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SOLVABILITY OF A THREE-DIMENSIONAL BOUNDARY VALUE PROBLEM WITH A FREE SURFACE FOR THE STATIONARY NAVIER–STOKES SYSTEM

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1. Introduction

In this paper the following stationary free surface problem is considered. A heavy viscous non-compressible capillary fluid slowly moves in a container under the influence of sources and drains which are concentrated on the bottom. The free boundary of the liquid and the vector of velocity are sought for.

Let us give a mathematical formulation of the problem. By a container we mean a modified semi-cylinder $V \subset R^3$ with a smooth boundary $S \in C^{l+2}$, l being a non-integer > 0. We suppose that $V_+ = \{x \in V: x_3 > 0\} = \omega \times R_+$ $(x' = (x_1, x_2) \in \omega,\ x_3 > 0)$, $\omega \subset R^2$ and $V_- = V \setminus V_+$ are bounded domains and that the projection of V_+ onto the plane $x_3 = 0$ coincides with ω.

The gravitational force is supposed to be parallel to the vector $(0, 0, -1)$ so that the domain $\Omega \subset V$ occupied by the liquid is determined by the inequality $x_3 < \varphi(x')$, $x' \in \omega$.

The vector of velocity $\vec{v}(x) = (v_1(x), v_2(x), v_3(x))$ and the pressure $p(x)$ satisfy in Ω the Navier–Stokes system

$$-\nabla^2\vec{v} + (\vec{v}\cdot\nabla)\vec{v} + \nabla p = 0, \qquad \nabla\cdot\vec{v} = 0 \tag{1.1}$$

and the boundary conditions

$$\begin{gathered} \vec{v}\,|_{\Sigma} = \vec{a}, \\ \vec{v}\cdot\vec{n}\,|_{\Gamma} = 0, \qquad S(\vec{v})\vec{n} - \vec{n}\big(\vec{n}\cdot S(\vec{v})\vec{n}\big)\big|_{\Gamma} = 0, \end{gathered} \tag{1.2}$$

where

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right),$$

$$\vec{f}\cdot\vec{g} = f_1g_1+f_2g_2+f_3g_3, \qquad \Sigma = S\cap\partial\Omega,$$

$\Gamma = \partial\Omega\setminus\Sigma = \{x\in V\colon x_3 = \varphi(x')\}$ is a free surface, $\vec{n}$ is the unit outward normal vector to $\partial\Omega$, $S(\vec{v})$ is the matrix with elements $\dfrac{\partial v_i}{\partial x_j}+\dfrac{\partial v_j}{\partial x_i}$. We assume that

$$\operatorname{supp}\vec{a}\subset S_- \equiv \{x\in S,\ x_3<0\}$$

and

$$\int_{S_-}\vec{a}\cdot\vec{n}\,dS = 0.$$

The function $\varphi(x')$ satisfies the equation

$$(1.3)\qquad \nabla\cdot\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}-\beta\varphi = W\left(-p+\vec{n}\cdot S(\vec{v})\,\vec{n}\right)\big|_{x_3=\varphi(x')}, \qquad x'\in\omega$$

and the boundary conditions

$$(1.4)\qquad \left.\frac{\nabla\varphi\cdot\vec{\nu}}{\sqrt{1+|\nabla\varphi|^2}}\right|_{\partial\omega} = \cos\theta,$$

where $\nabla\varphi = \left(\dfrac{\partial\varphi}{\partial x_1}, \dfrac{\partial\varphi}{\partial x_2}\right)$, $\vec{\nu} = (\nu_1,\nu_2)$ is the unit outward normal to $\partial\omega$, W, β, $\varepsilon>0$ and $\theta\in(0,\pi)$ are given constants. Finally, the function φ is subject to the condition

$$(1.5)\qquad \int_\omega \varphi(x')\,dx' = h,$$

which is equivalent to prescribing the volume of the liquid.

The result of this paper is an existence theorem for the problem (1.1)–(1.5) with a sufficiently small Reynolds number ε.

The rigorous mathematical analysis of free surface problems for noncompressible fluids takes its origin in the classical investigations of certain ideal flows carried out by Helmholtz and Kirchhoff more than hundred years ago. The basic tool in the analysis of such problem is the theory of holomorphic functions.

More recent investigations of free boundary problems for viscous flows are based on the theory of elliptic boundary value problems (see for instance [1]–[3]) and on regularity theorems for weak solutions of boundary value problems for the Stokes system in domains with smooth boundaries [4]. The most convenient for mathematical treatment are flows in which the free surface of the liquid has no contact with the walls of the container. Such flows are considered in the papers [5]–[8], where the solvability of corresponding boundary value problems is established in the Hölder classes C^l. If the surfaces under consideration do have a contact, then the line of the contact is, as a rule, a ridge on $\partial\Omega$. In this case the function $p(x)$ and the derivatives of $\vec{v}(x)$ may have singularities depending on the angle θ of the contact. For $\theta = \pi/2$ the singularities vanish. The problem of this type is considered in [9].

The solvability of the problem (1.1)–(1.5) for arbitrary $\theta\in(0,\pi)$ was first established in the two-dimensional case [10], [11]. The three-dimensional case, that requires a somewhat more detailed analysis of a linearized problem, has been treated by the author [12] and by V. G. Maz'ya, B. A. Plamenevsky, L. I. Stupialis, whose paper is published in 1979 in the proceedings of a seminar on differential equations at the Mathematical Institute of the Academy of Science of Lithuanian SSR (Vilnius).

Important results in the general theory of linear elliptic boundary value problems in domains with singularities on the boundary have been obtained by V. A. Kondratiev [13], [14], V. G. Maz'ya and B. A. Plamenevsky [15]–[19]. Nevertheless, up to the recent time this theory was not able to deliver all the information that is necessary for treating free boundary problems. Therefore a significant part of the papers [11], [12] is devoted to the analysis of a linearized problem under appropriate restrictions on the boundary $\partial\Omega$.

A linearized problem is considered in Sections 3–6 of the present paper. In treating the reference problem in a bihedral angle we use the ideas of V. A. Kondratiev [13], [14], V. G. Maz'ya and B. A. Plamenevsky [19], [20] concerning the estimates of the solution in weighted L_2 and Hölder spaces and the definition and estimates of Green's matrix. Section 7 contains the main theorem.

The results of this paper are more general than those of [12]. A few propositions which have been only formulated in [12] are now supplied with proofs. Several proofs are simplified.

The results of the paper were formulated in the author's lecture at the Banach Center semester on partial differential equations in 1978. It is the hope of the author that the article will draw attention of mathematicians who are interested both in free surface problems and in the theory of elliptic boundary value problems in domains with singularities at the boundary.

2. Auxiliary propositions

2.1. Notations and definitions. The following notations are used in this paper:

V, S, Ω, ω, Σ, Γ are domains and surfaces defined in Section 1, $\mathscr{M} = \Sigma \cap \bar{\Gamma}$.

$d_\theta \subset R^2$ is the angle of the size $\theta \in (0, 2\pi)$, i.e. a domain $0 < \varphi < \theta$ where $\varphi = \arctan(x_2/x_1)$ denotes a standard polar angle; γ_0, γ_θ are the half-lines $\varphi = 0$ and $\varphi = \theta$.

$D_\theta = d_\theta \times R^1$ $(x' = (x_1, x_2) \in d_\theta,\ x_3 \in R^1)$ denotes a bihedral angle in R^3 with the sides $\Gamma_0 = \gamma_0 \times R^1$, $\Gamma_\theta = \gamma_\theta \times R^1$ and edge $M = \Gamma_0 \cap \Gamma_\theta = \{x_3 \in R^1,\ x_1 = x_2 = 0\}$.

$K_\varrho(\xi) = \{x \in D_\theta\colon |x - \xi| < \varrho\}$, $\xi \in D_\theta$.

$\Omega_\varrho(\xi) = \{x \in \Omega\colon |x - \xi| \leqslant \varrho\}$, $\xi \in \bar{\Omega}$.

$\zeta(t)$ is an infinitely differentiable monotone function with $\zeta(t) = 0$ for $t \geqslant 2$ and $\zeta(t) = 1$ for $t \leqslant 1$.

$C^l(G)$ $(l > 0$ being a non-integer) denotes the space of functions defined in a domain $G \subset R^n$ with a finite norm

$$|u|_G^{(l)} = [u]_G^l + \sum_{|\alpha| < l} |D^\alpha u|_G,$$

where

$$\alpha = (\alpha_1, \ldots, \alpha_n), \qquad D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}, \qquad |\alpha| = \alpha_1 + \ldots + \alpha_n,$$

$$|v|_G = \sup_{x \in G} |v(x)|, \qquad [v]_G^{(l)} = \sum_{|\alpha| = [l]} \sup_{x, y \in G} \frac{|D^\alpha v(x) - D^\alpha v(y)|}{|x-y|^{l-[l]}}.$$

$\mathring{C}_s^l(G, F)$ $(F \subset \bar{G}$ being an m-dimensional manifold, $m < n)$ denotes the space of functions defined in $G \setminus F$ and having a finite norm

$$|u|_{\mathring{C}_s^l(G,F)} = \sum_{|\alpha| < l} \sup_{x \in G \setminus F} \varrho^{|\alpha| - s}(x) |D^\alpha u(x)| + \sum_{|\alpha| = [l]} \sup_{x \in G \setminus F} \varrho^{l-s}(x) \sup_{|x-y| < \frac{1}{2}\varrho(x)} |D^\alpha u(x) - D^\alpha u(y)|\, |x-y|^{[l]-l}$$

where $\varrho(x) = \operatorname{dist}(x, F)$.

$C_s^l(G, F)$ $(l > s > 0$, l, s being non-integers) denotes the space of functions with a finite norm

$$|u|_{C_s^l(G,F)} = |u|_G^{(s)} + \sum_{s < |\alpha| < l} \sup_{x \in G \setminus F} \varrho^{|\alpha| - s}(x) |D^\alpha u(x)| + \sum_{|\alpha| = [l]} \sup_{x \in G \setminus F} \varrho^{l-s}(x) \sup_{|x-y| < \frac{1}{2}\varrho(x)} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^{l-[l]}}.$$

Clearly, $\mathring{C}_s^l(G, F)$ is the subspace of $C_s^l(G, F)$ consisting of functions vanishing on F with their derivatives up to order $[s]$. For $s < 0$ assume $C_s^l(G, F) = \mathring{C}_s^l(G, F)$.

$C_0^\infty(G, F)$ is the set of all infinitely differentiable functions vanishing near F and also for $|x| \gg 1$ (if G is unbounded).

$H_\mu^k(G, F)$ (k being an integer $\geqslant 0$) denotes the completion of $C_0^\infty(G, F)$ in the norm

$$\|u\|_{H_\mu^k(G,F)} = \Big(\sum_{0 \leqslant |\alpha| \leqslant k} \int_G |D^\alpha u|^2 \varrho^{2\mu - 2(l - |\alpha|)} dx \Big)^{1/2}.$$

$L_{2,\mu}(G, F) = H_\mu^0(G, F)$.

$H_\mu^l(G, F)$ denotes the completion of $C_0^\infty(G, F)$ in the norm

$$\|u\|_{H_\mu^l(G,F)} = \Bigg(\sum_{|\alpha| < l} \int_G |D^\alpha u|^2 \varrho^{2\mu - 2(l - |\alpha|)} dx + \sum_{|\alpha| = [l]} \int_G \varrho^{2\mu}(x) dx \int_{|x-y| \leqslant \frac{1}{2}\varrho(x)} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2 dy}{|x-y|^{n+2(l-[l])}} \Bigg)^{1/2}.$$

$H_\mu^{k,t}(D_\theta, M)$ (t being an integer > 0) denotes the completion of $C_0^\infty(D_\theta, M)$ in the norm

$$\|u\|_{H_\mu^{k,t}(D_\theta, M)} = \Bigg(\sum_{q=0}^{t} \Big\| \frac{\partial^q u}{\partial x_3^q} \Big\|^2_{H_\mu^k(D_\theta, M)} \Bigg)^{1/2}.$$

For the spaces whose elements are vector functions the same notations are used.

The spaces H_μ^k were introduced and used for the study of elliptic boundary value problems by V. A. Kondrat'ev [13], [14], the spaces $\mathring{C}_s^l$ by V. G. Maz'ya and B. A. Plamenevsky [16].

2.2. Imbedding and extension theorems, interpolation inequalities. In this section $D_\theta = R^{n-2} \times d_\theta \subset R^n$: $(x_1, \ldots, x_{n-2}) \in R^{n-2}$, $(x_{n-1}, x_n) \in d_\theta$, Γ_0, Γ_θ, M are the sides and the edge of D_θ (for $n = 2$ $D_\theta = d_\theta$, $\Gamma_0 = \gamma_0$, $\Gamma_\theta = \gamma_\theta$), $x' = (x_1, \ldots, x_{n-1})$, $L(x') \in C_0^\infty(R^{n-1})$,

$$\operatorname{supp} L \subset B_a \equiv \{x_1^2 + \ldots + x_{n-2}^2 + (x_{n-1} - 2a)^2 \leqslant a^2\}.$$

LEMMA 2.1. *If* $\varphi(x') \in C_s^l(\Gamma_0, M)$, *then* $v(x) = \int_{B_a} L(\xi) \varphi(x' + \xi x_n)\, d\xi \in C_s^l(D_{\pi/2}, M)$ *and the following estimate holds:*

$$(2.1) \qquad |v|_{C_s^l(D_{\pi/2}, M)} \leqslant c_1 |\varphi|_{C_s^l(\Gamma_0, M)}.$$

If $\varphi(x') \in C^l(R^{n-1})$, *then* $v(x) \in C_l^r(R_+^n, R^{n-1})$ *for all* $r > l$ (R_+^n *is a half-space* $x_n > 0$) *and*

$$|v|_{C_l^r(R_+^n, R^{n-1})} \leqslant c_2 |\varphi|_{C^l(R^{n-1})}. \tag{2.2}$$

Proof. For all β and σ, $|\sigma| > 0$, we have

$$D^\beta v(x) = \sum_{|\gamma|=|\beta|} \int_{B_a} D_{x'}^\gamma \varphi(x' + \xi x_n) L_\gamma(\xi)\, d\xi, \tag{2.3}$$

$$D^{\beta+\sigma} v(x) = x_n^{-|\sigma|} \sum_{|\gamma|=|\beta|} \int_{B_a} [D_{x'}^\gamma \varphi(x' + \xi x_n) - D_{x'}^\gamma \varphi(x')] L_{\gamma\sigma}(\xi)\, d\xi, \tag{2.4}$$

where L_γ, $L_{\gamma\sigma} \in C_0^\infty(B_a)$, $\int_{B_a} L_{\gamma\sigma}(\xi) d\xi = 0$. Since $\varrho(x) = \sqrt{x_{n-1}^2 + x_n^2} \approx \sup_{\xi \in B_a} (x_{n-1} + \xi x_n)$ for $x_{n-1}, x_n \geqslant 0$, we easily derive from (2.3) the estimate (2.1) and $|v|_{C^l(R_+^n)} \leqslant c_3 |\varphi|_{C^l(R^{n-1})}$ (if $\varphi \in C^l(R^{n-1})$).

Taking now in (2.4) $|\beta| = [l]$ we obtain

$$|D^{\beta+\sigma} v(x)| \leqslant c_4 x_n^{l-[l]-|\sigma|} [\varphi]_{R^{n-1}}^{(l)}$$

and consequently (2.2).

In the same way the following lemma can be proved

LEMMA 2.2. *If* $\varphi \in \overset{\circ}{C}{}_{s-j}^{l-j}(\Gamma_0, M)$ *with some integer* $j < l$, *then*

$$v_j = \frac{x_n^j}{j!} \int_{B_a} L(\xi) \varphi(x' + \xi x_n)\, d\xi \in \overset{\circ}{C}{}_s^l(D_{\pi/2}, M)$$

and

$$|v_j|_{\overset{\circ}{C}{}_s^l(D_{\pi/2}, M)} \leqslant c |\varphi|_{\overset{\circ}{C}{}_{s-j}^{l-j}(\Gamma_0, M)}.$$

For $n = 2$ estimates of this type are established in [11].

THEOREM 2.1. *For any* $\psi_j^{(0)} \in \overset{\circ}{C}{}_{s-j}^{l-j}(\Gamma_0, M)$, $\psi_j^{(\theta)} \in \overset{\circ}{C}{}_{s-j}^{l-j}(\Gamma_\theta, M)$, $j = 0, \ldots, [l]$, *there exists a function* $u \in \overset{\circ}{C}{}_s^l(D_\theta, M)$ *satisfying the conditions*

$$\left(\frac{\partial}{\partial n}\right)^j u|_{\Gamma_0} = \psi_j^{(0)}, \quad \left(\frac{\partial}{\partial n}\right)^j u|_{\Gamma_\theta} = \psi_j^{(\theta)}$$

and the inequality

$$|u|_{\overset{\circ}{C}{}_s^l(D_\theta, M)} \leqslant c \sum_{j=0}^{[l]} \left(|\psi_j^{(0)}|_{\overset{\circ}{C}{}_{s-j}^{l-j}(\Gamma_0, M)} + |\psi_j^{(\theta)}|_{\overset{\circ}{C}{}_{s-j}^{l-j}(\Gamma_\theta, M)}\right) \tag{2.5}$$

with a constant c *independent of* $\psi_j^{(0)}$, $\psi_j^{(\theta)}$.

Proof. We may assume (without loss of generality) that $\psi_j^{(\theta)} = 0$. Take

$$u(x) = f(\varphi) \sum_{j=0}^{[l]} v_j(x), \tag{2.6}$$

where $f(\varphi)$ is a smooth function which equals one for $0 < \varphi < \varepsilon$ and zero for $\varphi > \min(\pi/2, \theta) - \varepsilon$ ($\varepsilon < \frac{1}{3}\min(\pi/2, \theta)$),

$$v_j(x) = \frac{x_n^j}{j!} \int_{B_a} L(\xi) \varphi_j(x' + \xi x_n)\, d\xi, \quad \varphi_j = \psi_j^{(0)} - \sum_{i=0}^{j-1} \frac{\partial^j v_i}{\partial x_n}\bigg|_{x_n=0}, \quad \varphi_0 = \psi_0^{(0)}.$$

The function (2.6) satisfies the necessary boundary conditions. The estimate (2.5) follows from Lemma 2.2.

THEOREM 2.2 [18]. *If* $u \in H_\mu^k(D_\theta, M)$ (k *being an integer* > 0) *and* $|\alpha| < k$, *then*

$$D^\alpha u|_{\Gamma_0} \in H_\mu^{k-|\alpha|-1/2}(\Gamma_0, M) \quad \text{and} \quad \|D^\alpha u\|_{H_\mu^{k-|\alpha|-1/2}(\Gamma_0, M)} \leqslant c \|u\|_{H_\mu^k(\Gamma_0, M)}.$$

For any $\psi_j^{(0)} \in H_\mu^{k-j-1/2}(\Gamma_0, M)$, $\psi_j^{(\theta)} \in H_\mu^{k-j-1/2}(\Gamma_\theta, M)$, $j = 1, \ldots, k-1$, *there exists a function* $u(x) \in H_\mu^k(D_\theta, M)$ *satisfying the conditions*

$$\left(\frac{\partial}{\partial n}\right)^j u\bigg|_{\Gamma_0} = \psi_j^{(0)}, \quad \left(\frac{\partial}{\partial n}\right)^i u\bigg|_{\Gamma_\theta} = \psi_j^{(\theta)}$$

and the inequality

$$\|u\|_{H_\mu^k(D_\theta, M)} \leqslant c_1 \sum_{j=0}^{k-1} \left(\|\psi_j^{(0)}\|_{H_\mu^{k-j-1/2}(\Gamma_0, M)} + \|\psi_j^{(\theta)}\|_{H_\mu^{k-j-1/2}(\Gamma_\theta, M)}\right).$$

For the proof of the second part of the theorem formula (2.6) may be used.

THEOREM 2.3. *Suppose* $v \in H_\mu^k(K_r(\xi), M)$, $u \in \overset{\circ}{C}{}_s^l(K_r(\xi), M)$. *Then for any* $\varepsilon \in (0, r)$, $j < k$, $i < l$ *we have*

$$\|v\|_{H_\mu^{k-j}(K_r(\xi), M)} \leqslant \varepsilon^j \|v\|_{H_\mu^k(K_r(\xi), M)} + c_1 \varepsilon^{j-k} \|v\|_{L_{2,\mu}(K_r(\xi), M)}, \tag{2.7}$$

$$|u|_{\overset{\circ}{C}{}_{s-i}^{l-i}(K_r(\xi), M)} \leqslant \varepsilon^i |u|_{\overset{\circ}{C}{}_s^l(K_r(\xi), M)} + c_2 \varepsilon^{i-s-n/2} \left[1 + \left(\frac{r + |\xi|}{\varepsilon}\right)^{l-s}\right] \|u\|_{L_2(K_r)}. \tag{2.8}$$

Proof. The norms

$$\left(\sum_{|\alpha| \leqslant k} \left\| D^\alpha \left(\varrho^{\mu+|\alpha|-k}(x) v\right) \right\|_{L_2(K_r(\xi))}^2\right)^{1/2},$$

$$[u\varrho^{l-s}]_{K_r(\xi)}^{(l)} + \sum_{s < |\alpha| < l} |D^\alpha(\varrho^{|\alpha|-s} u)|_{K_r(\xi)} + [u]_{K_r(\xi)}^{(s)}$$

(if $s < 0$, $[u]^{(s)}_{K_r}$ should be omitted) are equivalent to the norms $H^k_\mu(K_r(\xi))$ and $\mathring{C}^l_s(K_r(\xi), M)$. Therefore (2.7), (2.8) follow from the well-known interpolation inequalities [21]

$$\Big(\sum_{|\alpha|=m} \|D^\alpha w_1\|^2_{L_2(K_r(\xi))}\Big)^{1/2} \leqslant \varepsilon^j \Big(\sum_{|\alpha|=m+j} \|D^\alpha w_1\|^2_{L_2(K_r(\xi))}\Big)^{1/2} + c_3 \varepsilon^{-m} \|w_1\|_{L_2(K_r)},$$

$$[w_2]^{(l-i)}_{K_r(\xi)} \leqslant \varepsilon^i [w_2]^{(l)}_{K_r(\xi)} + c_4 \varepsilon^{-(l-i-n/2)} \|w_2\|_{L_2(K_r(\xi))},$$

$$\sum_{|\alpha|=m} |D^\alpha w_3|_{K_r(\xi)} \leqslant \varepsilon^i \sum_{|\alpha|=m+i} |D^\alpha w_3|_{K_r(\xi)} + c_5 \varepsilon^{-m-n/2} \|w_3\|_{L_2(K_r(\xi))},$$

$$[u]^{(s-i)}_{K_r(\xi)} \leqslant \varepsilon^i [u]^{(s)}_{K_r(\xi)} + c_6 \|u\|_{L_2(K_r(\xi))} \varepsilon^{i-s-n/2},$$

where $w_1 = v\varrho^{\mu+m+j-k}$, $w_2 = u\varrho^{l-s}$, $w_3 = u\varrho^{m+i-s}$.

THEOREM 2.4. *For arbitrary $u \in C_0^\infty(D_\theta, M)$, $D_\theta \subset R^3$ the inequality*

$$|D^\beta u|\,|x'|^a \leqslant c\|u\|_{H^{k,t}_\mu(D_\theta;M)} \tag{2.9}$$

holds under the conditions

$$a > \mu - \varkappa, \qquad \varkappa = k - \beta_1 - \beta_2 - (\beta_3 + 1/2)\frac{k}{t+k} - 1 > 0$$

(*the constant c may depend on* $\operatorname{supp} u$).

Proof. It follows from the estimate (15) in the paper [22] (with $n = 3$, $q = \infty$, $p_0 = p_1 = p_2 = p_3 = 2$, $l_1 = l_2 = k$, $l_3 = k+t$) that for arbitrary $v \in C_0^\infty(D_\theta, M)$, $\varkappa' < \varkappa$ we have

$$|D^\beta v(x)| \leqslant c_1 |x'|^{\varkappa'} \Big[\int_{D_\theta} \Big(\sum_{|\gamma|\leqslant k} |D^\gamma_{x'} v(x)|^2 + \Big|\frac{\partial^{k+t} v}{\partial x_3^{k+t}}\Big|^2 + |v|^2\Big) dx\Big]^{1/2}.$$

Setting $v = u|x'|^\mu$ we get (2.9) after simple calculations.

2.3. Certain transformations of the domain Ω. Suppose that $S = \partial V \in C^{l_1}$, $\varphi \in C^{l_2}_\sigma(\omega, \partial\omega)$, $l_2 > l_1 > \sigma > 1$.

THEOREM 2.5. *For a certain $d > 0$ and for every $\xi \in \mathscr{M}$ there exists a transformation T of the domain $\Omega_d(\xi)$, onto a subdomain of the bihedral angle $\mathfrak{D}$ formed by the planes tangent to Σ and Γ at the point ξ, with the properties:*

$$T \in C^{l_1}_\sigma(\Omega_d, \mathscr{M}), \qquad T|_{x_3=\varphi(x')} \in C^{l_2}_\sigma(\omega_d, \partial\omega),$$

$$T(\xi) = \xi, \qquad \frac{\partial T}{\partial x}\Big|_{x=\xi} = I, \tag{2.10}$$

where $\partial T/\partial x$ is the Jacobian matrix of the transformation T, I is the identity transformation, $\omega_\delta \subset \omega$ is the projection of $\Gamma_\delta = \Gamma \cap \partial\Omega_\delta(\xi)$ onto ω.

Proof. Assume that $\xi = 0$ and that the x_2-axis is parallel to the normal vector to Σ at this point. Let the domain V be defined in a neighbourhood of the origin by $x_2 > h(x_1) \in C^{l_1}(\Delta)$ where $h(0) = h'(0) = 0$, $\Delta = [-d_1, d_1]$. Extend $h(x_1)$ at first to R^1 (so that $h \in C^{l_1}(R^1)$) and then to R^2_+ by the formula

$$h(x_1, x_2) = \int_{-\infty}^{\infty} L(t) h(x_1 + tx_2)\,dt$$

where $L(t)$ is the same function as in subsection 2; $\operatorname{supp} L \subset [a, 3a]$. Suppose that a is so small that

$$\int_a^{3a} |L(t)|\,t\,dt \sup_{\xi\in R^1} |h'(\xi)| < 1; \tag{2.11}$$

then $|\partial h/\partial x_2| < 1$.

Consider the transformation V_1 given by

$$x_1 = y_1, \qquad x_2 = y_2 + h(y_1, y_2). \tag{2.12}$$

By Lemma 2.1, $V_1 \in C^{l_3}_{l_1}(R^2_+, R^1)$ for any $l_3 > l_1$ and in virtue of (2.11) V_1 maps R^2_+ onto the domain $x_2 > h(x_1)$ and is invertible. Let $T_1 x = (V_1^{-1}x', x_3)$; $T_1 \in C^{l_3}_{l_1}(\Omega_d, S)$. This transformation maps $\Sigma_d = \Sigma \cap \partial\Omega_d(\xi)$ onto a subdomain Σ' of the tangent plane to Σ and Γ_d onto the surface Γ' given by the equation $y_3 = \varphi_1(y') = \varphi(V_1 y') \in C^{l_2}_\sigma(B^+, \Delta)$, where $B^+ = \{y' \in R^2_+ : |y'| \leqslant d_2\}$, $d_2 \leqslant d_1$ and Δ is a rectilinear part of ∂B^+. Let us extend the function φ_1 at first to R^2_+ and then to $D_{\pi/2} = \{y_1 \in R^1, y_2, y_3 > 0\}$ setting

$$\varphi_1(y') = \zeta\Big(\frac{y'}{d_2}\Big) \sum_{k=1}^{1+[l_2]} \lambda_k \varphi_1\Big(\frac{y'}{|y'|}\Big(d_2 - \frac{|y'| - d_2}{k+1}\Big)\Big), \qquad |y'| \geqslant d_2,$$

$$\varphi_1(y) = \int_{B_a} \varphi_1(y' + \xi y_3) L_a(\xi)\,d\xi, \qquad y' \in R^2_+,\ y_3 > 0,$$

where $\int_{B_a} L(\xi)\xi_i\,d\xi = 0$, $i = 1, 2$, and λ_k satisfy the equations

$$\sum_{k=1}^{1+[l_2]} \lambda_k \Big(-\frac{1}{k+1}\Big)^s = 1, \qquad s = 0, \ldots, [l_2].$$

It is easy to verify that $\varphi_1 \in C^{l_2}_\sigma(D_{\pi/2}, R^1)$, $\varphi_1(0) = \varphi_{1y_3}(0) = 0$ and

$$|\varphi_1|_{C^{l_2}_\sigma(D_{\pi/2}, R^1)} \leqslant c|\varphi_1|_{C^{l_2}_\sigma(B^+, \Delta)}.$$

Let the number $a > 0$ be small enough in order that

$$\sum_{i=1}^{2} \int_{B_a} |L(\xi)|\,|\xi_i|\,d\xi \sup_{R^2_+} \Big|\frac{\partial\varphi_1(y')}{\partial y_i}\Big| < 1.$$

Then the mapping V_2 given by

$$z_1 = y_1, \quad z_2 = y_2,$$

$$z_3 = y_3 - \sum_{i=1}^{2} y_i \frac{\partial\varphi_1(\xi)}{\partial\xi_i}\bigg|_{\xi=0} + \varphi_1\Big(y_1, y_2, \sum_{i=1}^{2} y_i \frac{\partial\varphi_1}{\partial\xi_i}\bigg|_{\xi=0} - y_3\Big)$$

maps $\mathfrak{D} = \Big\{y \in R^3\colon y_2 > 0,\ y_3 < \sum_{i=1}^{2} y_i \frac{\partial\varphi_1}{\partial\xi_i}\Big|_{\xi=0}\Big\}$ onto the domain $\{z_1 \in R^1,$ $z_2 > 0,\ z_3 < \varphi_1(z')\}$ and has an inverse $V_2^{-1} = T_2$.

The transformation $T = T_2 \circ T_1$ is the required one; (2.10) follows from the fact that $V_1(0) = 0$, $\frac{\partial V_1}{\partial x}\Big|_{x=0} = I$, $V_2(0) = 0$, $\frac{\partial V_2}{\partial x}\Big|_{x=0} = I$.

Theorem 2.5 permits us to define in a standard way the spaces $C_s^l(\Sigma, \mathcal{M})$, $C_s^l(\Gamma, \mathcal{M})$, $\mathring{C}_s^l(\Sigma, \mathcal{M})$, $\mathring{C}_s^l(\Gamma, \mathcal{M})$. With the help of the results of subsection 2 we can prove the following

THEOREM 2.6. *Let $l < l_1$, $s < \sigma$. For any $\psi_j \in \mathring{C}_{s-j}^{l-j}(\Sigma, \mathcal{M})$, $\psi_j' \in \mathring{C}_{s-j}^{l-j}(\Gamma, \mathcal{M})$, $j = 0, \ldots, [l]$, there exists a function $u \in \mathring{C}_s^l(\Omega, \mathcal{M})$ satisfying the conditions*

$$\frac{\partial^j u}{\partial n^j}\bigg|_\Sigma = \sum_{i_1,\ldots,i_j=1}^{3} n_{i_1}\ldots n_{i_j} \frac{\partial^j u}{\partial x_{i_1}\ldots\partial x_{i_j}}\bigg|_\Sigma = \psi_j, \qquad \frac{\partial^j u}{\partial n^j}\bigg|_\Gamma = \psi_j'$$

and the inequality

$$|u|_{C_s^l(\Omega;\mathcal{M})} \leqslant c \sum_{j=0}^{[l]} \big(|\psi_j|_{C_{s-j}^{l-j}(\Sigma;\mathcal{M})} + |\psi_j'|_{C_{s-j}^{l-j}(\Gamma;\mathcal{M})}\big) \tag{2.13}$$

with a constant c independent of ψ_j, ψ_j'.

THEOREM 2.7. *Let $\varphi, \varphi' \in C_\sigma^{l_2}(\omega, \partial\omega)$, $\varphi, \varphi' \geqslant \lambda > 0$, $\Omega = \{x \in V\colon x_3 < \varphi(x')\}$, $\Omega' = \{x \in V\colon x_3 < \varphi'(x')\}$. If $|\varphi - \varphi'|_{C_\sigma^{l_2}(\omega,\partial\omega)} < \delta_0$, then there exists a transformation $X \in C_\sigma^{l_2}(\Omega', \mathcal{M}')\colon \Omega' \to \Omega$ such that $Xy = y$ for $y \in S_-$ and*

$$|X - I|_{C_\sigma^{l_2}(\Omega',\mathcal{M}')} \leqslant c|\varphi - \varphi'|_{C_\sigma^{l_2}(\omega,\partial\omega)}. \tag{2.14}$$

Proof. Let $V_+ = \omega \times (0, \infty)$ and let $\Phi(\xi)$, $\Phi'(\xi) \in C_\sigma^{l_2}(V_+, \partial\omega_0)$, $\omega_0 = \{x \in \overline{V}_+\colon x_3 = 0\}$, be extensions of φ and φ' from ω onto V_+ satisfying the conditions

$$\left|\frac{\partial\Phi}{\partial\xi_3}\right|, \left|\frac{\partial\Phi'}{\partial\xi_3}\right| < 1, \qquad |\Phi - \Phi'|_{C_\sigma^{l_2}(V_+,\partial\omega_0)} \leqslant c_1|\varphi - \varphi'|_{C_\sigma^{l_2}(\omega,\partial\omega)}$$

(these extensions can be defined with the help of operators $\int_{B_a} L(\eta)\varphi(z' + \eta z_3)\,d\eta$ with a small $a > 0$ and with use of partition of unity in ω).

If δ_0 is sufficiently small relative to δ_1, then the function $\Psi(\xi) = \Phi'\zeta\left(\frac{\xi_3}{\delta_1}\right) + \Phi\left(1 - \zeta\left(\frac{\xi_3}{\delta}\right)\right)$ which coincides with Φ' for $y_3 < \delta_1$ and with Φ for $y_3 > 2\delta_1$, also satisfies the condition $\left|\frac{\partial\Psi}{\partial\xi_3}\right| < 1$.

The transformations U and U' given by

$$x_1 = \xi_1,\ x_2 = \xi_2,\ x_3 = \Phi(\xi) - \xi_3,$$
$$x_1 = \xi_1,\ x_2 = \xi_2,\ x_3 = \Psi(\xi) - \xi_3$$

are invertible and map V_+ onto the domains $x_3 < \varphi(x')$ and $x_3 < \varphi'(x')$, $x' \in \omega$. For $\xi_3 > 2\delta_1$ we have $U\xi = U'\xi$. The transformation $X = UU'^{-1}$ with an appropriate δ_1 is that which is sought; the estimate (2.14) follows from the formula $X - I = (U - U')U'^{-1}$.

3. Auxiliary problems on the plane

Consider in $d_\theta \subset R^2$ the boundary value problems:

$$\nabla^2 u = f, \quad u|_{\gamma_\theta} = 0, \quad \frac{\partial u}{\partial x_2}\bigg|_{\gamma_0} = 0, \tag{3.1}$$

$$-\nabla^2\vec{v} + \nabla p = \vec{f}, \quad \nabla\cdot\vec{v} = r,$$
$$\vec{v}|_{\gamma_\theta} = 0, \quad v_2|_{\gamma_0} = \frac{\partial v_1}{\partial x_2}\bigg|_{\gamma_0} = 0. \tag{3.2}$$

THEOREM 3.1. *For arbitrary $f \in H_\mu^k(d_\theta, 0)$ (k being an integer $\geqslant 0$) the problem (3.1) has a unique solution*

$$u \in H_\mu^{2+k}(d_\theta, 0)$$

and

$$\|u\|_{H_\mu^{2+k}(d_\theta,0)} \leqslant c\|f\|_{H_\mu^k(d_\theta,0)}, \tag{3.3}$$

(c does not depend on f), provided $1 + k - \mu \neq (m + \frac{1}{2})\frac{\pi}{\theta}$ with an integer m.

THEOREM 3.2. *For arbitrary $\vec{f} = (f_1, f_2) \in H_\mu^k(d_\theta, 0)$, $r \in H_\mu^{k+1}(d_\theta, 0)$ the problem (3.2) has a unique solution*

$$\vec{v} = (v_1, v_2) \in H_\mu^{k+2}(d_\theta, 0), \quad p \in H_\mu^k(d_\theta, 0)$$

and

$$\|\vec{v}\|_{H_\mu^{k+2}(d_\theta,0)} + \|p\|_{H_\mu^{k+1}(d_\theta,0)} \leqslant c(\|\vec{f}\|_{H_\mu^k(d_\theta,0)} + \|r\|_{H_\mu^{k+1}(d_\theta,0)}), \tag{3.4}$$

provided $1 + k - \mu \neq \operatorname{Re}\sigma$ where $\sigma \neq 0$ is a root of the equation

$$\sin 2\theta\sigma = \sigma\sin 2\theta. \tag{3.5}$$

Both theorems can be proved just in the same way as Theorem 1.1 in [13]. We give a short proof of Theorem 3.2. Write the problem (3.2) in polar coordinates $\varrho \in (0, \infty)$, $\varphi \in (0, \theta)$ (see [23], Ch. 2.) Applying the Mellin transform with respect to ϱ,

$$\tilde{f}(\lambda, \varphi) = \int_0^\infty \varrho^{i\lambda-1} f(\varrho, \varphi)\, d\varrho,$$

we obtain a boundary value problem on the interval $\varphi \in (0, \theta)$

$$(3.6)\qquad \begin{aligned} &-\frac{d^2\tilde{v}}{d\varphi^2} + (1+\lambda^2)\tilde{v} + 2\frac{d\tilde{w}}{d\varphi} - (1+i\lambda)\tilde{q} = \tilde{F}_1,\\ &-\frac{d^2 w}{d\varphi^2} + (1+\lambda^2)\tilde{w} - 2\frac{d\tilde{v}}{d\varphi} + \frac{d\tilde{q}}{d\varphi} = \tilde{F}_2,\\ &\frac{d\tilde{w}}{d\varphi} + (1-i\lambda)\tilde{v} = \tilde{R}, \end{aligned}$$

$$(3.7)\qquad \tilde{v}|_{\varphi=\theta} = \tilde{w}|_{\varphi=\theta} = 0, \quad \tilde{w}|_{\varphi=0} = \frac{d\tilde{v}}{d\varphi}\Big|_{\varphi=0} = 0$$

where $\tilde{v}$, $\tilde{w}$, $\tilde{q}$, $\tilde{F}_1$, $\tilde{F}_2$, $\tilde{R}$ are the Mellin transforms of v_ϱ, v_φ, ϱp, $\varrho^2 f_\varrho$, $\varrho^2 f_\varphi$, r correspondingly, $v_\varrho = v_1\cos\varphi + v_2\sin\varphi$, $v_\varphi = -v_1\sin\varphi + v_2\cos\varphi$.

The system (3.6) can be reduced to

$$(3.6')\qquad dX/d\varphi = AX + P,$$

where $X = (\tilde{v}, \tilde{w}, \tilde{q}, \tilde{z} = d\tilde{v}/d\varphi)$, $P = (0, \tilde{R}, \tilde{F}_2 + d\tilde{R}/d\varphi, 2\tilde{R} - \tilde{F}_1)$.

The corresponding homogeneous problem $(P = 0)$ has non-trivial solutions if $\sigma = -i\lambda \neq 0$ is a root of (3.5). If $\lambda \neq i\sigma$, the problem (3.6′), (3.7) has a solution for arbitrary P. Suppose that the line $\operatorname{Im}\lambda = h$ is free of points $\lambda = i\sigma$; we show that in this case

$$(3.8)\qquad \begin{aligned} &\sum_{i=0}^{2+k} (1+|\lambda|^{2(2+k-i)}) \int_0^\theta \left(\left|\frac{d^i\tilde{v}}{d\varphi^i}\right|^2 + \left|\frac{d^i\tilde{w}}{d\varphi^i}\right|^2 \right) d\varphi + \\ &\quad + \sum_{i=0}^{1+k} (1+|\lambda|^{2(1+k-i)}) \int_0^\theta \left|\frac{d^i q}{d\varphi^i}\right|^2 d\varphi \\ &\leqslant c_1 \Big\{ \sum_{i=0}^{1+k} (1+|\lambda|^{2(1+k-i)}) \int_0^\theta \left|\frac{d^i\tilde{R}}{d\varphi^i}\right|^2 d\varphi + \\ &\quad + \sum_{i=0}^{k} (1+|\lambda|^{2(k-i)}) \int_0^\theta \left(\left|\frac{d^i\tilde{F}_1}{d\varphi^i}\right|^2 + \left|\frac{d^i\tilde{F}_2}{d\varphi^i}\right|^2 \right) d\varphi \Big\} = c_1 N_k \end{aligned}$$

where c_1 does not depend on $\operatorname{Re}\lambda$.

It is sufficient to get this inequality in the case $k = 0$, since the estimate of the derivatives

$$\frac{d^{i+1}\tilde{w}}{d\varphi^{i+1}}, \quad \frac{d^{i+1}\tilde{v}}{d\varphi^{i+1}}, \quad \frac{d^i\tilde{q}}{d\varphi^i}, \quad i \geqslant 2$$

can be obtained from the equations (3.6).

For $|\operatorname{Re}\lambda| \leqslant R$ ($R \geqslant 1$ is arbitrary) (3.8) follows from the representation of solutions in terms of Green's matrix [24]. In the case $|\operatorname{Re}\lambda| > R$ we make use of the "energetic relation"

$$\int_0^\theta \left[\left|\frac{d\tilde{v}}{d\varphi}\right|^2 + \left|\frac{d\tilde{w}}{d\varphi}\right|^2 + (1+\lambda^2)(|\tilde{v}|^2 + |\tilde{w}|^2) + 2\left(\frac{d\tilde{w}}{d\varphi}\tilde{v}^* - \frac{d\tilde{v}}{d\varphi}\tilde{w}^* \right) - \tilde{q}\tilde{R}^* \right] d\varphi = \int_0^\theta (\tilde{F}_1\tilde{v}^* + F_2\tilde{w}^*)\, d\varphi$$

where a^* is the complex conjugate to a. From this relation and from (3.6) we easily get

$$(3.9)\qquad \int_0^\theta \left[\sum_{i=0}^{2} \left|\frac{d^i\tilde{w}}{d\varphi^i}\right|^2 (1+|\lambda|^{2(2-i)}) + \sum_{i=0}^{1} (1+|\lambda|^{2(2-i)}) \left|\frac{d^i\tilde{v}}{d\varphi^i}\right|^2 + \left|\frac{d\tilde{q}}{d\varphi}\right|^2 \right] d\varphi \leqslant c_2 \left[N_0 + \sqrt{N_0}\sqrt{1+|\lambda|^2} \left(\int_0^\theta |\tilde{q}|^2 d\varphi \right)^{1/2} \right] \equiv c_2 N.$$

Now multiply the first equation of (3.6) by $d[\mu(\varphi)dv^*/d\varphi]/d\varphi$,

$$\mu(\varphi) = \begin{cases} 1 & \text{for} \quad 0 \leqslant \varphi \leqslant \theta - 1/|\lambda|, \\ |\lambda|(\theta-\varphi) & \text{for} \quad \theta - 1/|\lambda| \leqslant \varphi \leqslant \theta \end{cases}$$

and integrate with respect to $\varphi \in [0, \theta]$. After simple transformations we find that

$$\int_0^{\theta-1/|\lambda|} \left|\frac{d^2\tilde{v}}{d\varphi^2}\right|^2 d\varphi \leqslant c_3 N$$

and consequently

$$\int_0^{\theta-1/|\lambda|} |q|^2 d\varphi (1+|\lambda|^2) \leqslant c_4 N.$$

Further,

$$(1+|\lambda|^2)^{1/2} \left(\int_{\theta-1/|\lambda|}^{\theta} |q|^2 d\varphi \right)^{1/2} \leqslant (1+|\lambda|^2)^{1/2} \left(\int_{\theta-2/|\lambda|}^{\theta-1/|\lambda|} |q|^2 d\varphi \right)^{1/2} + (1+|\lambda|^2)^{1/2} \left(\int_{\theta-1/|\lambda|}^{\theta} d\varphi \left| \int_0^{1/|\lambda|} \frac{d\tilde{q}(\varphi-\xi, \lambda)}{d\varphi} d\xi \right|^2 \right)^{1/2} \leqslant c_5\sqrt{N}.$$

Hence $\int\limits_0^\theta |q|^2 d\varphi(1+|\lambda|^2) \leqslant (c_4+c_5^2)N$. This inequality and (3.9) imply (3.8). Integrating (3.8) over the line $\operatorname{Im}\lambda = h$ and using the Parseval formula we get (3.4) with $\mu = 1+k-h$.

4. Reference problem in spaces $H_\mu^k(D_\theta, M)$

We now proceed to investigate the reference problem

$$-\nabla^2\vec{v} + \nabla p = \vec{f}, \quad \nabla\cdot\vec{v} = r, \tag{4.1}$$

$$\vec{v}|_{\Gamma_\theta} = \vec{a}, \quad v_2|_{\Gamma_0} = b, \quad \frac{\partial v_1}{\partial x_2}\Big|_{\Gamma_0} = d_1, \quad \frac{\partial v_3}{\partial x_2}\Big|_{\Gamma_0} = d_2 \tag{4.2}$$

in a bihedral angle $D_\theta \subset R^3$. It is equivalent to the problem with the boundary conditions

$$\vec{v}|_{\Gamma_\theta} = \vec{a}, \quad v_2|_{\Gamma_0} = b, \quad S(\vec{v})\vec{n} - \vec{n}\left(\vec{n}\cdot S(\vec{v})\vec{n}\right)|_{\Gamma_0} = \vec{d}'$$
$$(d_2' = 0,\ d_1' = -d_1 - \partial b/\partial x_1,\ d_3' = -d_2 - \partial b/\partial x_3).$$

4.1. Weak solution. Consider the problem with homogeneous boundary conditions

$$\vec{v}|_{\Gamma_\theta} = 0, \quad v_2|_{\Gamma_0} = 0, \quad \frac{\partial v_1}{\partial x_2}\Big|_{\Gamma_0} = \frac{\partial v_3}{\partial x_2}\Big|_{\Gamma_0} = 0. \tag{4.3}$$

Let $\mathfrak{M}(D_\theta)$ denote the space of real-valued vector functions in $H_0^1(D_\theta, M)$ satisfying the boundary conditions $\vec{v}|_{\Gamma_\theta} = 0$, $v_2|_{\Gamma_0} = 0$ and let $\mathfrak{J}(D_\theta)$ be the subspace of divergencefree vectors $\vec{v} \in \mathfrak{M}(D_\theta)$. Define a weak solution of (4.1), (4.3) as a vector $\vec{v} \in \mathfrak{M}(D_\theta)$ satisfying the equation $\nabla\cdot\vec{v} = r$ and the integral identity

$$[\vec{v}, \vec{\eta}] = \sum_{i=1}^3 \int\limits_{D_\theta} \frac{\partial\vec{v}}{\partial x_i}\cdot\frac{\partial\vec{\eta}}{\partial x_i}\,dx = \int\limits_{D_\theta} \vec{f}\cdot\vec{\eta}\,dx \tag{4.4}$$

for arbitrary $\vec{\eta} \in \mathfrak{J}(D_\theta)$.

THEOREM 4.1. *For arbitrary $\vec{f} \in L_{2,1}(D_\theta, M)$, $r \in L_2(D_\theta)$ the problem (4.1), (4.3) has a unique weak solution; moreover, there exists a unique function $p \in L_2(D_\theta)$ such that*

$$[\vec{v}, \vec{\varphi}] - \int\limits_{D_\theta} \vec{f}\cdot\vec{\varphi}\,dx = \int\limits_{D_\theta} p\nabla\cdot\vec{\varphi}\,dx, \quad \forall\vec{\varphi} \in \mathfrak{M}(D_\theta). \tag{4.5}$$

For $\vec{v}$ and p the following estimate holds:

$$\|\vec{v}\|_{H_0^1(D_\theta,M)} + \|p\|_{L_2(D_\theta)} \leqslant c(\|\vec{f}\|_{L_{2,1}(D_\theta,M)} + \|r\|_{L_2(D_\theta)}). \tag{4.6}$$

Proof. If follows from the Hardy inequality

$$\|\vec{v}\|^2_{L_{2,-1}(D_\theta,M)} \leqslant 4[\vec{v}, \vec{v}], \quad \forall\vec{v} \in \mathfrak{M}(D_\theta),$$

that $\int\limits_{D_\theta} \vec{f}\cdot\vec{\eta}\,dx$ is a continuous linear functional in $\mathfrak{M}(D_\theta)$ and $[\vec{v}, \vec{\varphi}]$ may be considered as a scalar product in $\mathfrak{M}(D_\theta)$. Therefore the existence of a weak solution of the problem (4.1), (4.3) in the case $r = 0$ is a consequence of the Riesz representation theorem (see [25]). Taking $\vec{\eta} = \vec{v}$ in (4.4) we get

$$\|\vec{v}\|_{H_0^1(D_\theta)} \leqslant c_1\|\vec{f}\|_{L_{2,1}(D_\theta,M)}.$$

Let $\vec{f} = 0$. It is possible to prove just in the same way as in [26] (Lemma 2.5) that any linear functional $l(\vec{\varphi})$, $\vec{\varphi} \in \mathfrak{M}(D_\theta)$ vanishing for $\vec{\varphi} \in \mathfrak{J}(D_\theta)$ may be represented in the form $l(\vec{\varphi}) = \int\limits_{D_\theta} q\nabla\cdot\vec{\varphi}\,dx$, $q \in L_2(D_\theta)$ and its norm is equivalent to $\|q\|_{L_2(D_\theta)}$. Hence the formula $[\vec{u}, \vec{\varphi}] = \int\limits_{D_\theta} A\vec{u}\cdot\nabla\cdot\vec{\varphi}\,dx$ defines a linear bounded invertible operator $A: \mathfrak{M}(D_\theta)\ominus\mathfrak{J}(D_\theta) \to L_2(D_\theta)$ whose range is $L_2(D_\theta)$. Consequently there exists a vector $\vec{v} \in \mathfrak{M}(D_\theta)\ominus\mathfrak{J}(D_\theta)$ satisfying for all $\vec{u} \in \mathfrak{M}(D_\theta)\ominus\mathfrak{J}(D_\theta)$ the relation

$$\int\limits_{D_\theta} A\vec{u}\cdot r\,dx = [\vec{u}, \vec{v}] = \int\limits_{D_\theta} A\vec{u}\,\nabla\cdot\vec{v}\,dx.$$

It is easily seen that $\vec{v}$ is a weak solution of (4.1), (4.3) with $\vec{f} = 0$ and that $[\vec{v}, \vec{v}] \leqslant c_2\|r\|^2_{L_2(D_\theta)}$. Hence, the existence of a weak solution is proved. Now the functional in the left-hand side of (4.5) vanishes for $\vec{\varphi} = \vec{\eta} \in \mathfrak{J}(D_\theta)$; therefore (4.5) holds with some $p \in L_2(D_\theta)$. The estimate (4.6) follows from the boundedness of A.

THEOREM 4.2. *If $\partial^m\vec{f}/\partial x_3^m \in L_{2,1}(D_\theta, M)$, $\partial^m r/\partial x_3^m \in L_2(D_\theta)$, $m = 1, \ldots, t$, then*

$$\frac{\partial^m\vec{v}}{\partial x_3^m} \in H_0^1(D_\theta, M), \quad \frac{\partial^m p}{\partial x_3^m} \in L_2(D_\theta)$$

and

$$\left\|\frac{\partial^t\vec{v}}{\partial x_3^t}\right\|^2_{H_0^1(D_\theta,M)} + \left\|\frac{\partial^t p}{\partial x_3^t}\right\|^2_{L_2(D_\theta)} \leqslant c_1\left(\left\|\frac{\partial^t\vec{f}}{\partial x_3^t}\right\|^2_{L_{2,1}(D_\theta,M)} + \left\|\frac{\partial^t r}{\partial x_3^t}\right\|^2_{L_2(D_\theta)}\right). \tag{4.7}$$

If $\vec{f} \in H^k_{k+1}(D_\theta, M)$, $r \in H^{k+1}_{k+1}(D_\theta, M)$ *for some integer* $k \geqslant 0$, *then*

$$\vec{v} \in H^{k+2}_{k+1}(D_\theta, M), \qquad p \in H^{k+1}_{k+1}(D_\theta, M)$$

and

$$\|\vec{v}\|^2_{H^{k+2}_{k+1}(D_\theta,M)} + \|p\|^2_{H^{k+1}_{k+1}(D_\theta,M)} \leqslant c_2(\|\vec{f}\|^2_{H^k_{k+1}(D_\theta,M)} + \|r\|^2_{H^{k+1}_{k+1}(D_\theta,M)}). \tag{4.8}$$

Proof. (4.7) follows from

$$\|\Delta^t_3(h)\vec{v}\|^2_{H^1_0(D_\theta,M)} + \|\Delta^t_3(h)p\|^2_{L_2(D_\theta)} \leqslant c_1(\|\Delta^t_3(h)\vec{f}\|^2_{L_2(D_\theta,M)} + \|\Delta^t_3(h)r\|^2_{L_2(D_\theta)}), \tag{4.9}$$

where $\Delta^t_3(h)f(x) = \sum_{j=0}^{t}(-1)^{t-j}\binom{t}{j} f(x_1, x_2, x_3+jh)$ is the finite difference of order t with respect to x_3; $h > 0$ is arbitrary.

To prove (4.8) we use a "local estimate" ([3], Theorem 2.11)

$$\sum_{|\alpha|=i+2}\int_{\Omega_1}|D^\alpha\vec{v}|^2dx + \sum_{|\alpha|=i+1}\int_{\Omega_1}|D^\alpha p|^2dx \leqslant c_3\Big(\sum_{|\alpha|\leqslant i}\delta^{-2(i-|\alpha|)}\int_{\Omega_2}|D^\alpha\vec{f}|^2dx + \sum_{|\alpha|\leqslant i+1}\delta^{-2(i+1-|\alpha|)}\int_{\Omega_2}|D^\alpha r|^2dx + \delta^{-2(i+2)}\int_{\Omega_2}|\vec{v}|^2dx + \delta^{-2(i+1)}\int_{\Omega_2}|p|^2dx\Big) \tag{4.10}$$

where $\Omega_1 \subset \Omega_2 \subset D_\theta$, $\partial\Omega_2 \cap M = \emptyset$, $\delta = \operatorname{dist}(\Omega_1, D_\theta\setminus\Omega_2) > 0$, $i \leqslant k$ (the boundedness of the integrals in the left-hand side is a consequence of regularity theorems for elliptic boundary-value problems and of the results of the paper [4]). Let

$$\Omega_1 = U_{lm} = \{x \in D_\theta\colon 2^l \leqslant |x'| < 2^{l+1},\ m2^l \leqslant x_3 < (m+1)2^l\},$$
$$\Omega_2 = U'_{lm} = \{x \in D_\theta\colon 2^{l-1} \leqslant |x'| < 2^{l+2},\ (m-1)2^l \leqslant x_3 < (m+2)2^l\},$$
$$\delta = 2^{l-1}.$$

By a linear transformation of arguments it is not hard to prove that c_3 does not depend on l, m. Multiplynig (4.10) by 2^{2li}, summing with respect to $l, m = 0, \pm1, \ldots$, $i = 0, \ldots, k$, and taking into account (4.6), we obtain (4.8).

4.2. Solvability of the problem (4.1), (4.3) in $H^k_\mu(D_\theta, M)$.

THEOREM 4.3. *Let the numbers* k = *an integer* $\geqslant 0$ *and* $\mu \geqslant 0$ *satisfy the condition*

$$\min(\operatorname{Re}\sigma_0, \pi/2\theta) > 1+k-\mu \geqslant 0, \tag{4.11}$$

where σ_0 *is the root of* (3.5) *with the minimal positive real part.*

Then any solution $\vec{v} \in H^{2+k}_\mu(D_\theta, M)$, $p \in H^{1+k}_\mu(D_\theta, M)$ *of the problem* (4.1), (4.3) *satisfies the inequality*

$$X \equiv \|\vec{v}\|^2_{H^{2+k}_\mu(D_\theta,M)} + \|p\|^2_{H^{1+k}_\mu(D_\theta,M)} \leqslant c(\|\vec{f}\|^2_{H^k_\mu(D_\theta,M)} + \|r\|^2_{H^{k+1}_\mu(D_\theta,M)}) \equiv cY. \tag{4.12}$$

Proof. Suppose that $\mu \in [0, 1]$. By applying the Fourier transformation with respect to x_3

$$\tilde{f}(x', \xi) = \int_{-\infty}^{\infty} f(x', x_3)e^{-i\xi x_3}dx_3$$

the problem (4.1), (4.3) takes the form

$$\begin{aligned} -\nabla'^2\tilde{v}_\alpha + \xi^2\tilde{v}_\alpha + \frac{\partial\tilde{p}}{\partial x_\alpha} &= \tilde{f}_\alpha, \quad \alpha = 1, 2,\\ -\nabla'^2\tilde{v}_3 + \xi^2\tilde{v}_3 + i\xi\tilde{p} &= \tilde{f}_3,\\ \frac{\partial\tilde{v}_1}{\partial x_1} + \frac{\partial\tilde{v}_2}{\partial x_2} + i\xi\tilde{v}_3 &= \tilde{r}, \end{aligned} \tag{4.13}$$

$$\tilde{v}_1|_{\gamma_\theta} = \tilde{v}_2|_{\gamma_\theta} = \tilde{v}_3|_{\gamma_\theta} = 0, \quad \tilde{v}_2|_{\gamma_0} = 0, \quad \frac{\partial\tilde{v}_1}{\partial x_2}\Big|_{\gamma_0} = \frac{\partial\tilde{v}_3}{\partial x_2}\Big|_{\gamma_0} = 0 \tag{4.14}$$

with $\nabla' = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$.

Transpose in (4.13) the members containing ξ and consider (4.13), (4.14) as problems (3.1) for $\tilde{v}_3$ and (3.2) for $\tilde{v}_1, \tilde{v}_2, \tilde{p}$. In virtue of (3.3) and (3.4), for all $j = 0, \ldots, k$ we have

$$\|\vec{\tilde{v}}\|^2_{H^{2+j}_\mu(d_\theta,0)} + \|\tilde{p}\|^2_{H^{1+j}_\mu(d_\theta,0)} \leqslant c_1(\|\vec{\tilde{f}}\|^2_{H^j_\mu(d_\theta,0)} + \|\tilde{r}\|^2_{H^{1+j}_\mu(d_\theta,0)} + \xi^2\|\vec{\tilde{v}}\|^2_{H^{1+j}_\mu(d_\theta,0)} + \xi^4\|\vec{\tilde{v}}\|^2_{H^j_\mu(d_\theta,0)} + \xi^2\|\tilde{p}\|^2_{H^j_\mu(d_\theta,0)});$$

hence

$$\begin{aligned} X &\leqslant c_2\Big[\sum_{j=0}^{k+2}\int_{-\infty}^{\infty}\xi^{2(k+2-j)}\|\vec{\tilde{v}}\|^2_{H^j_\mu(d_\theta,0)}d\xi + \sum_{j=0}^{k+1}\int_{-\infty}^{\infty}\xi^{2(k+1-j)}\|\tilde{p}\|^2_{H^j_\mu(d_\theta,0)}d\xi\Big]\\ &\leqslant c_3\Big[\|\vec{f}\|^2_{H^k_\mu(D_\theta,M)} + \|r\|^2_{H^{k+1}_\mu(D_\theta,M)} + \int_{-\infty}^{\infty}\xi^{2(k+1)}d\xi\int_{d_\theta}(|\vec{\tilde{v}}_x|^2 + \xi^2|\vec{\tilde{v}}|^2 + |x'|^{-2}|\vec{\tilde{v}}|^2 + |\tilde{p}|^2)|x'|^{2\mu}dx'\Big] \end{aligned} \tag{4.15}$$

where $|\vec{\tilde{v}}_x|^2 = |\vec{\tilde{v}}_{x_1}|^2 + |\vec{\tilde{v}}_{x_2}|^2$.

Note that $\vec{w} = \vec{\tilde{v}}|x'|^{\mu}$, $q = \tilde{p}\,|x'|^{\mu}$ satisfy the system

$$-\nabla'^2 w_a + \xi^2 w_a + \frac{\partial q}{\partial x_a} = g_a, \qquad a = 1, 2,$$
$$-\nabla'^2 w_3 + \xi^2 w_3 + i\xi q = g_3,$$
$$\frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} + i\xi w_3 = h$$

with

$$g_a = |x'|^{\mu}\Big[\tilde{f}_a + |x'|^{-2}\mu\Big(x_a\tilde{p} - \mu\tilde{v}_a - 2\sum_{\beta=1}^{2} x_\beta \frac{\partial \tilde{v}_a}{\partial x_\beta}\Big)\Big],$$
$$g_3 = |x'|^{\mu}\Big[\tilde{f}_3 - |x'|^{-2}\mu\Big(\mu\tilde{v}_3 + 2\sum_{\beta=1}^{2} x_\beta \frac{\partial \tilde{v}_3}{\partial x_\beta}\Big)\Big],$$
$$h = |x'|^{\mu}\Big[r + |x'|^{-2}\mu\sum_{\beta=1}^{2} x_\beta \tilde{v}_\beta\Big]$$

and the boundary condition (4.14). Therefore for every complex-valued vector $\vec{\eta} = (\eta_1, \eta_2, \eta_3) \in L_2(d_\theta)$ having derivatives $\dfrac{\partial\vec{\eta}}{\partial x_a} \in L_2(d_\theta)$ and satisfying the boundary conditions $\vec{\eta}\,|_{\gamma_\theta} = 0$, $\eta_2|_{\gamma_0} = 0$ the integral identity

$$(4.16)\qquad \int_{d_\theta}\Big[\sum_{\beta=1}^{2}\frac{\partial\vec{w}}{\partial x_\beta}\cdot\frac{\partial\vec{\eta}}{\partial x_\beta} + \xi^2\vec{w}\cdot\vec{\eta} - q\Big(\frac{\partial\eta_1^*}{\partial x_1} + \frac{\partial\eta_2^*}{\partial x_2} - i\xi\eta_3^*\Big)\Big]dx' = \int_{d_\theta}\vec{g}\cdot\vec{\eta}\,dx'$$

is valid (for complex-valued vectors $\vec{g}\cdot\vec{\eta} = g_1\eta_1^* + g_2\eta_2^* + g_3\eta_3^*$). To estimate the right member of (4.15) we substitute into (4.16) $\vec{\eta} = \vec{w}$ and then $\vec{\eta} = \vec{\tilde{u}}$, where $\vec{u}$ is a weak solution of (4.1), (4.3) with $\vec{f} = 0$ and $r = q$. Making use of the inequality

$$\Big|\int_{-\infty}^{\infty}\xi^{2(k+1)}d\xi\int_{d_\theta}\vec{g}\cdot\vec{\eta}\,dx'\Big|$$
$$\leqslant\Big(\int_{-\infty}^{\infty}\xi^{2k}d\xi\int_{d_\theta}|\vec{\tilde{f}}|^2|x'|^{2\mu}dx'\Big)^{1/2}\Big(\int_{-\infty}^{\infty}\xi^{2k+4}d\xi\int_{d_\theta}|\vec{\eta}|^2dx'\Big)^{1/2} +$$
$$+\Big[\int_{-\infty}^{\infty}\xi^{2(k+1)-2\mu}d\xi\int_{d_\theta}\big(4|\vec{\tilde{v}}_x|^2 + |\tilde{p}|^2 + |x'|^{-2}|\vec{\tilde{v}}|^2\big)dx'\Big]^{1/2}\times$$
$$\times\Big[\int_{-\infty}^{\infty}\xi^{2(k+1)+2\mu}d\xi\int_{d_\theta}|\vec{\eta}|^2|x'|^{2\mu-2}dx'\Big]^{1/2}$$

and taking into account that, in virtue of (4.7),

$$\int_{-\infty}^{\infty}\xi^{2(k+1)}d\xi\int_{d_\theta}\big(|\vec{\tilde{u}}_x|^2 + \xi^2|\vec{\tilde{u}}|^2 + |x'|^{-2}|\vec{\tilde{u}}|^2\big)dx'$$
$$\leqslant c_4\int_{-\infty}^{\infty}\xi^{2(k+1)}d\xi\int_{d_\theta}|q|^2dx',$$

we obtain (after elementary calculations based on the Hölder inequality) the estimate

$$(4.17)\qquad \int_{-\infty}^{\infty}\xi^{2(k+1)}d\xi\int_{d_\theta}\big(|\vec{w}_x|^2 + \xi^2|\vec{w}|^2 + |q|^2\big)dx'$$
$$\leqslant c_5\Big\{\int_{-\infty}^{\infty}\xi^{2k}d\xi\int_{d_\theta}\big(|\vec{\tilde{f}}|^2 + \xi^2|\tilde{r}|^2\big)|x'|^{2\mu}dx' +$$
$$+X^{1/2}\Big[\int_{-\infty}^{\infty}\xi^{2(k+1)-2\mu}d\xi\int_{d_\theta}\big(|\vec{\tilde{v}}|^2 + |\tilde{p}|^2 + |x'|^{-2}|\vec{\tilde{v}}|^2\big)dx'\Big]^{1/2}\Big\}.$$

Since

$$\int_{-\infty}^{\infty}\xi^{2(k+1)-2\mu}d\xi\int_{d_\theta}|\vec{\tilde{v}}|^2|x'|^{-2}dx'$$
$$\leqslant X^{1/(2-\mu)}\Big(\int_{-\infty}^{\infty}\xi^{2k+4-2\mu}d\xi\int_{d_\theta}|\vec{\tilde{v}}|^2dx'\Big)^{(1-\mu)/(2-\mu)},$$

the inequalities (4.15) and (4.17) yield

$$X \leqslant c_6\Big[\|\vec{f}\|^2_{H^k_\mu(D_\theta,M)} + \|r\|^2_{H^{k+1}_\mu(D_\theta,M)} +$$
$$+\int_{-\infty}^{\infty}\xi^{2(k+1)-2\mu}d\xi\int_{d_\theta}\big(|\vec{\tilde{v}}_x|^2 + \xi^2|\vec{\tilde{v}}|^2 + |\tilde{p}|^2\big)dx'\Big].$$

The functions $\vec{\tilde{v}}$ and $\tilde{p}$ also satisfy the identity (4.16) (with $\vec{\tilde{f}}$ instead of $\vec{g}$ in the right member). Taking in this identity $\vec{\eta} = \vec{\tilde{v}}$ and then $\vec{\eta} = \vec{\tilde{\psi}}$, where $\vec{\psi}$ is a weak solution of (4.1), (4.3) with $\vec{f} = 0$, $r = p$ and repeating the above arguments we get

$$\int_{-\infty}^{\infty}\xi^{2(k+1)-2\mu}d\xi\int_{d_\theta}\big(|\vec{\tilde{v}}_x|^2 + \xi^2|\vec{\tilde{v}}|^2 + |\tilde{p}|^2\big)dx'$$
$$\leqslant c_7\Big[\|\vec{f}\|^2_{H^k_\mu(D_\theta,M)} + \|\vec{f}\|_{H^k_\mu(D_\theta,M)}\Big(\int_{-\infty}^{\infty}\xi^{2(k+2-2\mu)}d\xi\int_{d_\theta}|\vec{\tilde{v}}|^2|x'|^{-2\mu}dx'\Big)^{1/2} +$$
$$+\|r\|_{H^{k+1}_\mu(D_\theta,M)}\Big(\int_{-\infty}^{\infty}\xi^{2(k+1)-2\mu}d\xi\int_{d_\theta}|\tilde{p}|^2dx'\Big)^{1/2}\Big].$$

Thus

$$X \leqslant c_8(Y + X^{1/2} Y^{1/2}),$$

which implies (4.12).

Let $\mu > 1$, the numbers $\mu' = \mu - [\mu] \in [0, 1)$ and $k' = k - [\mu]$ satisfy (4.11); therefore

$$\|\vec{v}\|^2_{H^{2+k'}_{\mu'}(D_\theta, M)} + \|p\|^2_{H^{1+k'}_{\mu'}(D_\theta, M)} \leqslant c_9 Y.$$

The higher order derivatives of $\vec{v}$, p are estimated in the same way as in Theorem 4.2; this leads to (4.12).

THEOREM 4.4. *Let* (4.11) *be satisfied. For arbitrary* $\vec{f} \in H^k_\mu(D_\theta, M)$, $r \in H^{k+1}_\mu(D_\theta, M)$, $\vec{a} \in H^{k+3/2}_\mu(\Gamma_\theta, M)$, $b \in H^{k+3/2}_\mu(\Gamma_0, M)$, $\vec{d} \in H^{k+1/2}_\mu(\Gamma_0, M)$ *the problem* (4.1), (4.2) *has a unique solution* $\vec{v} \in H^{2+k}_\mu(D_\theta, M)$, $p \in H^{1+k}_\mu(D_\theta, M)$ *and*

(4.18) $$\|\vec{v}\|_{H^{k+2}_\mu(D_\theta, M)} + \|p\|_{H^{k+1}_\mu(D_\theta, M)} \leqslant c\big(\|\vec{f}\|_{H^k_\mu(D_\theta, M)} + \|r\|_{H^{k+1}_\mu(D_\theta, M)} + \|\vec{a}\|_{H^{k+3/2}_\mu(\Gamma_\theta, M)} + \|b\|_{H^{k+3/2}_\mu(\Gamma_0, M)} + \|\vec{d}\|_{H^{k+1/2}_\mu(\Gamma_0, M)}\big).$$

Proof. Theorem 2.2 allows us to reduce (4.1), (4.2) to the homogeneous problem (4.1), (4.3); it remains to prove its solvability at least for $\vec{f} \in C^\infty_0(D_\theta, M)$, $r \in C^\infty_0(D_\theta, M)$. By Theorem 4.1, the problem (4.1), (4.3) has a solution and $\partial^t\vec{v}/\partial x_3^t \in H^{j+2}_{j+1}(D_\theta, M)$, $\partial^t p/\partial x_3^t \in H^{j+1}_j(D_\theta, M)$ for all $t \geqslant 0$ and $j = 0, \ldots, k$. Therefore

$$\frac{\partial^{t+2}\vec{v}}{\partial x_i \partial x_3^{t+1}}, \ \frac{\partial^{t+1} p}{\partial x_3^{t+1}} \in L_{2,1}(D_\theta, M) \cap L_2(D_\theta) \subset L_{2,\mu'}(D_\theta, M), \qquad \mu' = \mu - [\mu].$$

From (4.13), (4.14) (which should be considered as problems (3.1) for $\tilde{v}_3$ and (3.2) for $\tilde{v}_1$, $\tilde{v}_2$, $\tilde{p}$) we may conclude that $\vec{v} \in H^{2+k'}_{\mu'}(D_\theta, M)$, $p \in H^{1+k'}_{\mu'}(D_\theta, M)$ with $k' = k - [\mu]$.

Finally, repeating the arguments of Theorem 4.2 we prove that $\vec{v} \in H^{2+k}_\mu(D_\theta, M)$, $p \in H^{1+k}_\mu(D_\theta, M)$.

4.3. Local estimates.

THEOREM 4.5. *For a solution* $\vec{v} \in H^{k+2}_\mu(D_\theta, M)$, $p \in H^{k+1}_\mu(D_\theta, M)$ *of* (4.1), (4.2) *with* k, μ *satisfying* (4.11) *the estimate*

(4.19) $$\|\vec{v}\|_{H^{k+2}_\mu(K_{\varrho-\varkappa\varrho}(\xi), M)} + \|p\|_{H^{k+1}_\mu(K_{\varrho-\varkappa\varrho}(\xi), M)} \leqslant c_1(\varkappa) T_0 + c_2(\varkappa)\big(\varrho^{-2-k}\|\vec{v}\|_{L_{2,\mu}(K_\varrho(\xi), M)} + \varrho^{-1-k}\|p\|_{L_{2,\mu}(K_\varrho(\xi), M)}\big)$$

is valid in an arbitrary domain

$$K_\varrho(\xi) = \{x \in D_\theta \colon |x - \xi| < \varrho\}$$

with $\xi \in D_\theta$, $\varrho > c_0|\xi'| \geqslant 0$ *and with arbitrary* $\varkappa \in (0, 1)$. *In this estimate*

$$T_0 = \|\vec{f}\|_{H^k_\mu(K_\varrho(\xi), M)} + \|r\|_{H^{k+1}_\mu(K_\varrho(\xi), M)} + \|\vec{a}\|_{H^{k+3/2}_\mu(\partial K_\varrho \cap \Gamma_\theta, M)} + \|b\|_{H^{k+3/2}_\mu(\partial K_\varrho \cap \Gamma_0, M)} + \|\vec{d}\|_{H^{k+3/2}_\mu(\partial K_\varrho \cap \Gamma_0, M)}.$$

Proof. We establish (4.19) by means of a method presented in [27], § 19. Let $\psi_\lambda \in C^\infty_0(R^3)$ be a function satisfying the conditions $\psi_\lambda(x) = 1$ for $|x - \xi| \leqslant \varrho - \lambda$, $\psi_\lambda(x) = 0$ for $|x - \xi| \geqslant \varrho - \lambda/2$, $|D^\alpha \psi_\lambda(x)| \leqslant c_\alpha \lambda^{-|\alpha|}$ for a fixed $\lambda \in (0, \varrho\varkappa)$. The functions $\vec{u} = \vec{v}\psi_\lambda$, $q = p\psi_\lambda$ are a solution of a problem (4.1), (4.2) with right members $\psi_\lambda \vec{f} - 2(\nabla\psi_\lambda \cdot \nabla)\vec{v} - \vec{v}\,\nabla^2\psi_\lambda$ etc. Applying to $\vec{u}$, q the preceding theorem we obtain

(4.20) $$\|\vec{v}\|_{H^{k+2}_\mu(K_{\varrho-\lambda}, M)} + \|p\|_{H^{k+1}_\mu(K_{\varrho-\lambda}, M)} \leqslant \|\vec{u}\|_{H^{k+2}_\mu(D_\theta, M)} + \|q\|_{H^{k+1}_\mu(D_\theta, M)} \leqslant c_3\Big\{T(\lambda) + \sum_{j=0}^{k+1} \lambda^{j-k-2}\|\vec{v}\|_{H^j_\mu(K_{\varrho-\lambda/2}, M)} + \sum_{j=0}^{k} \lambda^{j-k-1}\|p\|_{H^j_\mu(K_{\varrho-\lambda/2}, M)}\Big\},$$

where

$$T(\lambda) = \sum_{j=0}^{k} \lambda^{j-k}\|\vec{f}\|_{H^j_\mu(K_\varrho, M)} + \sum_{j=0}^{k+1} \lambda^{j-k-1}\|r\|_{H^j_\mu(K_\varrho, M)} + \sum_{j=0}^{k+1} \lambda^{j-k-1}\|\vec{a}\|_{H^{j+1/2}_\mu(K_\varrho \cap \Gamma_\theta, M)} + \sum_{j=0}^{k+1} \lambda^{j-k-1}\|b\|_{H^{j+1/2}_\mu(K_\varrho \cap \Gamma_0, M)} + \sum_{j=0}^{k} \lambda^{j-k}\|\vec{d}\|_{H^{j+1/2}_\mu(K_\varrho \cap \Gamma_0, M)}.$$

To estimate the norms of $\vec{v}$ and p in the right-hand side we use interpolation inequalities (2.7). For any $\varepsilon \in (0, 1)$ we have

$$\sum_{j=0}^{k+1} \lambda^{j-k-2}\|\vec{v}\|_{H^j_\mu(K_{\varrho-\lambda/2}, M)} \leqslant \varepsilon\|\vec{v}\|_{H^{k+2}_\mu(K_{\varrho-\lambda/2}, M)} + c_4(\varepsilon)\lambda^{-k-2}\|\vec{v}\|_{L_{2,\mu}(K_{\varrho-\lambda/2}, M)},$$

$$\sum_{j=0}^{k} \lambda^{j-k-1}\|p\|_{H^j_\mu(K_{\varrho-\lambda/2}, M)} \leqslant \varepsilon\|p\|_{H^{k+1}_\mu(K_{\varrho-\lambda/2}, M)} + c_5(\varepsilon)\lambda^{-k-1}\|p\|_{L_{2,\mu}(K_{\varrho-\lambda/2}, M)}.$$

For sufficiently small ε this leads to the inequality $F(\lambda) \leqslant \frac{1}{2}F(\lambda/2) + K(\lambda)$ for the functions

$$F(\lambda) = \lambda^{k+2}(\|\vec{v}\|_{H_\mu^{k+2}(K_{\varrho-\lambda}, M)} + \|p\|_{H_\mu^{k+1}(K_{\varrho-\lambda}, M)})$$

and

$$K(\lambda) = c_6\big(\lambda^{k+2}T(\lambda) + \|\vec{v}\|_{L_{2,\mu}(K_\varrho, M)} + \|p\|_{L_{2,\mu}(K_\varrho, M)}\,\lambda\big).$$

Since $K(\lambda)$ is non-decreasing, this inequality yields $f(\lambda) \leqslant 2K(\lambda)$. Taking $\lambda = \varkappa\varrho$ we get (4.19).

THEOREM 4.6. *Let the conditions of Theorem 4.5 be satisfied and let* $\dfrac{\partial^m \vec{v}}{\partial x_3^m} \in H_\mu^{k+2}(D_\theta, M)$, $\dfrac{\partial^m p}{\partial x_3^m} \in H_\mu^{k+1}(D_\theta, M)$ *for* $m = 0, \ldots, t$. *Then*

$$(4.21)\quad \left\|\frac{\partial^t \vec{v}}{\partial x_3^t}\right\|_{H_\mu^{k+2}(K_{\varrho-\varkappa\varrho}(\xi), M)} + \left\|\frac{\partial^t p}{\partial x_3^t}\right\|_{H_\mu^{k+1}(K_{\varrho-\varkappa\varrho}, M)}$$
$$\leqslant c_1 \sum_{m=0}^{t} \varrho^{m-t} T_m + c_2\big(\varrho^{-2-k-t}\|\vec{v}\|_{L_{2,\mu}(K_\varrho(\xi), M)} + \varrho^{-1-k-t}\|p\|_{L_{2,\mu}(K_\varrho(\xi), M)}\big)$$

where

$$T_m = \left\|\frac{\partial^m \vec{f}}{\partial x_3^m}\right\|_{H_\mu^k(K_\varrho(\xi), M)} + \left\|\frac{\partial^m r}{\partial x_3^m}\right\|_{H_\mu^{k+1}(K_\varrho(\xi), M)} +$$
$$+ \left\|\frac{\partial^m \vec{a}}{\partial x_3^m}\right\|_{H_\mu^{k+3/2}(K_\varrho \cap \Gamma_\theta, M)} + \left\|\frac{\partial^m b}{\partial x_3^m}\right\|_{H_\mu^{k+3/2}(K_\varrho \cap \Gamma_0, M)} + \left\|\frac{\partial^m \vec{d}}{\partial x_3^m}\right\|_{H_\mu^{k+1/2}(K_\varrho \cap \Gamma_0, M)}.$$

Proof. Let $1 > \varkappa > \varkappa_1 > \varkappa_2 > 1/2$ and let $R_t(K_{\varrho-\varkappa\varrho})$ denote the left member in (4.21). In virtue of (4.19),

$$R_t(K_{\varrho-\varkappa\varrho}) \leqslant c_3 T_t + c_4 R_{t-1}(K_{\varrho-\varkappa_1\varrho})\,\varrho^{-1} \leqslant \ldots$$
$$\leqslant c_5 \sum_{q=1}^{t} \varrho^{q-t} T_q + c_6 \varrho^{-t} R_0(K_{\varrho-\varkappa_2\varrho})$$
$$\leqslant c_7 \sum_{q=0}^{t} \varrho^{q-t} T_q + c_8\big(\varrho^{-2-k-t}\|\vec{v}\|_{L_{2,\mu}(K_\varrho, M)} + \varrho^{-1-k-t}\|p\|_{L_{2,\mu}(K_\varrho, M)}\big).$$

Remark. For a fixed $K_\varrho(\xi)$ the estimates (4.19), (4.21) are valid under the weaker assumptions that $\vec{v} \in H_\mu^{k+2}(K_\varrho, M)$, $p \in H_\mu^{k+1}(K_\varrho, M)$ or $\vec{v} \in H_\mu^{k+2,t}(K_\varrho, M)$, $p \in H_\mu^{k+1,t}(K_\varrho, M)$.

5. Reference problem in $\mathring{C}_s^l(D_\theta, M)$

5.1. Green's matrix for the problem (4.1), (4.3). In this section we define Green's matrix for the problem (4.1), (4.3) and obtain exact pointwise estimates for its elements with the aid of Theorem 4.6. The idea of using local estimates for solutions of elliptic boundary value problems in the investigation of their Green's matrices is quite familiar (see for instance [23]).

The estimates of Green's functions for reference elliptic boundary-value problems in a bihedral angle were first obtained by V. G. Maz'ya and B. A. Plamenevsky [19]. In the article [20] these authors announced estimates for Green's matrix of a boundary value problem for the Stokes system with the boundary conditions $\vec{u}|_{\partial\Omega} = 0$.

In what follows we give an extended exposition of the contents of Section 6 in [12].

We recall that the fundamental matrix of the Stokes system is the matrix of fourth order with the elements

$$Z_{kj}(x) = \frac{1}{8\pi}\left(\frac{\delta_{kj}}{|x|} + \frac{x_j x_k}{|x|^3}\right), \quad k, j = 1, 2, 3,$$

$$Z_{4j}(x) = Z_{j4}(x) = \frac{x_j}{4\pi|x|^3}, \quad Z_{44}(x) = \delta(x)$$

($\delta(x)$ is the Dirac function) satisfying the relations

$$-\nabla^2 Z_{kj} + \frac{\partial Z_{4j}}{\partial x_k} = \delta_{kj}\delta(x), \quad \sum_{k=1}^{3}\frac{\partial Z_{kj}}{\partial x_k} = 0,$$
$$-\nabla^2 Z_{4k} + \frac{\partial Z_{44}}{\partial x_k} = 0, \quad \sum_{k=1}^{3}\frac{\partial Z_{k4}}{\partial x_k} = \delta(x).$$

Define elements G_{ab} of Green's matrix $\mathcal{G}$ by the formula

$$(5.1)\quad G_{ab}(x, y) = \psi(x, y)Z_{ab}(x-y) + G'_{ab}(x, y), \quad a, b = 1, 2, 3, 4,$$

where ψ is an infinitely differentiable function of both variables; for each $y \in D_\theta$, $\psi(x, y) = 1$ in a neighbourhood of y and $\psi(x, y) = 0$ near M and for big values of $|x|$. The functions G'_{ab} are solutions of the problem

$$-\nabla^2 \vec{G}'_b(x, y) + \nabla G'_{4b} = 2(\nabla\psi \cdot \nabla)\vec{Z}_b(x-y) + \nabla^2\psi \vec{Z}_b - Z_{4b}\nabla\psi,$$
$$\nabla \cdot \vec{G}'_b = -\nabla\psi \cdot \vec{Z}_b(x-y),$$
$$\vec{G}'_b|_{x\in\Gamma_\theta} = -\psi\vec{Z}_b(x-y)|_{x\in\Gamma_\theta}, \quad G'_{2b}|_{x\in\Gamma_0} = -\psi Z_{2b}|_{x\in\Gamma_0},$$

$$(5.2)\quad \left.\frac{\partial G'_{1b}}{\partial x_2}\right|_{x\in\Gamma_0} = -\left.\frac{\partial\psi Z_{1b}}{\partial x_2}\right|_{x\in\Gamma_0}, \quad \left.\frac{\partial G'_{3b}}{\partial x_2}\right|_{x\in\Gamma_0} = -\left.\frac{\partial\psi Z_{3b}}{\partial x_2}\right|_{x\in\Gamma_0},$$

where $\vec{G}_b(x, y) = (G_{1b}, G_{2b}, G_{3b})$, $\vec{Z}_b = (Z_{1b}, Z_{2b}, Z_{3b})$, $Z_{44}\nabla\psi = 0$, ∇ contains derivatives with respect to x.

The following properties of G_{ab} may be easily deduced from the definition.

1. $\mathscr{G}$ and G'_{44} do not depend on ψ.

2. $G_{kj}(\lambda x, \lambda y) = \lambda^{-1}G_{kj}(x, y)$, $j, k = 1, 2, 3$, $G_{4j}(\lambda x, \lambda y) = \lambda^{-2}G_{4j}(x, y)$, $G_{j4}(\lambda x, \lambda y) = \lambda^{-2}G_{j4}(x, y)$, $G'_{44}(\lambda x, \lambda y) = \lambda^{-3}G'_{44}(x, y)$.

3. $G_{ab}(x, y)$ are infinitely differentiable with respect to both arguments everywhere except the sets $x = y$, $x \in M$, $y \in M$ and satisfy the relations

$$(5.3)\qquad \begin{gathered} -\nabla^2\vec{G}_b(x, y) + \nabla G_{4b}(x, y) = 0, \\ \nabla\cdot\vec{G}_b = 0 \quad (x, y \in D_\theta,\ x \neq y), \\ \vec{G}_b|_{x\in\Gamma_\theta} = 0, \quad G_{2b}|_{x\in\Gamma_0} = 0, \quad \frac{\partial G_{1b}}{\partial x_2}\bigg|_{x\in\Gamma_0} = \frac{\partial G_{3b}}{\partial x_2}\bigg|_{x\in\Gamma_0} = 0. \end{gathered}$$

Take in (5.1) $\psi(x, y) = \zeta\left(\dfrac{2|x-y|}{|y'|}\right)$. Then in virtue of (4.18)

$$(5.4)\qquad \begin{aligned} &\|D_y^\beta\vec{G}'_j(x, y)\|_{H_\mu^{2+k}(D_\theta, M)} + \|D_y^\beta G'_{4j}(x, y)\|_{H_\mu^{1+k}(D_\theta, M)} \\ &\qquad\qquad \leqslant c_1 d(y)^{-3/2-|\beta|+\mu-k}, \quad j < 4, \\ &\|D_y^\beta\vec{G}'_4(x, y)\|_{H_\mu^{2+k}(D_\theta, M)} + \|D_y^\beta G'_{44}(x, y)\|_{H_\mu^{1+k}(D_\theta, M)} \\ &\qquad\qquad \leqslant c_2 d(y)^{-5/2-|\beta|+\mu-k}, \end{aligned}$$

where $d(y) = \operatorname{dist}(y, \partial D_\theta)$. These estimates are useful if $d(y) \approx |y'|$. If y is close to Γ_0 or Γ_θ it is convenient to use certain other representations of $\mathscr{G}$, namely,

$$(5.5)\qquad \begin{aligned} \mathscr{G}(x, y) &= \zeta\left(\frac{2|x-y|}{|y'|}\right)\mathscr{G}^{(0)}(x, y) + \mathscr{G}'^{(0)}(x, y), \quad y \in D^{(0)}, \\ \mathscr{G}(x, y) &= \zeta\left(\frac{2|x-y|}{|y'|}\right)\mathscr{G}^{(1)}(x, y) + \mathscr{G}'^{(1)}(x, y), \quad y \in D^{(1)}. \end{aligned}$$

Here $D^{(0)} = D_{\theta/3}$, $D^{(1)} = D_\theta \setminus D_{2\theta/3}$, $\mathscr{G}^{(0)}$ and $\mathscr{G}^{(1)}$ are Green's matrices of the boundary value problems for the Stokes system in the half-spaces $R^{(0)} = \{0 < \varphi < \pi\}$ and $R^{(1)} = \{\theta - \pi < \varphi < \theta\}$ (φ is the polar angle in R^2) with the boundary conditions

$$G_{2b}^{(0)}|_{x\in\partial R^{(0)}} = 0, \quad \frac{\partial G_{1b}^{(0)}}{\partial x_2}\bigg|_{x\in\partial R^{(0)}} = \frac{\partial G_{3b}^{(0)}}{\partial x_2}\bigg|_{x\in\partial R^{(0)}} = 0, \quad \vec{G}_b^{(1)}|_{x\in\partial R^{(1)}} = 0.$$

Green's matrices for elliptic boundary value problems in a half-space are studied in [28]. In particular, it follows from Lemma 2.4 of [28] that for any $x, y \in R^{(i)}$, $x \neq y$, $j, k = 1, 2, 3$,

$$(5.6)\qquad \begin{gathered} |D_x^\alpha D_y^\beta G_{jk}^{(i)}(x, y)| \leqslant c_3|x-y|^{-1-|\alpha|-|\beta|}, \\ |D_x^\alpha D_y^\beta G_{j4}^{(i)}(x, y)| + |D_x^\alpha D_y^\beta G_{4j}^{(i)}(x, y)| \leqslant c_4|x-y|^{-2-|\alpha|-|\beta|}, \\ |D_x^\alpha D_y^\beta G_{44}^{(i)}(x, y)| \leqslant c_5|x-y^{(i)}|^{-3-|\alpha|-|\beta|} \end{gathered}$$

($y^{(i)}$ is symmetric to y with respect to the plane $\partial R^{(i)}$).

The elements of the matrices $\mathscr{G}'^{(i)}$ are solutions of the boundary value problems

$$\begin{gathered} -\nabla^2\vec{G}_b'^{(i)} + \nabla G_{4b}'^{(i)} = 2(\nabla\zeta\cdot\nabla)\vec{G}_b^{(i)} + \vec{G}_b^{(i)}\nabla^2\zeta - G_{4b}^{(i)}\nabla\zeta, \\ \nabla\cdot\vec{G}_b'^{(i)} = -\nabla\zeta\cdot\vec{G}_b^{(i)}, \\ \vec{G}_b'^{(i)}|_{x\in\Gamma_\theta} = -\zeta\vec{G}_b^{(i)}|_{x\in\Gamma_\theta}, \quad G_{2b}'^{(i)}|_{x\in\Gamma_0} = -\zeta G_{2b}^{(i)}|_{x\in\Gamma_0'}, \\ \frac{\partial G_{1b}'^{(i)}}{\partial x_2}\bigg|_{x\in\Gamma_0} = -\frac{\partial\zeta G_{1b}^{(i)}}{\partial x_2}\bigg|_{x\in\Gamma_0}, \quad \frac{\partial G_{3b}'^{(i)}}{\partial x_2}\bigg|_{x\in\Gamma_0} = -\frac{\partial\zeta G_{3b}^{(i)}}{\partial x_2}\bigg|_{x\in\Gamma_0}. \end{gathered}$$

Taking into account the boundary conditions for $\vec{G}^{(i)}$, it is not hard to prove by means of Theorem 4.4 that for $y \in D^{(i)}$

$$(5.7)\qquad \|D_y^\beta\vec{G}'^{(i)}(x, y)\|_{H_\mu^{2+k}(D_\theta, M)} + \|D_y^\beta G_{4j}'^{(i)}(x, y)\|_{H_\mu^{1+k}(D_\theta, M)} \leqslant c_6|y'|^{-3/2-|\beta|+\mu-k},$$

$$(5.8)\qquad \|D_y^\beta\vec{G}_4'^{(i)}(x, y)\|_{H_\mu^{2+k}(D_\theta, M)} + \|D_y^\beta G_{44}'^{(i)}(x, y)\|_{H_\mu^{1+k}(D_\theta, M)} \leqslant c_7|y'|^{-5/2-|\beta|+\mu-k}.$$

Note that the formulas (5.5) have a sense, since for $y \in D^{(i)}$

$$\operatorname{supp}\zeta\left(\frac{2|x-y|}{|y'|}\right) \subset R^{(i)}.$$

THEOREM 5.1. *For all $z, y \in D_\theta$, $k, j = 1, 2, 3$, we have*

$$(5.9)\qquad \begin{gathered} G_{jk}(z, y) = G_{kj}(y, z), \quad G_{j4}(z, y) = -G_{4j}(y, z), \\ G'_{44}(z, y) = G'_{44}(y, z). \end{gathered}$$

Proof. Let $D_\theta^\varepsilon = \{x \in D_\theta\colon |x-y| > \varepsilon,\ |x-z| > \varepsilon\}$. Take in Green's identity

$$(5.10)\qquad \int_{D_\theta^\varepsilon} \{[\vec{u}\cdot(\nabla^2\vec{v} - \nabla p) - \vec{v}\cdot(\nabla^2\vec{u} - \nabla q)] + (q\nabla\cdot\vec{v} - p\nabla\cdot\vec{u})\}\,dx$$

$$= \int_{\partial D_\theta^\varepsilon} \left\{\vec{u}\cdot\left(\frac{\partial\vec{v}}{\partial n} - \vec{n}p\right) - \vec{v}\cdot\left(\frac{\partial\vec{u}}{\partial n} - \vec{n}q\right)\right\} dS,$$

$$\vec{v} = \vec{G}_b(x, z), \qquad p = G_{4b}(x, z),$$

$$\vec{u} = \vec{G}_a(x, y)\,\zeta\left(\frac{|x'|}{R}\right)\zeta\left(\frac{|x_3|}{R}\right), \qquad q = G_{4a}(x, y)\,\zeta\left(\frac{|x'|}{R}\right)\zeta\left(\frac{|x_3|}{R}\right).$$

Each term in the left-hand side integral contains derivatives of ζ, and so, for big R, $G_{mn} = G'_{mn}$ in the left integrand. Taking into account the fact that $G'_{4a} \in L_2(D_\theta)$, $\vec{G}'_a \in \mathring{H}^1_0(D_\theta, M)$ it is easy to verify that the left-hand side of (5.10) tends to zero as $R \to \infty$, which leads to

$$\int_{|x-y|=\varepsilon} \left\{\vec{G}_a(x, y)\cdot\left(\frac{\partial\vec{G}_b(x, z)}{\partial n} - \vec{n}G_{4b}(x, z)\right) - \right.$$

$$\left. - \vec{G}_b(x, z)\cdot\left(\frac{\partial\vec{G}_a(x, y)}{\partial n} - \vec{n}G_{4a}(x, y)\right)\right\} dS$$

$$= \int_{|x-z|=\varepsilon} \left\{\vec{G}_b(x, z)\cdot\left(\frac{\partial\vec{G}_a(x, y)}{\partial n} - \vec{n}G_{4a}(x, y)\right) - \right.$$

$$\left. - \vec{G}_a(x, y)\cdot\left(\frac{\partial\vec{G}_b(x, z)}{\partial n} - \vec{n}G_{4b}(x, z)\right)\right\} dS.$$

Now letting $\varepsilon \to 0$ we obtain (5.9).

THEOREM 5.2. *Let $\vec{f} \in L_{2,\mu}(D_\theta, M)$ and $r \in H^1_\mu(D_\theta, M)$ $(\mu \in [0, 1])$ have compact supports. Then for the solutions $\vec{v} \in H^2_\mu(D_\theta, M)$, $p \in H^1_\mu(D_\theta, M)$ of* (4.1), (4.3) *the following representation formulas hold:*

$$(5.11)\qquad \begin{aligned} v_k(z) &= \sum_{j=1}^{3} \int_{D_\theta} G_{kj}(z, x) f_j(x)\,dx + \int_{D_\theta} G_{k4}(z, x) r(x)\,dx, \\ p(z) &= \sum_{j=1}^{3} \int_{D_\theta} G_{4j}(z, x) f_j(x)\,dx + r(z) + \int_{D_\theta} G'_{44}(z, x) r(x)\,dx. \end{aligned}$$

Proof. We again make use of Green's identity (5.10) with

$$D_\theta^\varepsilon = \{x \in D_\theta\colon |x-z| > \varepsilon\}, \qquad \vec{u} = \vec{G}_b(x, z)\,\zeta\left(\frac{|x'|}{R}\right)\zeta\left(\frac{|x_3|}{R}\right),$$

$$q = G_{4b}(x, z)\,\zeta\left(\frac{|x'|}{R}\right)\zeta\left(\frac{|x_3|}{R}\right).$$

Letting $R \to \infty$ we get

$$(5.10')\qquad \int_{D_\theta^\varepsilon} \big(G_{4b}(x, z) r(x) - \vec{G}_b(x, z)\cdot\vec{f}(x)\big)\,dx$$

$$= \int_{|x-z|=\varepsilon} \left[\vec{G}_b(x, z)\cdot\left(\frac{\partial\vec{v}}{\partial n} - \vec{n}p\right) - \vec{v}(x)\cdot\left(\frac{\partial\vec{G}_b(x, z)}{\partial n} - \vec{n}G_{4b}(x, z)\right)\right] dS$$

(for $b = 4$, G_{4b} should be replaced by G'_{44}).

Now consider both sides of (5.10) as elements of $L_{1,\mathrm{loc}}(D_\theta)$ and pass to the limit as $\varepsilon \to 0$. It can be shown that in the limit we get (5.11).

THEOREM 5.3. *For all β, γ and all $x, y \in D_\theta$, $x \neq y$, the estimates*

$$(5.12)\qquad |D_x^\beta D_y^\gamma G_{ij}(x, y)| \leqslant c_1\left(\frac{|x'|}{|x'| + |x-y|}\right)^{\lambda-\beta_1-\beta_2}\left(\frac{|y'|}{|y'| + |x-y|}\right)^{\lambda-\gamma_1-\gamma_2}|x-y|^{-1-|\beta|-|\gamma|},$$

$$(5.13)\qquad |D_x^\beta D_y^\gamma G_{i4}(x, y)| \leqslant c_2\left(\frac{|x'|}{|x'| + |x-y|}\right)^{\lambda-\beta_1-\beta_2}\left(\frac{|y'|}{|y'| + |x-y|}\right)^{\lambda-1-\gamma_1-\gamma_2}|x-y|^{-2-|\beta|-|\gamma|},$$

$$(5.14)\qquad |D_x^\beta D_y^\gamma G_{4j}(x, y)| \leqslant c_3\left(\frac{|x'|}{|x'| + |x-y|}\right)^{\lambda-1-\beta_1-\beta_2}\left(\frac{|y'|}{|y'| + |x-y|}\right)^{\lambda-\gamma_1-\gamma_2}|x-y|^{-2-|\beta|-|\gamma|},$$

$$(5.15)\qquad |D_x^\beta D_y^\gamma G_{44}(x, y)| \leqslant c_4\left(\frac{|x'|}{|x'| + |x-y|}\right)^{\lambda-1-\beta_1-\beta_2}\left(\frac{|y'|}{|y'| + |x-y|}\right)^{\lambda-1-\gamma_1-\gamma_2}|x-y|^{-3-|\beta|-|\gamma|}$$

hold with any $\lambda \in [0, \min(\operatorname{Re}\sigma_0, \pi/2\theta))$, $i, j = 1, 2, 3$.

Proof. In view of the homogeneity of G_{ab} it suffices to prove (5.12)–(5.15) for $|x-y| = 1$. Suppose that $|y'| \geqslant 2$. Then $|x'| \geqslant |y'| - |x'-y'| \geqslant 1$ and (5.12)–(5.15) are equivalent to the boundedness of $D_x^\beta D_y^\gamma G_{ab}$, which follows from the formulas (5.1), (5.5) and estimates (5.4), (5.7), (5.8).

Now let $|y'| \leqslant 2$. In virtue of Theorems 4.6 and 2.4,

$$(5.16)\quad |D_x^\beta D_y^\gamma \vec{G}_b(x,y)|\,|x'|^{-h} + |D_x^\beta D_y^\gamma G_{4b}(x,y)|\,|x'|^{-\tilde{h}+1}$$
$$\leqslant c_5 \big(\|D_y^\gamma \vec{G}_b\|_{H_\mu^{2+k,t}(K_{1/8}(x),M)} + \|D_y^\gamma G_{4b}\|_{H_\mu^{1+k,t}(K_{1/8}(x),M)}\big)$$
$$\leqslant c_6 \big(\|D_y^\gamma \vec{G}_b\|_{L_{2,\mu}(K_{1/4}(x),M)} + \|D_y^\gamma G_{4b}\|_{L_{2,\mu}(K_{1/4}(x),M)}\big)$$

with $h < \varkappa - \mu$, $\tilde{h} < \tilde{\varkappa} - \mu$, $\varkappa = k+1-\beta_1-\beta_2-(\beta_3+\frac{1}{2})\dfrac{k+2}{k+2+t} > 0$,
$\tilde{\varkappa} = k+1-\beta_1-\beta_2-(\beta_3+\frac{1}{2})\dfrac{k+1}{k+1+t}$, t an integer >0, k, μ satisfying (4.11).

Consider the problem (4.1), (4.3) with $\vec{f}(z) = D_y^\gamma \vec{G}_b(z,y)\,\zeta(2|x-z|)$, $r(z) = -\zeta(2|x-z|)\,D_y^\gamma G_{4b}(z,y)$. For its solution a similar estimate holds; namely, we have

$$|D_y^\gamma \vec{v}(y)|\,|y'|^{-h_1} + |D_y^\gamma p(y)|\,|y'|^{-\tilde{h}_1+1} \leqslant c_7 \big(\|\vec{v}\|_{L_{2,\mu_1}(K_{1/4}(y),M)} + \|p\|_{L_{2,\mu_1}(K_{1/4}(y),M)}\big)$$

with $h_1 < \varkappa_1 - \mu_1$, $\tilde{h}_1 < \tilde{\varkappa}_1 - \mu_1$, $\varkappa_1 = k_1+1-\beta_1-\beta_2-(\beta_3+\frac{1}{2})\dfrac{k_1+2}{k_1+2+t}$,
$\tilde{\varkappa}_1 = k_1+1-\beta_1-\beta_2-(\beta_3+\frac{1}{2})\dfrac{k_1+1}{k_1+2+t}$, $\mu_1 > 0$.

Making use of (4.6), we obtain

$$(5.17)\quad |D_y^\gamma \vec{v}(y)|\,|y'|^{-h_1} + |D_y^\gamma p(y)|\,|y'|^{-\tilde{h}_1+1}$$
$$\leqslant c_8 \Big(\int_{K_{1/2}(x)} |D_y^\gamma \vec{G}_b(z,y)|^2 \zeta(2|x-z|)^2\,dz +$$
$$+ \int_{K_{1/2}(x)} |D_y^\gamma G_{4b}(z,y)|^2 \zeta(2|x-z|)\,dz\Big)^{1/2}.$$

From (5.11) and (5.9) we conclude that the right-hand side of (5.17) equals $c_8|D_y^\gamma v_b(y)|^{1/2}$ for $b<4$ and $c_8|D_y^\gamma p(y)|^{1/2}$ for $b=4$. Therefore it does not exceed $c_9|y'|^{h_1}$ for $b<4$ and $c_9|y'|^{\tilde{h}_1-1}$ for $b=4$ and

$$|D_x^\beta D_y^\gamma \vec{G}_j(x,y)|\,|x'|^{-h}|y|^{-h_1} + |D_x^\beta D_y^\gamma G_{4j}(x,y)|\,|x'|^{-\tilde{h}+1}|y'|^{-h_1} \leqslant c_{10},$$
$$|D_x^\beta D_y^\gamma \vec{G}_4(x,y)|\,|x'|^{-h}|y|^{-\tilde{h}_1+1} + |D_x^\beta D_y^\gamma G_{44}(x,y)|\,|x'|^{-\tilde{h}+1}|y'|^{-\tilde{h}_1+1} \leqslant c_{11}$$

($j = 1,2,3$). This proves (5.12)–(5.15), since by choosing t, k, μ, k_1, μ_1 in an appropriate way we may make h, $\tilde{h}$ and h_1, $\tilde{h}_1$ arbitrarily close to $\min(\operatorname{Re}\sigma_0, \pi/2\theta) - \beta_1 - \beta_2$ and $\min(\operatorname{Re}\sigma_0, \pi/2\theta) - \gamma_1 - \gamma_2$, respectively.

5.2. Solvability of the problem (4.1), (4.2) and estimates of the solution.

THEOREM 5.4. *Let*

$$(5.18)\qquad 0 < s < \min(\operatorname{Re}\sigma_0, \pi/2\theta).$$

For arbitrary $\vec{f} \in \mathring{C}^l_{s-2}(D_\theta, M)$, $r \in \mathring{C}^{l+1}_{s-1}(D_\theta, M)$, $\vec{a} \in \mathring{C}^{l+2}_s(\Gamma_\theta, M)$, $b \in \mathring{C}^{l+2}_s(\Gamma_\theta, M)$, $\vec{d} \in \mathring{C}^{l+1}_{s-1}(\Gamma_\theta, M)$ *with compact supports the problem* (4.1), (4.2) *has a unique solution* $\vec{v} \in \mathring{C}^{l+2}_s(D_\theta, M)$, $p \in \mathring{C}^{l+1}_{s-1}(D_\theta, M)$ *and*

$$(5.19)\quad |\vec{v}|_{\mathring{C}^{l+2}_s(D_\theta,M)} + |p|_{\mathring{C}^{l+1}_{s-1}(D_\theta,M)} \leqslant c\big(|\vec{f}|_{\mathring{C}^l_{s-2}(D_\theta,M)} +$$
$$+ |r|_{\mathring{C}^{l+1}_{s-1}(D_\theta,M)} + |\vec{a}|_{\mathring{C}^{l+2}_s(\Gamma_\theta,M)} + |b|_{\mathring{C}^{l+2}_s(\Gamma_\theta,M)} + |\vec{d}|_{\mathring{C}^{l+1}_{s-1}(\Gamma_\theta,M)}\big).$$

Proof. In virtue of Theorem 2.1, it is sufficient to consider the case $\vec{a} = 0$, $b = 0$, $\vec{d} = 0$. Since $\vec{f} \in L_{2,\mu}(D_\theta, M)$, $r \in H^1_\mu(D_\theta, M)$ with any $\mu > 1-s$, there exists a solution $\vec{v} \in H^2_\mu(D_\theta, M)$, $p \in H^1_\mu(D_\theta, M)$ of (4.1), (4.3) with $1-s < \mu \leqslant 1$. It is given by (5.11).

Let us now prove the estimates

$$(5.20)\quad \sup_{D_\theta} |x'|^{-s}|\vec{v}(x)| \leqslant c_1\big(\sup_{D_\theta}|x'|^{2-s}|\vec{f}(x)| + \sup_{D_\theta}|x'|^{1-s}|r(x)|\big),$$

$$(5.21)\quad \sup_{D_\theta} |x'|^{1-s}|p(x)| \leqslant c_2\big(\sup_{D_\theta}|x'|^{2-s}|\vec{f}(x)| + |r|_{\mathring{C}^\delta_{s-1}(D_\theta,M)}\big)$$

with an arbitrary small $\delta > 0$.

Let $x \in D_\theta$, $D' = \{y \in D_\theta\colon |y'| \leqslant 2|x'|\}$, $D'' = D_\theta \setminus D'$. In virtue of (5.12), (5.13) we have

$$\Big|\int_{D_\theta} G_{ij}(x,y) f_j(y)\,dy\Big|$$
$$\leqslant c_3 \sup_{D_\theta}|y'|^{2-s}|f_j(y)| \Big\{ |x'|^{\lambda_1}\int_{D'} \frac{|y'|^{s-2}}{|x-y|^{1+\lambda_1}}\,dy + |x'|^{\lambda_2}\int_{D''}\frac{|y'|^{s-2}}{|x-y|^{1+\lambda_2}}\,dy\Big\}$$
$$\leqslant c_4 |x'|^s \sup_{D_\theta}|y'|^{2-s}|\vec{f}(y)|,$$

$$\Big|\int_{D_\theta} G_{i4}(x,y) r(y)\,dy\Big|$$
$$\leqslant c_5 \sup_{D_\theta}|y'|^{1-s}|r(y)| \Big\{ |x'|^{\lambda_1}\int_{D'} \frac{|y'|^{s-2}}{|x-y|^{2+\lambda_1}}\,(|y'| + |x-y|)\,dy +$$
$$+ |x'|^{\lambda_2}\int_{D''} (|x-y| + |y'|)\frac{|y'|^{s-2}}{|x-y|^{2+\lambda_2}}\,dy\Big\}$$
$$\leqslant c_6 |x'|^s \sup_{D_\theta}|y'|^{1-s}|r(y)|$$

with $0 < \lambda_1 < \min(\operatorname{Re}\sigma_0, \pi/2\theta)$, $s < \lambda_2 < \min(\operatorname{Re}\sigma_0, \pi/2\theta)$. Thus (5.20) is proved. Furthermore,

$$\Big|\int_{D_\theta} G_{4j}(x, y) f_j(y)\,dy\Big|$$
$$\leqslant c_7 \sup_{D_\theta} |y'|^{2-s}|\vec{f}_j(y)|\Big\{|x'|^{\lambda_1-1}\int_{D'}(|x-y|+|x'|)^{1-\lambda_1}\frac{|y'|^{s-2}}{|x-y|^2}\,dy+$$
$$+|x'|^{\lambda_2-1}\int_{D''}\frac{|y'|^{s-2}}{|x-y|^2(|x-y|+|x'|)^{\lambda_2-1}}\,dy\Big\}$$
$$\leqslant c_8|x'|^{s-1}\sup_{D_\theta}|y'|^{2-s}|\vec{f}(y)|,$$

(5.22)
$$\Big|\int_{D_\theta} G'_{44}(x, y) r(y)\,dy\Big|$$
$$\leqslant\Big|\int_{\mathscr{K}(x)} G'_{44}(x, y)[r(x)-r(y)]\,dy\Big|+$$
$$+|r(x)|\Big|\int_{\mathscr{K}(x)} G'_{44}(x, y)\,dy\Big|+\Big|\int_{D_\theta\setminus\mathscr{K}(x)} G'_{44}(x, y) r(y)\,dy\Big|,$$

where

$$\mathscr{K}(x) = \{y \in D_\theta\colon |x-y| \leqslant \eta|x'|\}, \qquad 1/4 > \eta > 0.$$

We now show that for small η we have

$$\Big|\int_{\mathscr{K}(x)} G'_{44}(x, y)\,dy\Big| \leqslant c_6. \tag{5.23}$$

Assume that $\operatorname{dist}(x, \partial D_\theta) \geqslant 2\eta|x'|$. Then, in virtue of (5.4),

$$|G'_{44}(x, y)| \leqslant c_9|x'|^{-3} \quad \text{and} \quad \Big|\int_{\mathscr{K}(x)} G'_{44}\,dy\Big| \leqslant c_9|x'|^{-3}|\mathscr{K}(x)| \leqslant c_6.$$

If the point x is close to Γ_0 or Γ_θ, we use the appropriate formula (5.5), i.e.

$$G'_{44}(x, y) = g^{(i)}_{44}(x, y) + G'^{(i)}_{44}(x, y),$$

where $g^{(i)}_{44} = G^{(i)}_{44}(x, y) - \delta(x, y)$ is the regular part of $G^{(i)}_{44}$.

It follows from the formulas (2.28), (2.29) in the paper [28] that

$$\Big|\int_{\mathscr{K}(x)} g^{(i)}_{44}(x, y)\,dy\Big| \leqslant c_{10}$$

(the left-hand side may be transformed into an integral over the spherical part of $\partial\mathscr{K}(x)$ with the integrand not exceeding $c_{11}|x-y|^{-2}$). Now (5.23) follows from the estimate

$$|G'^{(i)}_{44}(x, y)| \leqslant c_{12}|x'|^{-3}.$$

The estimate (5.23) is proved for any $x \in D_\theta$. The other terms of (5.22) can be easily estimated by means of (5.15), which leads to

$$\Big|\int_{D_\theta} G'_{44}(x, y) r(y)\,dy\Big| \leqslant c_{13}|x'|^{s-1}|r|_{\mathring{C}^\delta_{s-1}(D_\theta, M)}.$$

The proof of (5.20), (5.21) is completed.

The estimate of higher order derivatives of the solution is based on the inequality

(5.24)
$$[\vec{v}]^{(l+2)}_{\Omega_1} + [p]^{(l+1)}_{\Omega_1} \leqslant c_{14}\Big([\vec{f}]^{(l)}_{\Omega_2} + \sum_{|\alpha|<l}\delta^{|\alpha|-l}|D^\alpha\vec{f}|_{\Omega_2} +$$
$$+[r]^{(l+1)}_{\Omega_2} + \sum_{|\alpha|<l+1}\delta^{|\alpha|-l-1}|D^\alpha r|_{\Omega_2} + \delta^{-l-2}|\vec{v}|_{\Omega_2} + \delta^{-l-1}|p|_{\Omega_2}\Big),$$

where $\Omega_1 \subset \Omega_2 \subset D_\theta$, $\delta = \operatorname{dist}(\Omega_1, D_\theta\setminus\Omega_2) > 0$ (see [3], Theorem 2.2). Just as in Theorem 4.2 let $\Omega_1 = U_{nm}$, $\Omega_2 = U'_{nm}$, $\delta = 2^{n-1}$ and multiply (5.24) by δ^{l+2-s}. Maximizing both sides with respect to m, n and taking into account (5.20), (5.21), we get (5.19) for the functions (5.11).

It remains to prove the uniqueness of solution. Let $\vec{v} \in \mathring{C}^{l+2}_s(D_\theta, M)$, $p \in \mathring{C}^{l+1}_{s-1}(D_\theta, M)$ be solutions of (4.1), (4.3) with $r = 0$, $\vec{f} = 0$. Then for $\vec{v}(x)\zeta(|x|/R)$, $p(x)\zeta(|x|/R)$ the formulas (5.11) hold with

$$\vec{f} = -2(\nabla\zeta\cdot\nabla)\vec{v} - \vec{v}\nabla^2\zeta + p\nabla\zeta, \qquad r = \nabla\zeta\cdot\vec{v}.$$

For fixed $x \in D_\theta$ we have

$$|\vec{v}(x)| \leqslant c_{15}|x'|^{\lambda_2}\int_{K_{2R}(0)\setminus K_R(0)}|x-y|^{-1-\lambda_2}(R^{-1}|y'|^{s-1}+R^{-2}|y'|^s)\,dy \xrightarrow[R\to\infty]{} 0,$$

which completes the proof of the theorem.

THEOREM 5.5. *For the solution of* (4.1), (4.2) *the estimate*

(5.25)
$$|\vec{v}|_{\mathring{C}^{l+2}_s(K_\varrho(\xi), M)} + |p|_{\mathring{C}^{l+1}_{s-1}(K_\varrho(\xi), M)} \leqslant c(|\vec{f}|_{\mathring{C}^l_{s-2}(K_\varrho(\xi), M)} +$$
$$+|r|_{\mathring{C}^{l+1}_{s-1}(K_{2\varrho}(\xi), M)} + |\vec{a}|_{\mathring{C}^{l+2}_s(\Gamma_\theta\cap K_{2\varrho}, M)} + |b|_{\mathring{C}^{l+2}_s(\Gamma_0\cap K_{2\varrho}, M)} +$$
$$+|\vec{d}|_{\mathring{C}^{l+1}_{s-1}(\Gamma_0\cap K_{2\varrho}, M)} + \varrho^{l+1/2-s}\|\vec{v}\|_{L_2(K_{2\varrho})} + \varrho^{l-1/2-s}\|p\|_{L_2(K_{2\varrho})})$$

holds with any $\varrho > 0$, $\xi \in M$.

The proof is just the same as in Theorem 4.5, but instead of (2.7) the interpolation inequality (2.8) should now be used.

6. The linear problem in the domain Ω

Suppose that $S \in C^{l+2}$, l is a non-integer > 0, and that the function $\varphi(x_1, x_2)$ which determines the surface Γ belongs to $C^{l+3}_{s+1}(\omega, \partial\omega)$ with $s \in (0, l+2)$. Consider in Ω the linear boundary value problem

(6.1) $$-\nabla^2\vec{v} + \nabla p = \vec{f}, \qquad \nabla\cdot\vec{v} = r,$$

(6.2) $$\vec{v}|_\Sigma = 0, \qquad \vec{v}\cdot\vec{n}|_\Gamma = 0, \qquad S(\vec{v})\vec{n} - \vec{n}\big(\vec{n}\cdot S(\vec{v})\vec{n}\big)|_\Gamma = 0.$$

Let $\mathfrak{M}(\Omega)$ denote the space of vectors $\vec{v} \in H^1_0(\Omega, \mathscr{M})$ satisfying the boundary conditions $\vec{v}|_\Sigma = 0$, $\vec{v}\cdot\vec{n}|_\Gamma = 0$ and let $\mathfrak{J}(\Omega)$ denote the subspace of divergence-free vectors in $\mathfrak{M}(\Omega)$. Define a weak solution of (6.1), (6.2) as a vector $\vec{v} \in \mathfrak{M}(\Omega)$ satisfying the equation $\nabla\cdot\vec{v} = r$ and the integral identity

$$\frac{1}{2}\int_\Omega \sum_{i,j=1}^{3} S_{ij}(\vec{v})S_{ij}(\vec{\eta})\,dx = \int_\Omega \vec{f}\cdot\vec{\eta}\,dx, \qquad \forall \vec{\eta} \in \mathfrak{J}(\Omega).$$

THEOREM 6.1. *For arbitrary $\vec{f} \in L_{2,1}(\Omega, \mathscr{M})$, $r \in L_2(\Omega)$ such that $\int_\Omega r\,dx = 0$ the problem (6.1), (6.2) has a unique weak solution and there exists a unique function $p \in L_2(\Omega)$, satisfying the condition $\int_\Omega p\,dx = 0$ and the identity*

$$\frac{1}{2}\int_\Omega \sum_{i,j=1}^{3} S_{ij}(\vec{v})S_{ij}(\vec{\varphi})\,dx - \int_\Omega \vec{f}\cdot\vec{\varphi}\,dx - \int_\Omega r\nabla\cdot\vec{\varphi}\,dx = \int_\Omega p\nabla\cdot\vec{\varphi}\,dx$$

for any $\vec{\varphi} \in \mathfrak{M}(\Omega)$. Moreover,

(6.3) $$\|\vec{v}\|_{H^1_0(\Omega,\mathscr{M})} + \|p\|_{L_2(\Omega)} \leqslant c(\|\vec{f}\|_{L_{2,1}(\Omega,\mathscr{M})} + \|r\|_{L_2(\Omega)}).$$

The proof is the same as that of Theorem 4.1 (see also [4]).

THEOREM 6.2. *For any $\vec{f} \in \overset{\circ}{C}{}^{l}_{s-2}(\Omega, \mathscr{M})$, $r \in \overset{\circ}{C}{}^{l+1}_{s-1}(\Omega, \mathscr{M})$ $(\int_\Omega r\,dx = 0)$ the problem (6.1), (6.2) has a unique solution $\vec{v} \in \overset{\circ}{C}{}^{l+2}_{s}(\Omega, \mathscr{M})$, $p \in \overset{\circ}{C}{}^{l+1}_{s-1}(\Omega, \mathscr{M})$ satisfying the condition $\int_\Omega p\,dx = 0$, and the inequality*

(6.4) $$|\vec{v}|_{\overset{\circ}{C}{}^{l+2}_{s}(\Omega,\mathscr{M})} + |p|_{\overset{\circ}{C}{}^{l+1}_{s-1}(\Omega,\mathscr{M})} \leqslant c(|\vec{f}|_{\overset{\circ}{C}{}^{l}_{s-2}(\Omega,\mathscr{M})} + |r|_{\overset{\circ}{C}{}^{l+1}_{s-1}(\Omega,\mathscr{M})})$$

provided $0 < s < \min(\operatorname{Re}\sigma_0, \pi/2\theta)$, $\varphi \in C^{l+3}_{s+1}(\omega, \partial\omega)$.

Proof. As shown in [4], for a weak solution of (6.1), (6.2) the inequality

(6.5) $$|\vec{v}|_{C^{l+2}(\Omega')} + |p|_{C^{l+1}(\Omega')} \leqslant c_1(|\vec{f}|_{C^{l}(\Omega'')} + |r|_{C^{l+1}(\Omega'')} + \|\vec{f}\|_{L_{2,1}(\Omega,\mathscr{M})} + \|r\|_{L_2(\Omega)})$$

holds in arbitrary $\Omega' \subset \Omega'' \subset \Omega$; $\operatorname{dist}(\Omega'', \mathscr{M}) > 0$. To estimate the solution near $\mathscr{M}$, introduce in the domain $\Omega_d(\xi)$, $\xi \in \mathscr{M}$ (we may assume that $\xi = 0$), new variables $z = Tx$, the transformation T being defined in Theorem 2.5. It transforms the operator $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ into

$$\hat{\nabla} = A\nabla = \left(\sum_{m=1}^{3} a_{1m}\frac{\partial}{\partial z_m}, \sum_{m=1}^{3} a_{2m}\frac{\partial}{\partial z_m}, \sum_{m=1}^{3} a_{3m}\frac{\partial}{\partial z_m}\right)$$

where

$$a_{kl}(z) = \frac{\partial T_l(x)}{\partial x_k}\bigg|_{x=T^{-1}z}, \qquad A = \{a_{kl}\}.$$

Let $\hat{f}(z) = f(T^{-1}z)$. The functions

$$\vec{u}(z) = \hat{\vec{v}}(z)\zeta\left(\frac{2|z|}{\delta}\right), \qquad q(z) = \hat{p}(z)\zeta\left(\frac{2|z|}{\delta}\right)$$

satisfy the following relations in the bihedral angle $\mathfrak{D}$ whose sides are the planes Σ_0 and Γ_0 tangent to Σ and Γ at the point ξ:

(6.6) $$\begin{gathered} -\nabla^2\vec{u} + \nabla q = L_1(\vec{u}, q) + \vec{g} + \zeta\hat{\vec{f}}, \qquad \nabla\cdot\vec{u} = L_2(\vec{u}) + h + \zeta\hat{r}, \\ \vec{u}|_{\Sigma_0} = 0, \qquad \vec{u}\cdot\vec{n}_0|_{\Gamma_0} = L_3(\vec{u}), \\ S(\vec{u})\vec{n}_0 - \vec{n}_0(\vec{n}_0\cdot S(\vec{u})\vec{n}_0)|_{\Gamma_0} = L_4(\vec{u}) + \vec{d}. \end{gathered}$$

In these relations

$$g = -2(\hat{\nabla}\zeta\cdot\hat{\nabla})\hat{\vec{v}} - \hat{\vec{v}}\,\hat{\nabla}^2\zeta + \hat{p}\hat{\nabla}\zeta, \qquad h = \hat{\nabla}\zeta\cdot\hat{\vec{v}},$$

$$\vec{d} = \hat{\nabla}\zeta(\hat{\vec{v}}\cdot\hat{\vec{n}}) + \hat{\vec{v}}(\hat{\nabla}\zeta\cdot\hat{\vec{n}}) - 2(\hat{\vec{v}}\cdot\hat{\vec{n}})(\hat{\nabla}\zeta\cdot\hat{\vec{n}})\hat{\vec{n}},$$

$\vec{n}_0$ is the unit normal vector to Γ_0 (it is easy to verify that

$$\hat{\vec{n}}(z) = \frac{A(z)\vec{n}_0}{|A(z)\vec{n}_0|} = B(z)\vec{n}_0),$$

$$L_1(\vec{u}, q) = [(\hat{V}^2 - V^2)\vec{u} + (V - \hat{V})q]\zeta\left(\frac{|z|}{\delta}\right),$$

$$L_2(\vec{u}) = \zeta\left(\frac{|z|}{\delta}\right)(V - \hat{V})\vec{u},$$

(6.7)
$$L_3(\vec{u}) = \zeta\left(\frac{|z|}{\delta}\right)\vec{u}\cdot(I - B(z))\vec{n}_0,$$

$$L_4(\vec{u}) = \zeta\left(\frac{|z|}{\delta}\right)\{[S(\vec{u})\vec{n} - \hat{S}(\vec{u})\vec{n}_0] - [\vec{n}_0(\vec{n}_0\cdot S(\vec{u})\vec{n}_0) - \\ - \hat{\vec{n}}(\hat{\vec{n}}\cdot\hat{S}(\vec{u})\hat{\vec{n}})]\},$$

$\hat{S}(\vec{u})$ is the matrix with elements

$$\hat{S}_{ij}(\vec{u}) = \sum_{l=1}^{3}\left(a_{il}\frac{\partial u_j}{\partial z_l} + a_{jl}\frac{\partial u_i}{\partial z_l}\right).$$

Since $\operatorname{supp}\vec{u}$, $\operatorname{supp}\zeta \subset K_\delta(0)$, the factor $\zeta(|z|/\delta)$ in (6.7) may be omitted, but we are going to consider the operators L_i for arbitrary $\vec{u}$, q. The coefficients of L_i contain factors $a_{kl}(z) - \delta_{kl}$ or $\partial a_{kl}/\partial z_i$; therefore for any $\vec{u} \in H^2_\mu(\mathfrak{D}, \mathfrak{M})$, $q \in H^1_\mu(\mathfrak{D}, \mathfrak{M})$ ($\mathfrak{M}$ is the edge of $\mathfrak{D}$) we have

$$\|L_1(\vec{u}, q)\|_{L_{2,\mu}(\mathfrak{D},\mathfrak{M})} \leqslant c_1\delta^{s_1}(\|\vec{u}\|_{H^2_\mu(\mathfrak{D},\mathfrak{M})} + \|q\|_{H^1_\mu(\mathfrak{D},\mathfrak{M})}),$$

(6.8)
$$\|L_2(\vec{u})\|_{H^1_\mu(\mathfrak{D},\mathfrak{M})} + \|L_3(\vec{u})\|_{H^{3/2}_\mu(\Gamma_0,\mathfrak{M})} + \|L_4(\vec{u})\|_{H^{1/2}_\mu(\Gamma_0,\mathfrak{M})} \leqslant c_2\delta^{s_1}\|\vec{u}\|_{H^2_\mu(\mathfrak{D},\mathfrak{M})}$$

where c_i are independent of δ and $s_1 \leqslant s$. Moreover, for arbitrary $\vec{u} \in \mathring{C}^{l+2}_s(\mathfrak{D}, \mathfrak{M})$, $q \in \mathring{C}^{l+1}_{s-1}(\mathfrak{D}, \mathfrak{M})$ we have

$$|L_1(\vec{u}, q)|_{\mathring{C}^l_{s-2}(\mathfrak{D},\mathfrak{M})} \leqslant c_3\delta^{s_1}(|\vec{u}|_{\mathring{C}^{l+2}_s(\mathfrak{D},\mathfrak{M})} + |q|_{\mathring{C}^{l+1}_{s-1}(\mathfrak{D},\mathfrak{M})}),$$

(6.9)
$$|L_2(\vec{u})|_{\mathring{C}^{l+1}_{s-1}(\mathfrak{D},\mathfrak{M})} + |L_3(\vec{u})|_{\mathring{C}^{l+2}_s(\Gamma_0,\mathfrak{M})} + |L_4(\vec{u})|_{\mathring{C}^{l+1}_{s-1}(\Gamma_0,\mathfrak{M})} \leqslant c_4\delta^{s_1}|\vec{u}|_{\mathring{C}^{l+2}_s(\mathfrak{D},\mathfrak{M})}.$$

Taking δ sufficiently small and repeating the arguments of Theorem 4.2, we easily show that $\vec{u} = \zeta\hat{\vec{v}} \in H^2_1(\mathfrak{D}, \mathfrak{M})$, $q = \zeta\hat{p} \in H^1_1(\mathfrak{D}, \mathfrak{M})$ and

(6.10)
$$\|\vec{u}\|_{H^2_1(\mathfrak{D},\mathfrak{M})} + \|q\|_{H^1_1(\mathfrak{D},\mathfrak{M})} \leqslant c_5(\|\vec{g}\|_{L_{2,1}(\mathfrak{D},\mathfrak{M})} + \|h\|_{H^1_1(\mathfrak{D},\mathfrak{M})} + \|\vec{d}\|_{H^{1/2}_1(\Gamma_0,\mathfrak{M})} + \|\zeta\vec{f}\|_{L_{2,1}(\mathfrak{D},\mathfrak{M})} + \|\zeta\hat{r}\|_{H^1_1(\mathfrak{D},\mathfrak{M})}).$$

In virtue of (6.8), this solution of the problem (6.6) is unique. Approximate $\vec{g}$, h, $\vec{d}$ in $L_{2,1}(\mathfrak{D}, \mathfrak{M})$, $H^1_1(\mathfrak{D}, \mathfrak{M})$, $H^{1/2}_1(\mathfrak{D}, \mathfrak{M})$ by smooth functions $\vec{g}^{(n)}$, $h^{(n)}$, $\vec{d}^{(n)}$ with compact supports vanishing in $K_{\delta/2}(0)$ and replace in (6.6) $\vec{g}$, h, $\vec{d}$ by $\vec{g}^{(n)}$, $h^{(n)}$, $\vec{d}^{(n)}$. It follows from (6.8), (6.9) that for sufficiently small δ the resulting problem has a unique solution

$$\vec{u}^{(n)} \in H^2_1(\mathfrak{D}, \mathfrak{M}) \cap \mathring{C}^{l+2}_s(\mathfrak{D}, \mathfrak{M}), \qquad q^{(n)} \in H^1_1(\mathfrak{D}, \mathfrak{M}) \cap \mathring{C}^{l+1}_{s-1}(\mathfrak{D}, \mathfrak{M})$$

and that $\vec{u}^{(n)} \to \vec{u}$ in $H^2_1(\mathfrak{D}, \mathfrak{M})$, $q^{(n)} \to q$ in $H^1_1(\mathfrak{D}, \mathfrak{M})$ as $n \to \infty$.

By Theorem 5.5,

$$|\vec{u}^{(n)}|_{\mathring{C}^{l+2}_s(K_{\delta/4}(0),\mathfrak{M})} + |q^{(n)}|_{\mathring{C}^{l+1}_{s-1}(K_{\delta/4}(0),\mathfrak{M})} \leqslant c_6(|\zeta\vec{f}|_{\mathring{C}^l_{s-2}(K_{\delta/2}(0),\mathfrak{M})} + |\zeta\hat{r}|_{\mathring{C}^{l+1}_{s-1}(K_{\delta/2}(0),\mathfrak{M})}) + \\ + c_7(\delta)(\|\vec{u}^{(n)}\|_{L_2(K_{\delta/2}(0))} + \|q^{(n)}\|_{L_2(K_{\delta/2}(0))}).$$

Consequently, in virtue of (6.10),

$$|\vec{u}|_{\mathring{C}^{l+2}_s(K_{\delta/4}(0),\mathfrak{M})} + |q|_{\mathring{C}^{l+1}_{s-1}(K_{\delta/4}(0),\mathfrak{M})} \leqslant c_8(|\vec{f}|_{\mathring{C}^l_{s-2}(\Omega,\mathscr{M})} + |r|_{\mathring{C}^{l+1}_{s-1}(\Omega,\mathscr{M})}).$$

Since the point $\xi \in \mathscr{M}$ is arbitrary, this estimate and the inequality (6.5) prove (6.4).

Consider the non-homogeneous problem

(6.11)
$$-V^2\vec{v} + Vp = \vec{f}, \quad V\cdot\vec{v} = r, \\ \vec{v}|_\Sigma = \vec{a}, \quad \vec{v}\cdot\vec{n}|_\Gamma = b,$$

(6.12)
$$S(\vec{v})\vec{n} - \vec{n}(\vec{n}\cdot S(\vec{v})\vec{n})|_\Gamma = \vec{d}.$$

THEOREM 6.3. *Let* $0 < s < \min(\operatorname{Re}\sigma_0, \pi/2\theta)$. *For arbitrary* $\vec{f} \in \mathring{C}^l_{s-2}(\Omega, \mathscr{M})$, $r \in \mathring{C}^{l+1}_{s-1}(\Omega, \mathscr{M})$, $\vec{a} \in \mathring{C}^{l+2}_s(\Sigma, \mathscr{M})$, $b \in \mathring{C}^{l+2}_s(\Gamma, \mathscr{M})$, $\vec{d} \in \mathring{C}^{l+1}_{s-1}(\Gamma, \mathscr{M})$ *satisfying the conditions* $\int_\Omega r\,dx = \int_\Sigma \vec{a}\cdot\vec{n}\,dS + \int_\Gamma b\,dS$, $\vec{d}\cdot\vec{n} = 0$ *the problem* (6.11), (6.12) *has a unique solution* $\vec{v} \in \mathring{C}^{l+2}_s(\Omega, \mathscr{M})$, $p \in \mathring{C}^{l+2}_{s-1}(\Omega, \mathscr{M})$ *with* $\int_\Omega p\,dx = 0$; *moreover,*

(6.13)
$$|\vec{v}|_{\mathring{C}^{l+2}_s(\Omega,\mathscr{M})} + |p|_{\mathring{C}^{l+1}_{s-1}(\Omega,\mathscr{M})} \leqslant c(|\vec{f}|_{\mathring{C}^l_{s-2}(\Omega,\mathscr{M})} + |r|_{\mathring{C}^{l+1}_{s-1}(\Omega,\mathscr{M})} + |\vec{a}|_{\mathring{C}^{l+2}_s(\Sigma,\mathscr{M})} + |b|_{\mathring{C}^{l+2}_s(\Gamma,\mathscr{M})} + \\ + |\vec{d}|_{\mathring{C}^{l+1}_{s-1}(\Gamma,\mathscr{M})}).$$

Proof. It is not hard to verify that a vector $\vec{w}$ satisfying the boundary conditions

$$\vec{w}|_{\Sigma} = \vec{a}, \quad w_1|_{x_3=\varphi(x')} = w_2|_{x_3=\varphi(x')} = 0, \quad w_3|_{x_3=\varphi(x')} = \frac{b}{n_3}\Big|_{x_3=\varphi(x')},$$

$$\frac{\partial w_3}{\partial x_3}\Big|_{x_3=\varphi(x')} = 0,$$

$$(6.14)\qquad \frac{\partial w_1}{\partial x_3}\Big|_{x_3=\varphi(x')} = \frac{1-n_2^2}{n_3^2}h_1 + \frac{n_1 n_2}{n_3^2}h_2,$$

$$\frac{\partial w_2}{\partial x_3}\Big|_{x_3=\varphi(x')} = \frac{n_1 n_2}{n_3^2}h_1 + \frac{1-n_1^2}{n_3^2}h_2$$

with

$$h_i = n_3 d_i + n_3^2\Big(2n_i \sum_{j=1}^{2} n_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_i}\Big)\frac{b}{n_3}\Big|_{x_3=\varphi(x')}$$

also satisfies (6.12). Theorem 2.6 asserts that there exists a vector satisfying (6.14) and the inequality

$$|\vec{w}|_{C_s^{l+2}(\Omega,\mathcal{M})} \leqslant c_1(|\vec{a}|_{C_s^{l+2}(\Sigma,\mathcal{M})} + |b|_{C_s^{l+2}(\Gamma,\mathcal{M})} + |\vec{d}|_{C_{s-1}^{l+1}(\Gamma,\mathcal{M})})$$

with a constant c_1 independent on $\vec{a}$, b, $\vec{d}$.

Hence the problem (6.11), (6.12) is reduced to (6.1), (6.2). Theorem 6.3 follows from Theorem 6.2.

7. The non-linear problem

7.1. Neumann problem for an elliptic equation of the second order. Consider in ω the boundary value problem

$$(7.1)\qquad \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} A_{ij}(x) \frac{\partial u}{\partial x_j} - \beta u = f(x), \qquad \sum_{i,j=1}^{2} \nu_i(x) A_{ij} \frac{\partial u}{\partial x_j}\Big|_{\partial\omega} = g(x)$$

where $\nu_i(x)$ are the components of the unit outward normal vector to $\partial\omega$, and A_{ij} satisfy the condition of ellipticity

$$\sum_{i,j=1}^{2} A_{ij}\xi_i\xi_j \geqslant \mu\xi^2, \qquad \forall \xi \in R^2.$$

THEOREM 7.1. *Suppose that $\partial\omega \in C^{l+2}$ and $A_{ij} \in C_s^{l+2}(\omega, \partial\omega)$ with a non-integer $s \in (0, l+1)$. Then for arbitrary $f \in C_{s-1}^{l+1}(\omega, \partial\omega)$, $g \in C^s(\partial\omega)$ the problem*

(7.1) has a unique solution $u \in C_{s+1}^{l+3}(\omega, \partial\omega)$. It is subject to the inequality

$$(7.2)\qquad |u|_{C_{s+1}^{l+3}(\omega,\partial\omega)} \leqslant c(|f|_{C_{s-1}^{l+1}(\omega,\partial\omega)} + |g|_{C^s(\partial\omega)}),$$

the constant c being independent of f, g.

Proof. If $s > 1$, then the problem (7.1) has a solution $u \in C^{s+1}(\omega)$ and

$$(7.3)\qquad |u|_{C^{s+1}(\omega)} \leqslant c_1(|f|_{C^{s-1}(\omega)} + |g|_{C^s(\partial\omega)}).$$

In the case of $s < 1$ we write f in a divergence form: $f = \nabla\cdot\vec{F}$,

$$F_i = \frac{1}{2\pi}\int_{\omega} \frac{x_i - y_i}{|x-y|^2} f(y)\,dy.$$

It is not hard to prove that

$$|\vec{F}|_{C^s(\omega)} \leqslant c_2 \sup_{\omega} |f(y)|\,\mathrm{dist}(y, \partial\omega)^{1-s}.$$

As shown in [1, I], Ch. III,

$$(7.4)\qquad |u|_{C^{s+1}(\omega)} \leqslant c_3(|\vec{F}|_{C^s(\omega)} + |g|_{C^s(\partial\omega)}) \leqslant c_4(|f|_{C_{s-1}^{l+1}(\omega,\partial\omega)} + |g|_{C^s(\partial\omega)})$$

(the solvability of (7.1) can be proved in this case with the aid of approximation of f and g by smooth functions).

Let us estimate the higher order derivatives of u. Since $f \in C_{s-1}^{l+1}(\omega, \partial\omega)$, we have $u \in C^{l+3}(\omega')$ for arbitrary $\omega' \subset \omega$ with $\mathrm{dist}(\omega', \partial\omega) > 0$. According to E. Stein [29], Ch. 6, $\omega = \bigcup_k Q_k$, where Q_k is a square such that $\mathrm{dist}(Q_k, \partial\omega) \approx \mathrm{diam}\,Q_k$ (i.e. $c_5\,\mathrm{dist}(Q_k, \partial\omega) \leqslant \mathrm{diam}\,Q_k \leqslant c_6\,\mathrm{dist}(Q_k, \partial\omega)$, c_5, c_6 do not depend on k). Let

$$w = u - \sum_{|\alpha|\leqslant[s]+1} D^\alpha u(x^{(k)}) \frac{(x-x^{(k)})^\alpha}{\alpha!},$$

where $x^{(k)}$ is the centre of Q_k, $\alpha! = \alpha_1!\alpha_2!$, $(x-x^{(k)})^\alpha = (x_1 - x_1^{(k)})^{\alpha_1} \times (x_2 - x_2^{(k)})^{\alpha_2}$. Clearly,

$$(7.5)\qquad \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} A_{ij} \frac{\partial w}{\partial x_j} - \beta w = f + \sum_{|\beta|\leqslant|\alpha|\leqslant[s]+1} c_{\alpha\beta}(x) D^\alpha u(x^{(k)})(x-x^{(k)})^\beta$$

with $c_{\alpha\beta}(x) \in C_{s-1}^{l+1}(\omega, \partial\omega)$ for $|\beta| < |\alpha|$, $c_{\alpha\beta} = \mathrm{const}$ for $|\beta| = |\alpha|$. Let us now use a local Schauder estimate for (7.5) in a square $Q_k^* \supset Q_k$ such that $\mathrm{dist}(Q_k^*, \partial\omega) \approx \mathrm{dist}(Q_k, \omega\setminus Q_k^*) \approx \mathrm{diam}\,Q_k$. We have

$$[u]_{Q_k}^{(l+3)} = [w]_{Q_k}^{(l+3)} \leqslant c_7\Big([f]_{Q_k^*}^{(l+1)} + \sum_{|\alpha|<l+1} |D^\alpha f|_{Q_k^*} (\mathrm{diam}\,Q_k)^{-l-1+|\alpha|} + |u|_{C_{s+1}(\omega)} (\mathrm{diam}\,Q_k)^{s-l-2}\Big).$$

Now we multiply this estimate by $(\operatorname{diam} Q_k)^{l+2-s}$, maximize with respect to k and take into account (7.3), (7.4). The resulting estimate is equivalent to (7.2).

7.2. Solvability of the problem (1.1)–(1.5). We are now ready to prove our main results. The general scheme of the proof will be the same as in [5], [10], [11].

Consider the problem (1.1), (1.2) in a prescribed domain $\Omega = \{x \in V\colon x_3 < \varphi(x')\}$ with $\varphi \in C^{l+3}_{s+1}(\omega, \partial\omega)$, $s \in (0, \min(\operatorname{Re}\sigma_0, \pi/2\theta))$.

Let $\varphi > 0$, so that $\Gamma \cap S_- = \emptyset$.

THEOREM 7.2. *Suppose that $\vec{a} \in C^{l+2}(S_-)$ satisfies the condiiions of* §1. *If $\varepsilon \in (0, \varepsilon_1)$ and ε_1 is sufficiently small, then the problem* (1.1), (1.2) *has a unique solution which satisfies the condition $\int_\Omega p\,dx = 0$ and the inequality*

$$|\vec{v}|_{C^{l+2}_s(\Omega,\mathscr{M})} + |p|_{C^{l+1}_{s-1}(\Omega,\mathscr{M})} \leqslant c\varepsilon|\vec{a}|_{C^{l+2}(S_-)}, \tag{7.6}$$

the constant c being independent of ε.

This theorem follows from Theorem 6.3 and may be proved, for instance, by successive approximations.

Let $U_\delta(\varphi) \subset C^{l+3}_{s+1}(\omega, \partial\omega)$ denote a neighbourhood $|\psi - \varphi|_{C^{l+2}_{s+1}(\omega,\partial\omega)} < \delta$ of the function φ. Suppose that δ is so small that all functions in $U_\delta(\varphi)$ are positive. Let $\varphi' \in U_\delta(\varphi)$, $\Omega' = \{x \in V\colon x_3 < \varphi(x')\}$ and let $\vec{v}' \in \mathring{C}^{l+2}_s(\Omega, \mathscr{M})$, $p' \in \mathring{C}^{l+1}_{s-1}(\Omega', \mathscr{M}')$ be a solution of (1.1), (1.2) in Ω', satisfying the condition $\int_{\Omega'} p'\,dx = 0$.

THEOREM 7.3. *For arbitrary $\varepsilon \in (0, \varepsilon_2)$, $\varepsilon_2 \leqslant \varepsilon_1$ and $\varphi' \in U_{\delta_0}(\varphi)$ (δ_0 is the same as in Theorem* 2.7) *the inequality*

$$|h - h'|_{C^{l+1}_{s-1}(\omega,\partial\omega)} \leqslant c\varepsilon|\varphi - \varphi'|_{C^{l+3}_{s+1}(\omega,\partial\omega)} \tag{7.7}$$

holds for the functions

$$h(x') = -p + \vec{n}\cdot S(\vec{v})\vec{n}|_{x_3=\varphi(x')}, \qquad h'(x') = -p' + \vec{n}\cdot S(\vec{v}')\vec{n}|_{x_3=\varphi'(x')},$$

the constant c being independent on $\varphi' \in U_{\delta_0}$ and ε.

Proof. We map the domain Ω' onto Ω by means of the transformation $x = Xz$ defined in Theorem 2.7. The problem (1.1), (1.2) for $\vec{v}'$, p' in the new coordinates takes the form

$$-\nabla'^2\vec{v}' + \nabla'p' + (\vec{v}'\cdot\nabla')\vec{v}' = 0, \qquad \nabla'\cdot\vec{v}' = 0,$$

$$\vec{v}'|_\Sigma = \varepsilon\vec{a}, \qquad \vec{v}'\cdot\vec{n}'|_\Gamma = 0,$$

$$S'(\vec{v}')\vec{n}' - \vec{n}'(\vec{n}'\cdot S'(\vec{v}')\vec{n}')|_\Gamma = 0,$$

where

$$\nabla' = A(x)\nabla, \qquad \vec{n}'(x) = \frac{A\vec{n}}{|A\vec{n}|} = B(x)\vec{n},$$

$A(x)$ and $\vec{S}'(\vec{v}')$ are matrices with the elements

$$a_{kl} = \frac{\partial X_l}{\partial z_k}\Big|_{z=X^{-1}x}, \qquad S'_{ij} = \sum_{m=1}^{3}\Big(a_{jm}\frac{\partial v_i}{\partial x_m} + a_{im}\frac{\partial v_j}{\partial x_m}\Big).$$

The differences $\vec{v} - \vec{v}' = \vec{w}$, $p - p' = q$ satisfy the relations

$$\begin{aligned}
-\nabla^2\vec{w} + \nabla q &= (\nabla^2 - \nabla'^2)\vec{v}' - (\nabla - \nabla')p' - (\vec{w}\cdot\nabla)\vec{v} - (\vec{v}'\cdot\nabla - \nabla')\vec{v}' - (\vec{v}'\cdot\nabla)\vec{w},\\
\nabla\cdot\vec{w} &= (\nabla' - \nabla)\cdot\vec{v}',\\
\vec{w}|_\Sigma = 0, \qquad \vec{w}\cdot\vec{n}|_\Gamma &= \vec{v}'\cdot(\vec{n}' - \vec{n}) = \vec{v}'\cdot(B - I)\vec{n},\\
S(\vec{w})\vec{n} - \vec{n}(\vec{n}\cdot S(\vec{w})\vec{n})|_\Gamma &= [S'(\vec{v}') - S(\vec{v})]\vec{n} + \\
&\quad + S'(\vec{v}')(\vec{n}' - \vec{n}) + \vec{n}\,[\vec{n}\cdot(S'(\vec{v}') - S(\vec{v}'))\vec{n}] + \\
&\quad + [\vec{n}'(\vec{n}'\cdot S'(\vec{v}')\vec{n}') - \vec{n}(\vec{n}\cdot S'(\vec{v}')\vec{n})]|_\Gamma.
\end{aligned}$$

We estimate $\vec{w}$, q with the aid of Theorem 6.3, taking into account that both $\vec{v}$, p and $\vec{v}'$, p' satisfy (7.6) and that in virtue of (2.14)

$$|a_{kj}(x) - \delta_{kj}|_{C^{l+2}_s(\Omega,\mathscr{M})} \leqslant c_1|\varphi - \varphi'|_{C^{l+3}_{s+1}(\omega,\partial\omega)}.$$

The function $q = p - p'$ satisfies the condition

$$\int_\Omega q\,dx = -\int_\Omega p'\,dx = \int_\Omega p'(x)[\det A^{-1} - 1]\,dx.$$

Since

$$\Big|\int_\Omega p'(x)[\det A^{-1} - 1]\,dx\Big| \leqslant c_2\varepsilon|\varphi - \varphi'|_{C^{l+3}_{s+1}(\omega,\partial\omega)},$$

the inequality (6.13) yields

$$|\vec{w}|_{C^{l+2}_s(\Omega,\mathscr{M})} + |q|_{C^{l+1}_{s-1}(\Omega,\mathscr{M})} \leqslant c_3\varepsilon|\varphi - \varphi'|_{C^{l+3}_{s+1}(\omega,\partial\omega)} + c_4\varepsilon|\vec{a}|_{C^{l+2}(S_-)}|\vec{w}|_{C^{l+2}_s(\Omega,\mathscr{M})}.$$

Assumig that $c_4\varepsilon_2|\vec{a}|_{C^{l+2}(S_-)} \leqslant 1/2$ we obtain

$$|\vec{w}|_{C^{l+2}_s(\Omega,\mathscr{M})} + |q|_{C^{l+1}_{s-1}(\Omega,\mathscr{M})} \leqslant 2c_3\varepsilon|\varphi - \varphi'|_{C^{l+3}_{s+1}(\omega,\partial\omega)}$$

and hence (7.7).

THEOREM 7.4. *There exist numbers* $\varepsilon_0 \in (0, \varepsilon_2]$, $h_0 > 0$ *such that for arbitrary* $\varepsilon \in (0, \varepsilon_0)$, $h > h_0$ *and arbitrary* $\vec{a} \in C^{l+2}(S_-)$ *with* $\operatorname{supp}\vec{a} \in S_-$, $\int_{S_-} \vec{a}\cdot\vec{n}\,dS = 0$, *the problem* (1.1)–(1.5) *has a unique solution* $\varphi \in C^{l+3}_{s+1}(\omega, \partial\omega)$, $\vec{v} \in \overset{\circ}{C}{}^{l+2}_s(\Omega, \mathscr{M})$, $p \in \overset{\circ}{C}{}^{l+1}_{s-1}(\Omega, \mathscr{M})$ *with any* $s \in (0, \min(\operatorname{Re}\sigma_0, \pi/2\theta))$.

Proof. Consider first the case $\varepsilon = 0$. In this case $\vec{v} = 0$ and $p = \bar{p}$ is a constant which may be computed from (1.3)–(1.5):

$$-W\bar{p}|\omega| = \int_\omega \left(\nabla\cdot\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}} - \beta\varphi\right)dx' = |\partial\omega|\cos\theta - \beta h$$

($|\omega| = \operatorname{mes}\omega$). The function $\varphi = \varphi_0$ is a solution of the problem

$$\text{(7.8)}\qquad \begin{gathered}\nabla\cdot\frac{\nabla\varphi_0}{\sqrt{1+|\nabla\varphi_0|^2}} - \beta\varphi_0 = -W\bar{p} = \frac{|\partial\omega|\cos\theta-\beta h}{|\omega|},\\ \left.\frac{\nabla\varphi_0\cdot\vec{\nu}}{\sqrt{1+|\nabla\varphi_0|^2}}\right|_{\partial\omega} = \cos\theta.\end{gathered}$$

As shown by N. N. Uraltzeva [30], this problem has a unique solution $\varphi_0 \in C^{2+l}(\omega)$. It follows from Theorem 7.1 that $\varphi_0 \in C^{l+3}_{s+1}(\omega, \partial\omega)$. Let us fix the number $h_0 > 0$ in order that $\varphi_0(x) > 0$ for $h \geqslant h_0$.

We shall seek a solution $\varphi(x')$ of (1.3), (1.4) in $U_{\delta_1}(\varphi_0)$, $\delta_1 \leqslant \delta_0/2$. Since (1.3)–(1.5) determines a normalization of p,

$$\int_\omega p\big(x', \varphi(x')\big)dx' = \int_\omega \vec{n}\cdot S(\vec{v})\vec{n}|_{x_3=\varrho(x')}dx' + |\omega|\,\bar{p},$$

it is convenient to introduce a new function $\tilde{p} = p + c$ where

$$\int_\Omega \tilde{p}\,dx = 0, \qquad c = \frac{1}{|\omega|}\int_\omega \Big(\tilde{p}\big(x', \varphi(x')\big) - \vec{n}'S(\vec{v})\vec{n}|_{x_3=\varphi(x)}\Big)dx' - \bar{p}.$$

For the difference $\psi = \varphi - \varphi_0$ we have

$$\text{(7.9)}\qquad \begin{aligned}\nabla\cdot A(x)\nabla\psi - \beta\psi = {}& W[-\tilde{p} + \vec{n}\cdot S(\vec{v})\vec{n}]|_{x_3=\varphi_0(x')+\psi(x')} + \\ &+ \frac{W}{|\omega|}\int_\omega \big(\tilde{p} - \vec{n}\cdot S(\vec{v})\vec{n}\big)\big|_{x_3=\varphi_0(x')+\psi(x')}dx' - \\ &- \sum_{j=1}^{2}\frac{\partial}{\partial x_j}\Big[\nabla\psi\cdot\int_0^1 (1-t)B^{(j)}(\nabla\varphi_0 + t\nabla\psi)dt\nabla\psi\Big],\end{aligned}$$

$$\vec{\nu}\cdot A(x)\nabla\psi|_{\partial\omega} = -\sum_{j=1}^{2}\nu_j\Big[\nabla\psi\cdot\int_0^1 (1-t)B^{(j)}(\nabla\varphi_0 + t\nabla\psi)dt\nabla\psi\Big]\Big|_{\partial\omega},$$

where $A(x)$ and $B^{(j)}(\xi)$ are matrices with the elements

$$A_{kl} = \left.\frac{\partial F_k(\xi)}{\partial\xi_l}\right|_{\xi=\nabla\varphi_0(x')}, \qquad B^{(j)}_{kl}(\xi) = \frac{\partial^2 F_j(\xi)}{\partial\xi_k\partial\xi_l}$$

and $F_j(\xi) = \xi_j/\sqrt{1+\xi_1^2+\xi_2^2}$. The equivalent form of (7.9) is

$$\text{(7.10)}\qquad \psi = T_1(\psi) + T_2(\psi) \equiv T(\psi),$$

where

$$\begin{aligned}T_1(\psi) = R\Big[&W\big(-\tilde{p} + \vec{n}\cdot S(\vec{v})\vec{n}\big)|_{x_3=\varphi_0(x')+\psi(x)} + \\ &+ \frac{W}{|\omega|}\int_\omega \big(-\tilde{p} + \vec{n}\cdot S(\vec{v})\vec{n}\big)|_{x_3=\varphi_0(x')+\psi(x')}dx',\ 0\Big],\end{aligned}$$

$$\begin{aligned}T_2(\psi) = R\Big[&\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\Big(\nabla\psi\cdot\int_0^1 (1-t)B^{(j)}(\nabla\varphi_0 + t\nabla\psi)dt\nabla\psi\Big), \\ &\sum_{j=1}^{2}\nu_j\Big(\nabla\psi\cdot\int_0^1 (1-t)B^{(j)}(\nabla\varphi_0 + t\nabla\psi)dt\nabla\psi\Big)\Big]\end{aligned}$$

and $R[f, g]$ is the solution of (7.1). Theorem 7.1 asserts that the linear operator R: $C^{l+1}_{s-1}(\omega, \partial\omega)\times C^s(\partial\omega) \to C^{3+l}_{1+s}(\omega, \partial\omega)$ is bounded. Clearly, the non-linear operator T_2 satisfies the condition $T_2(0) = 0$ and for arbitrary ψ, $\psi' \in U_{\delta_1}(0)$ we have

$$|T_2(\psi) - T_2(\psi')|_{C^{l+3}_{s+1}(\omega,\partial\omega)} \leqslant c_1\delta_1(1+\delta_1)|\psi-\psi'|_{C^{l+3}_{s+1}(\omega,\partial\omega)}.$$

Finally, in virtue of Theorems 7.1–7.3,

$$|T_1(\psi)|_{C^{l+3}_{s+1}(\omega,\partial\omega)} \leqslant c_2\varepsilon,$$

$$|T_1(\psi) - T_1(\psi')|_{C^{l+3}_{s+1}(\omega,\partial\omega)} \leqslant c_3\varepsilon|\psi-\psi'|_{C^{l+3}_{s+1}(\omega,\partial\omega)}.$$

Hence, for ψ, $\psi' \in U_{\delta_1}(0)$ we have

$$|T(\psi)|_{C^{l+3}_{s+1}(\omega,\partial\omega)} \leqslant c_1\delta_1^2(1+\delta_1) + c_2\varepsilon,$$

$$|T(\psi) - T(\psi')|_{C^{l+3}_{s+1}(\omega,\partial\omega)} \leqslant [c_1\delta_1(1+\delta_1) + c_3\varepsilon]|\psi-\psi'|_{C^{l+3}_{s+1}(\omega,\partial\omega)}.$$

If $c_1\delta_1^2(1+\delta_1) + c_2\varepsilon < \delta_1$, $c_1\delta_1(1+\delta_1) + c_3\varepsilon < 1$, then T is a contraction operator in $U_{\delta_1}(0)$ and (7.10) has in $U_{\delta_1}(0)$ a unique solution.

The functions $\vec{v}$, p are now determined by (1.1), (1.2).

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