad C = \text{const} > 0.$$

To complete the proof $Pv_\lambda$ has to be evaluated in a domain of the type $0 < 2\eta \leqslant |x_1| + |t| \leqslant 2A$, where $A$, $\eta$ are some positive numbers. To obtain an upper bound for $|D_{x_1}^\alpha D_t^\beta w_\lambda(x_1, t)|$ when $2\eta \leqslant |x_1| + |t| \leqslant 2A$ and $|\alpha + \beta| \leqslant M + R$, $R = \operatorname{ord} P$, two cases will be considered: **I**: $|t| \geqslant \eta$, **II**: $|x_1| \geqslant \eta$.

In the case I we shall only use the fact that if $u \in S(\mathbf{R}^{n-1})$ then $D_t^\alpha u \in S(\mathbf{R}^{n-1})$. So

$$\begin{aligned} &\left|\int \psi(\lambda\varrho) e^{ix_1\lambda^2\varrho} (\varrho\lambda^2)^N (D_t^\alpha u) \big((\varrho\lambda^2)^{1/(1+\delta)} t\big)\, d\varrho\right| \\ &\qquad \leqslant \int \psi(\lambda\varrho)(\varrho\lambda^2)^N \frac{C_{\alpha,Q}}{|\varrho\lambda^2 t|^Q}\, d\varrho \leqslant \frac{C_{\alpha,Q}}{\eta^Q} \lambda^{N-Q} \int \frac{\psi(\varrho)\, d\varrho}{\varrho^Q} \\ &\qquad\qquad (C_{\alpha,Q} = \text{const} > 0,\ \lambda^2\varrho \in [\lambda, 2\lambda]). \end{aligned}$$

Note that $Q$ is an arbitrary positive integer, while $N \leqslant M + R$.

In the case II the following identity will be used:

$$\frac{d^Q}{d\varrho^Q} (e^{ix_1\varrho\lambda^2}) = (ix_1\lambda^2)^Q e^{ix_1\varrho\lambda^2}.$$

After $Q$ integrations by parts in the integral (18) we obtain the following quantity expressing the highest power of $\lambda$:

$$(19)\quad \int \psi(\lambda\varrho)(\lambda^2\varrho)^M u\big((\varrho\lambda^2)^{1/(1+\delta)}t\big)e^{ix_1\lambda^2\varrho}\,d\varrho$$
$$= \frac{1}{(ix_1\lambda)^Q}\int \frac{d^Q}{d\varrho^Q}\big[\psi(\varrho)(\varrho\lambda)^M u\big((\varrho\lambda)^{1/(1+\delta)}t\big)\big]e^{ix_1\lambda^2\varrho}\,d\varrho.$$

For $|x_1|\geqslant\eta$ it follows that

$$\Big|\int \psi(\lambda\varrho)(\lambda^2\varrho)^M u\big((\varrho\lambda^2)^{1/(1+\delta)}t\big)e^{ix_1\lambda^2\varrho}\,d\varrho\Big|\leqslant C_Q\lambda^{M-1-Q\delta/(1+\delta)},\qquad C_Q>0.$$

If $Q$ is taken large enough with respect to $2n$, $M+R$, $2M+2|\alpha_0|$, we conclude that (16) fails to hold when $\lambda\to+\infty$.

Thus the proof of Theorem 4 is accomplished.

In conclusion we are able to state that the problem of proving local nonsolvability for a class of quasihomogeneous second order partial differential equations in $\boldsymbol{R}^2$ is reduced to the following one: to find the spectrum and the eigenfunctions belonging to $L^2(\boldsymbol{R}^1)$ of a second order ordinary differential operator.

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