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THE USE OF CONJUGATE CONVEX FUNCTIONS IN COMPLEX ANALYSIS

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1. Introduction

The basis for many, if not all, applications of convexity theory to complex analysis lies in the observation that a subharmonic function f of a complex variable z which is independent of $\operatorname{Im} z$ is convex. This makes it meaningful to consider the function $\tilde{f}$ conjugate to f, for f can then be retrieved from $\tilde{f}$. However, in several complex variables it is often the case that a function f of n complex variables $z_1, \ldots, z_n$ is a convex function of some of the variables, say $z_{k+1}, \ldots, z_n$, when the others are kept fixed. In that case we may of course form the function which is conjugate to f in the variables $z_{k+1}, \ldots, z_n$, but this is not so useful unless we have some information on how the transformed function depends on the remaining variables $z_1, \ldots, z_k$. The minimum principle for plurisubharmonic functions furnishes precisely this kind of information. Before stating it, let us recall the concepts from convex analysis that we shall need.

If $f\colon \boldsymbol{R}^m \to [-\infty, +\infty]$ is any numerical function on $\boldsymbol{R}^m$ we define its *Legendre transform* $\tilde{f}$ (or *Young–Fenchel transform*, or *conjugate* or *polar* function), as

$$\tilde{f}(\eta) = \sup_{y \in \boldsymbol{R}^m} (\langle y, \eta \rangle - f(y)), \quad \eta \in \boldsymbol{R}^m.$$

The transform of $\tilde{f}$,

$$\tilde{\tilde{f}}(y) = \sup_{\eta \in \boldsymbol{R}^m} (\langle y, \eta \rangle - \tilde{f}(\eta)), \quad y \in \boldsymbol{R}^m,$$

is the largest minorant of f which is convex, lower semicontinuous and takes the value $-\infty$ only when it is $-\infty$ identically (the Fenchel–Moreau theorem). In particular, $\tilde{\tilde{f}} = f$ if and only if f itself has these three properties. As general references for convex analysis we mention Rockafellar [7] and Ioffe & Tihomirov [2].

Now if f is a convex function of $x_1, \ldots, x_n, y_1, \ldots, y_m$ and we regard $x_1, \ldots, x_n$ as parameters while performing the transformation, $\tilde{f}$ will be a concave function of $x_1, \ldots, x_n$ for fixed $\eta_1, \ldots, \eta_n$. The corresponding result in the complex case is the following theorem.

THEOREM 1.1. *Let v be a plurisubharmonic function in a pseudoconvex open set Ω in C^{n+m}. We write the variables of C^{n+m} as $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and assume that both v and Ω are independent of* $\operatorname{Im} y$ *in the sense that if $(x, y) \in \Omega$ and* $\operatorname{Re} y' = \operatorname{Re} y$, *then $(x, y') \in \Omega$ and $v(x, y') = v(x, y)$. Let us assume, this time for simplicity only, that $\pi^{-1}(x) \cap \Omega$ is connected, thus convex, for every $x \in \pi(\Omega)$, where π: $C^{n+m} \to C^n$ is the projection $\pi(x, y) = x$. Then $\pi(\Omega)$ is pseudoconvex and*

$$w(x) = \inf_{y \in C^m} v(x, y), \quad x \in \pi(\Omega), \tag{1.1}$$

is plurisubharmonic. The partial Legendre transform of v,

$$\tilde{v}(x, \eta) = \sup_{y \in C^m} \big(\langle \operatorname{Re} y, \eta \rangle - v(x, y)\big), \quad (x, \eta) \in C^n \times R^m, \tag{1.2}$$

is thus plurisuperharmonic as a function of $x \in \pi(\Omega)$ for fixed $\eta \in R^m$ and convex as a function of $\eta \in R^m$ for fixed $x \in C^n$. The transform of $\tilde{v}$ is

$$v(x, y) = \sup_{\eta \in R^m} \big(\langle \operatorname{Re} y, \eta \rangle - \tilde{v}(x, \eta)\big), \quad (x, y) \in \Omega. \tag{1.3}$$

In (1.1) we make the convention that $v = +\infty$ in the complement of Ω; hence y ranges effectively only over $\pi^{-1}(x) \cap \Omega$. We make similar conventions throughout the paper in order not to worry more than necessary about the domain of functions. Note, however, that in (1.3) equality does not necessarily hold on the boundary of Ω.

The fact that w is plurisubharmonic will be referred to as the *minimum principle*. For the general version of this result, as well as for its proof, we refer to [3]. The transformation (1.2) can be used to study both global and local properties of plurisubharmonic functions. To indicate this, let ω be an open set in C^n and let $f \in \mathrm{PSH}(\omega)$; by this we understand that f is plurisubharmonic in ω. Then the function

$$u(x, y, t) = f(x + e^{2\pi t} y), \quad (x, y, t) \in \omega \times C^n \times C,\ x + e^{2\pi t} y \in \omega,$$

is plurisubharmonic wherever it is defined, for the mapping $(x, y, t) \mapsto x + e^{2\pi t} y$ is holomorphic. In general it is not independent of the imaginary part of any of the variables, but if we form

$$v(x, t) = \int_{|y|=1} u(x, y, t)\, dS(y) \Big/ \int_{|y|=1} dS(y) = \operatorname*{mean}_{\substack{y \in C^n \\ |y|=1}} f(x + e^{2\pi t} y),$$

then v becomes independent of $\operatorname{Im} t$, and $v \in \mathrm{PSH}(\Omega)$ where

$$\Omega = \{(x, t) \in \omega \times C;\ d(x, C^n \setminus \omega) > |e^{2\pi t}|\}.$$

If ω is assumed to be pseudoconvex, then Ω is pseudoconvex and we may form the partial Legendre transform $\tilde{v}$ of v:

$$\tilde{v}(x, \tau) = \sup_{t \in C} \big(\operatorname{Re} t\tau - v(x, t)\big), \quad (x, \tau) \in C^n \times R,$$

(remember that we have agreed to take $v = +\infty$ outside Ω even though the integral defining it may have a sense there). All information about f is contained in $\tilde{v}$, for $-\tilde{v}(x, 0) = f(x)$. The function $f_\tau(x) = -\tilde{v}(x, \tau)$ has density (or Lelong number) $\nu_{f_\tau} = (\nu_f - \tau)^+$ everywhere in ω as we have shown in [4]. (This very regular behavior of $\tilde{v}$ as a function of τ is all the more remarkable as the corresponding result for the density of a convex function in R^n fails completely.) Thus the function $\tilde{v}$ can be used to study important local properties of f. In the next section we shall see that the domain of harmonicity of f, i.e. the open set $\omega_f = \omega \setminus \operatorname{supp} \Delta f$, is revealed in a very simple way by $\tilde{v}$, giving also the boundary distance in ω_f.

In Section 3 we shall consider the growth of convex functions, expressed as the order and type with respect to a given function. These concepts are in fact conjugate to each other under the Legendre transformation. The theory of relative order of plurisubharmonic functions, as developed by Lelong [5], is simplified a lot by the use of this symmetry, and we review it in Section 4, which also contains an existence theorem for functions of prescribed relative order.

Section 5 is similar in scope to Section 4 except that we consider questions of growth which are of interest for slowly growing functions.

As should be clear from the above, this paper is mainly expository. We emphasize a unified approach and give new short proofs of known results on the structure and growth of holomorphic or plurisubharmonic functions. However, the existence theorems (4.2 and 5.2) are exceptions: we believe they are new. Are they also inconceivable without conjugate convex functions?

2. The domain of harmonicity

To illustrate the use of the Legendre transformation we shall give a new proof of the following result due to Cegrell [1], p. 330.

THEOREM 2.1. *Let ω be a pseudoconvex open set in C^n and let $f \in \mathrm{PSH}(\omega)$. Then the largest open set ω_f in which f is harmonic is pseudoconvex.*

Proof. Consider the mean value of f over a sphere of radius $e^{\operatorname{Re} t}$:

$$v(x, t) = \operatorname*{mean}_{\substack{|y|=1 \\ y \in C^n}} f(x + e^t y), \quad (x, t) \in \Omega, \tag{2.1}$$

where

$$\Omega = \{(x, t) \in \omega \times C;\ |e^t| < d(x, C^n \setminus \omega)\}.$$

If f is harmonic, or what is the same thing, pluriharmonic, in a neighborhood of x, then $v(x, t) = f(x)$ for all t in some half-plane $\operatorname{Re} t < a$. Now this property is reflected in a nice way in the partial Legendre transform $\tilde{v}$ of v:

$$\tilde{v}(x, \tau) = \sup_{t \in R} \big(t\tau - v(x, t)\big), \quad (x, \tau) \in \omega \times R.$$

Define w as the right-hand derivative of $-\tilde{v}(x,\tau)$ with respect to τ at the origin, i.e.

$$w(x) = -\partial_\tau^+\tilde{v}(x,0) = \lim_{\tau\to 0+}\frac{-\tilde{v}(x,\tau)-f(x)}{\tau} = \sup_{\tau>0}\frac{-\tilde{v}(x,\tau)-f(x)}{\tau}.$$

Here we have used that $-\tilde{v}(x,0)=f(x)$ and, for the last equality, that $\tilde{v}(x,\tau)$ is convex in τ. Now f is pluriharmonic in ω_f so the last expression is, in view of the minimum principle, a supremum of plurisubharmonic functions; hence we have:

LEMMA 2.2. *The upper regularization w^* of w is plurisubharmonic wherever it is $<+\infty$ in ω_f.*

In fact, we shall see in a moment that $w^* = w < +\infty$ everywhere in ω_f. The next lemma shows this and gives the desired interpretation of w.

LEMMA 2.3. *For every $x\in\omega_f$ we have*

$$w^*(x) = w(x) = -\log d(x, \boldsymbol{C}^n\setminus\omega_f). \tag{2.2}$$

Proof. We introduce two auxiliary functions to estimate w:

$$v_1(x,t) = \begin{cases} f(x) & \text{if } e^t < d(x, \boldsymbol{C}^n\setminus\omega),\\ +\infty & \text{otherwise;}\end{cases}$$

$$v_2(x,t) = \begin{cases} f(x) & \text{if } e^t < d(x, \boldsymbol{C}^n\setminus\omega_f),\\ +\infty & \text{otherwise.}\end{cases}$$

Then $v_1\leqslant v\leqslant v_2$ and $\tilde{v}_2\leqslant\tilde{v}\leqslant\tilde{v}_1$. Assuming $\tau>0$ we obtain an estimate

$$-\log d(x, \boldsymbol{C}^n\setminus\omega)\leqslant\frac{-\tilde{v}(x,\tau)-f(x)}{\tau}\leqslant -\log d(x,\boldsymbol{C}^n\setminus\omega_f), \tag{2.3}$$

for the Legendre transform of v_1 is

$$\tilde{v}_1(x,\tau) = \begin{cases}\tau\log d(x,\boldsymbol{C}^n\setminus\omega)-f(x), & \tau\geqslant 0,\\ +\infty, & \tau<0,\end{cases}$$

and similarly for v_2. Taking the supremum over all $\tau>0$ in (2.3) we get

$$-\log d(x,\boldsymbol{C}^n\setminus\omega)\leqslant w(x)\leqslant -\log d(x,\boldsymbol{C}^n\setminus\omega_f), \tag{2.4}$$

proving one of the inequalities necessary to give (2.2).

Conversely we have for every ω_f and every positive τ in view of the convexity of $\tilde{v}$:

$$\frac{-\tilde{v}(x,\tau)-f(x)}{\tau}\leqslant w(x),$$

or equivalently, introducing an arbitrary real number t,

$$t\tau-\tilde{v}(x,\tau)\leqslant f(x)+t\tau+w(x)\tau.$$

Taking the supremum over all $\tau\geqslant 0$ we get v back, for $\tilde{v}(x,\tau)=+\infty$ when $\tau<0$ so negative values do not contribute to the supremum; thus

$$f(x)\leqslant v(x,t) = \sup_{\tau\geqslant 0}\big(t\tau-\tilde{v}(x,\tau)\big)\leqslant f(x)+\sup_{\tau\geqslant 0}\big(t+w(x)\big)\tau = f(x)$$

provided $(x,t)\in\Omega$ and $t+w(x)\leqslant 0$. This shows that f satisfies the mean value property with respect to all spheres of center x and radius r such that $r<d(x,\boldsymbol{C}^n\setminus\omega)$ and $r\leqslant\exp\big(-w(x)\big)$. Hence f is harmonic inside these spheres, i.e.

$$d(x,\boldsymbol{C}^n\setminus\omega_f)\geqslant\inf\big(d(x,\boldsymbol{C}^n\setminus\omega), e^{-w(x)}\big) = e^{-w(x)},$$

where the last equation holds in view of (2.4). Thus $-\log d(x,\boldsymbol{C}^n\setminus\omega_f)\leqslant w(x)$ and this concludes the proof of the lemma.

Combining the lemmas we see that $-\log d(x,\boldsymbol{C}^n\setminus\omega_f)$ is plurisubharmonic which means that ω_f is pseudoconvex.

Remark 2.4. A similar analysis can be made of the function

$$v(x,y,t) = \operatorname*{mean}_{\substack{|z|=1\\ z\in\boldsymbol{C}}} f(x+e^t zy),\qquad (x,y,t)\in\Omega,$$

where now

$$\Omega = \{(x,y,t)\in\omega\times\boldsymbol{C}^n\times\boldsymbol{C};\ |e^t|<d_y(x,\boldsymbol{C}^n\setminus\omega)\};$$

d_y denoting the distance in the direction $y\in\boldsymbol{C}^n$. This time, however, we have to use the upper regularization of $w(x,y) = -\partial_\tau^+\tilde{v}(x,y,0)$ and we obtain, as a result corresponding to Lemma 2.3,

$$w^*(x,y) = -\log d_y(x,\boldsymbol{C}^n\setminus\omega_f),\qquad (x,y)\in\omega_f\times\boldsymbol{C}^n.$$

3. Order and type

The classical growth scale for entire functions can be generalized in a natural way to the concepts of order and type with respect to a given convex function. It turns out that order and type are conjugate to each other.

DEFINITION 3.1. Let h and v be two numerical functions on the real line, i.e. $h, v\colon \boldsymbol{R}\to[-\infty,+\infty]$. We shall say that v has *finite h-order* if there is a number $\alpha>0$ such that

$$v(t)\leqslant\frac{1}{\alpha}h(\alpha t)\quad\text{ for all large } t.$$

We define the *h-order* of v as the infimum ϱ of all such numbers α and denote it by $\varrho = \operatorname{order}(v{:}h)\in[0,+\infty]$.

This definition would perhaps have looked more familiar had we required only that $v(t)\leqslant Ch(\alpha t)$ for some constant C and all sufficiently large t. Note, however, that when h increases so fast that $h\big((1+\varepsilon)t\big)/h(t)$ tends to $+\infty$ as $t\to+\infty$ for every $\varepsilon>0$, then this weaker estimate defines the same h-order. The reason for taking $C = 1/\alpha$ will be clear in a moment.

If f is an entire function on C^n we put

$$v(t) = \sup_{|z| \leqslant e^t} \log|f(z)| \quad \text{and} \quad h(t) = e^t$$

to get the usual order for such functions.

DEFINITION 3.2. For any two numerical functions h and v on R we define the *h-type* σ of v as the infimum of all real numbers $\beta \neq 0$ such that

$$v(t) \leqslant \beta h(t) \quad \text{for } t \text{ large enough}.$$

Note that σ may be negative. We shall use the notation $\sigma = \text{type}\,(v:h) \in [-\infty, +\infty]$.

PROPOSITION 3.3. *Let h and v be two numerical increasing convex functions on R and let*

$$\tilde{h}(\tau) = \sup_t \big(t\tau - h(t)\big), \quad \tau \in R,$$

denote the Legendre transform of h, and similarly for v. Assume that one of v and h is real-valued and that one of them grows faster than any linear function. Then

$$\text{order}(v:h) = \text{type}(\tilde{h}:\tilde{v}) \quad \text{and} \quad \text{type}(v:h) = \text{order}(\tilde{h}:\tilde{v}).$$

Proof. Assume first that both h and v are real-valued and grow faster than any linear function. Then $\tilde{h}$ and $\tilde{v}$ are of the same kind and it is easy to check that $v(t) \leqslant h(t)$ for large t if and only if $\tilde{v}(\tau) \geqslant \tilde{h}(\tau)$ for large τ.

Define $h_\alpha(t) = \alpha^{-1}h(\alpha t)$ for any $\alpha > 0$. Its transform is $\tilde{h}_\alpha(\tau) = \alpha^{-1}\tilde{h}(\tau)$. Now $\alpha > \text{order}(v:h)$ implies that $v(t) \leqslant h_\alpha(t)$ for large t and by the remark just made this is equivalent to $\tilde{v}(\tau) \geqslant \alpha^{-1}\tilde{h}(\tau)$ for large τ, which in turn implies that $\alpha \geqslant \text{type}(\tilde{h}:\tilde{v})$. Thus $\text{order}(v:h) \geqslant \text{type}(\tilde{h}:\tilde{v})$, and similarly we see that $\text{type}(\tilde{h}:\tilde{v}) \geqslant \text{order}(v:h)$. Applying the Legendre transformation we get the second relation: $\text{order}(\tilde{h}:\tilde{v}) = \text{type}(\tilde{\tilde{v}}:\tilde{\tilde{h}}) = \text{type}(v:h)$.

Next consider the case when one of the functions, say v, grows at most like a linear function whereas h grows faster than any linear function, possibly being $+\infty$ for large values of the argument. It is then easy to see that all orders and types listed in the statement of the proposition are zero. If conversely it is h that grows at most like a linear function, then all orders and types are $+\infty$.

Finally we have to look at the case when one of the functions, say v, is $+\infty$ to the right of some point. Then, however, $\tilde{v}$ grows at most like a linear function, but $\tilde{h}$ grows fast, so we are reduced to the case just considered. Hence the proposition is completely proved.

COROLLARY 3.4. *With h and v as in Proposition 3.3 the h-order ϱ of v is given by*

$$(3.1) \qquad -\frac{1}{\varrho} = \limsup_{\tau\to+\infty} \frac{-\tilde{v}(\tau)}{\tilde{h}(\tau)} = \text{type}(-\tilde{v}:\tilde{h}) \in [-\infty, 0].$$

Proof. By the proposition,

$$\varrho = \text{type}(\tilde{h}:\tilde{v}) = \limsup_{\tau\to+\infty} \tilde{h}(\tau)/\tilde{v}(\tau),$$

from which (3.1) follows immediately.

4. The relative order of partial functions

With the symmetry between order and type established in Section 3, properties of the relative order of partial functions of a plurisubharmonic function can easily be deduced. This subject has been developed by Lelong [5], Chapter 6, and we shall give a new proof of the main result. Then we shall prove an existence theorem for plurisubharmonic functions with prescribed orders for their partial functions.

Let u be a plurisubharmonic function of two groups of variables, $x \in C^n$ and $y \in C^m$. We shall consider the growth of the partial functions $y \mapsto u(x, y)$. Let us define

$$v(x, t) = \sup_{|y| \leqslant |e^t|} u(x, y) = \sup_{|z| \leqslant 1} u(x, e^t z), \quad (x, t) \in C^n \times C.$$

Since the mapping $(x, t) \mapsto (x, e^t z)$ is holomorphic for every fixed z the last expression shows that v is plurisubharmonic. All questions on the growth of $y \mapsto u(x, y)$ involving only x and $|y|$ will therefore be reduced to studying the function v which is independent of $\operatorname{Im} t$.

THEOREM 4.1. *Let $h\colon R \to R$ be convex and increasing faster than any linear function, let $v \in \text{PSH}(\omega \times C)$ be independent of $\operatorname{Im} t$, $t \in C$, and assume that $t \mapsto v(x, t)$ is uniformly of finite h-order for $x \in \omega$ in the sense that*

$$(4.1) \qquad v(x, t) \leqslant \alpha^{-1} h(\alpha t) \quad \textit{for some } \alpha \textit{ and all } t \geqslant t_0,$$

where t_0 does not depend on $x \in \omega$. Let $\varrho(x)$ denote the h-order of $t \mapsto v(x, t)$. Then $-1/\varrho^* \in \text{PSH}(\omega)$; as a consequence ϱ^* and $\log \varrho^*$ are also plurisubharmonic.

This result follows easily from Corollary 3.4: we know that $-\tilde{v}(x, \tau) = \inf\big(v(x, t) - t\tau\big)$ is plurisubharmonic in x by the minimum principle and $-\tilde{v}(x, \tau)/\tilde{h}(\tau) \leqslant 0$ for large τ if (4.1) holds, so an application of the "lim-sup-star theorem" to (3.1) gives the desired conclusion. Lelong ([5], Theorem 6.6.2) proved this using instead an inverse-function theorem. It follows from Theorem 4.1 that the set of $x \in \omega$ such that $\varrho^*(x) = 0$ is either polar or equal to ω (assuming ω to be connected), and that, in case ϱ^* is a constant, the set of $x \in \omega$ such that $\varrho(x) < \varrho^*$ is polar.

It is natural to ask if the regularized order $\varrho^*(x)$ has any other property than that expressed by Theorem 4.1. The answer is given by the following existence theorem.

THEOREM 4.2. *Let $h\colon R \to R$ be an increasing convex function which grows faster than any linear function, and let $u < 0$ be plurisubharmonic on a manifold ω.*

Then there exists $v \in \mathrm{PSH}(\omega\times C)$ whose partial functions $t\mapsto v(x,t)$ have h-order $\varrho(x) = -1/u(x)$, $x\in\omega$.

Proof. Let $\tilde{h}$ denote the Legendre transform of h,

$$\tilde{h}(\tau) = \sup_t \big(t\tau - h(t)\big), \quad \tau\in \boldsymbol{R},$$

and define

$$w(\alpha,\tau) = -\alpha\big(\tilde{h}(\tau)+h(0)\big), \quad \alpha<0, \quad \tau\in\boldsymbol{R}.$$

Then w is convex in τ for fixed $\alpha<0$ and linear and decreasing in α for fixed τ (for $\tilde{h}(\tau)\geqslant \inf\tilde{h} = -h(0)$). Now

$$h(t) = \tilde{\tilde{h}}(t) = \sup_\tau \big(t\tau-\tilde{h}(\tau)\big), \quad t\in\boldsymbol{R},$$

so that the partial transform of w with respect to τ is

$$\tilde{w}(\alpha,t) = \sup_\tau\big(t\tau - w(\alpha,\tau)\big) = -\alpha h(-t/\alpha)+\alpha h(0), \quad \alpha<0, \quad t\in\boldsymbol{R}.$$

Now $\tilde{w}$ is obviously convex and finite when $\alpha<0$, $t\in\boldsymbol{R}$; moreover it is increasing in α for fixed t. This implies that the composition

$$v(x,t) = \tilde{w}\big(u(x), \mathrm{Re}\, t\big), \quad (x,t)\in\omega\times\boldsymbol{C},$$

is plurisubharmonic as a function of (x,t). We claim that v solves our problem. Indeed, we can easily calculate its h-order using (3.1); the partial Legendre transform of v is

$$\tilde{v}(x,\tau) = \sup_{t\in R}\big(t\tau - v(x,t)\big) = w\big(u(x),\tau\big) = -u(x)\,\big(\tilde{h}(\tau)+h(0)\big),$$

so that (3.1) gives

$$-1/\varrho(x) = \limsup_{\tau\to+\infty} \frac{u(x)\,\big(\tilde{h}(\tau)+h(0)\big)}{\tilde{h}(\tau)} = u(x).$$

5. Functions of minimal growth

In this final section we present results analogous to those of Section 4 but now for functions of slow growth.

Let, as in the beginning of Section 4, u be plurisubharmonic of two groups of variables, $x\in\boldsymbol{C}^n$ and $y\in\boldsymbol{C}^m$. Define

$$v(x,t) = \operatorname*{mean}_{|z|=1} u(x, e^{2\pi t}z), \quad (x,t)\in\boldsymbol{C}^n\times\boldsymbol{R},$$

to be the mean of u over the sphere of radius $e^{2\pi t}$ and center at the origin. If $v(x,t)>-\infty$, its right-hand derivative

$$\partial^+ v(x,t) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{v(x,t+\varepsilon)-v(x,t)}{\varepsilon}, \quad (x,t)\in\boldsymbol{C}^n\times\boldsymbol{R},$$

is equal to the mean $(2m-2)$-dimensional density in the ball $e^{2\pi t}B$ of the Riesz mass of the partial function $y\mapsto u(x,y)$, in symbols

$$\partial_t^+ v(x,t) = \frac{\mu(e^{2\pi t}B)}{\lambda_{2m-2}(e^{2\pi t}B\cap\boldsymbol{C}^{m-1})},$$

where B is the closed unit ball, λ_{2m-2} Lebesgue measure in $\boldsymbol{R}^{2m-2}$ and $\mu = \Delta_y u$, the measure defined by u, x being a parameter. For a proof of this, see [4], p. 297. In particular, if $m=1$, $\partial_t^+ v$ is just the total mass. For functions of the form $u = (2\pi)^{-1}\log|f|$, f entire, $\partial_t^+ v$ is the mean $(2m-2)$-dimensional density of its zero set; for $m=1$ simply the total number of zeros in $|y|\leqslant e^{2\pi t}$.

Now define

$$M(x) = \lim_{t\to+\infty}\partial_t^+ v(x,t) = \lim_{t\to+\infty} v(x,t)/t \leqslant +\infty. \tag{5.1}$$

If $v(x,t) = -\infty$ for all t let us agree to set $M(x) = -\infty$.

Thus $M(x)$ is closely associated with the mass distribution of the partial functions $y\mapsto u(x,y)$. If $M(x)\leqslant 0$, then $u(x,y)$ is independent of y and generally speaking functions with $M(x)<+\infty$ have the slowest possible growth in y except for the constants, justifying the name minimal growth.

THEOREM 5.1. *Let ω be a pseudoconvex domain in $\boldsymbol{C}^n$ and let $v\in\mathrm{PSH}(\omega\times\boldsymbol{C})$ be independent of $\mathrm{Im}\, t$, $t\in\boldsymbol{C}$, the variables in $\omega\times\boldsymbol{C}$ being written as $(x_1,\ldots,x_n,t)$. Define $M(x)$ by (5.1). Then for every $\tau\in[0,+\infty]$ the sets*

$$E_\tau = \{x\in\omega;\, M(x)<\tau\} \tag{5.2}$$

and

$$E'_\tau = \{x\in\omega;\, M(x)\leqslant\tau\} \tag{5.3}$$

are either equal to ω or polar in ω.

Proof. Define

$$v_\tau(x) = \inf_{t>0}\big(v(x,t)-t\tau\big), \quad (x,\tau)\in\omega\times\boldsymbol{R}.$$

If $M(x)<\tau$ then $v_\tau(x) = -\infty$. Conversely, if $M(x)>\tau$ for some $\tau\in[0,+\infty[$ then $v_\tau(x)>-\infty$. Writing

$$P(g) = \{x\in\omega;\, g(x) = -\infty\}$$

or the polar set of a function we may express these implications as follows:

$$E_\tau\subset P(v_\tau)\subset E'_\tau, \quad 0\leqslant\tau<+\infty. \tag{5.4}$$

Now let $\tau\in\,]0,+\infty]$ and let $\tau_k\to\tau$, $\tau_k<\tau_{k+1}<\tau$. Applying (5.4) to τ_k we get

$$E_{\tau_k}\subset P(v_{\tau_k})\subset E'_{\tau_k}\subset E_{\tau_{k+1}};$$

thus $E_\tau = \bigcup P(v_{\tau_k})$. Now v_{τ_k} is plurisubharmonic by the minimum principle, so each $P(v_{\tau_k})$ is either polar or $=\omega$. Since the union of a denumerable family of polar sets is polar, we are done. Similarly, $E'_\tau = \bigcap P(v_{\tau_k})$ where $\tau\in[0,+\infty[$ and (τ_k) is a strictly decreasing sequence of numbers tending to τ.

Theorem 5.1 is due to Lelong [6], p. 177. Let $u = (2\pi)^{-1}\log|f|$, f entire in $C\times C$, and let v be related to u as in the beginning of this section ($n = m = 1$). Then the statement "E_0' is polar or equal to C" means that the partial functions $y \mapsto f(x, y)$ are either constant or have a zero except when x belongs to a set of logarithmic capacity zero. This special case of Theorem 5.1 was proved by Tsuji [8].

As a converse to Theorem 5.1 we shall now construct plurisubharmonic functions with the sets E_τ (almost) prescribed.

THEOREM 5.2. *Let ω be an open set in C^n and let $u_\tau \in \mathrm{PSH}(\omega)$ be given for $0 \leqslant \tau < +\infty$ as well as for $\tau = -\infty$ such that $u_\sigma(x) = -\infty$ if $u_\tau(x) = -\infty$ for some $\tau \leqslant \sigma$. Then there exists a plurisubharmonic function v in $\omega\times C$, independent of $\mathrm{Im}\, t$, such that the sets E_τ and E_τ' defined as in (5.2) and (5.3) satisfy*

$$E_\tau = \bigcup_{\sigma<\tau} P(u_\sigma) \subset P(u_\tau) \subset \bigcap_{\sigma>\tau} P(u_\sigma) = E_\tau', \qquad \tau \in [0, +\infty].$$

(In particular $E_0 = P(u_{-\infty})$ and $E_\infty = \bigcup P(u_\sigma)$.)

Proof. Let $(K_j)_0^\infty$ be a fundamental sequence of compact sets in ω. If τ is a positive number not of the form $k2^{-m}$ for any $k, m \in N$ we take v_τ as the constant $-\infty$; for $\tau = -\infty$ we take $v_\tau = u_\tau$; for other (dyadic) values of τ we take v_τ as u_τ plus a constant chosen so that

$$v_\tau(x) \leqslant -\tau^2 \quad \text{when } x \in K_j, j \leqslant \tau,$$

and that

$$v_\tau(x) \leqslant -m-\tau^2 \quad \text{when } x \in K_m,\ \tau \notin 2^{-m}N.$$

In particular, $P(v_\tau) = P(u_\tau)$ for dyadic $\tau \geqslant 0$ and for $\tau = -\infty$. Now define

$$v(x, t) = \sup_\tau \big(v_\tau(x)+\mathrm{Re}\, t\tau\big), \qquad (x, t) \in \omega\times C.$$

We claim that v is upper semicontinuous and $< +\infty$ in $\omega\times C$; this will show that $v \in \mathrm{PSH}(\omega\times C)$. Let (x^0, t^0) be an arbitrary point in $\omega\times C$ and let A be any real number. Fix j so that x^0 belongs to the interior of K_j.

Next fix an integer m and a number α such that

$$-m+(|\mathrm{Re}\, t^0|+1)^2/4 \leqslant A, \qquad m \geqslant j,$$

and

$$-\alpha^2/2 \leqslant A, \qquad \alpha \geqslant j, \qquad \alpha \geqslant 2|t^0|+2.$$

Now $v(x, t)$ is the supremum of a denumerable family of functions, indexed by dyadic numbers τ and $-\infty$. We divide the set of τ into three subsets T_1, T_2, T_3 defined by means of m and α as follows:

$$T_1 = \{\tau \in 2^{-m}N;\ \tau \leqslant \alpha\}\cup\{-\infty\},$$
$$T_2 = \{\tau \in 2^{-m}N;\ \tau > \alpha\},$$
$$T_3 = \{\tau \geqslant 0;\ \tau \notin 2^{-m}N\}.$$

The first set T_1 is finite, so the supremum over this set is an upper semicontinuous function $v_1 < +\infty$. When $\tau \in T_2$ we have, provided $x \in K_j$ and $t \leqslant |t^0|+1 \leqslant \alpha/2$,

$$v_\tau(x)+\mathrm{Re}\, t\tau \leqslant -\tau^2+\mathrm{Re}\, t\tau \leqslant -\tau^2+\alpha\tau/2 \leqslant -\tau^2/2 \leqslant -\alpha^2/2,$$

giving

$$v_2(x, t) = \sup_{\tau\in T_2}\big(v_\tau(x)+\mathrm{Re}\, t\tau\big) \leqslant -\alpha^2/2 \leqslant A.$$

Finally, when $\tau \in T_3$ we have the estimate

$$v_\tau(x)+\mathrm{Re}\, t\tau \leqslant -m-\tau^2+\mathrm{Re}\, t\tau \leqslant -m+(\mathrm{Re}\, t)^2/4.$$

This gives

$$v_3(x, t) = \sup_{\tau\in T_3}\big(v_\tau(x)+\mathrm{Re}\, t\tau\big) \leqslant -m+(\mathrm{Re}\, t)^2/4 \leqslant A.$$

Combining these inequalities we obtain

$$v(x, t) = \sup_{q=1,2,3}\ \sup_{\tau\in T_q}\big(v_\tau(x)+\mathrm{Re}\, t\tau\big) \leqslant \sup\big(v_1(x, t), A\big)$$

provided only $x \in K_j$ and $\mathrm{Re}\, t < |\mathrm{Re}\, t^0|+1$. Since v_1, as noted above, is less than $+\infty$, we do have $v < +\infty$ in a neighborhood of (x^0, t^0). Next, repeating the argument with A taken to be an arbitrary number greater than $v(x^0, t^0)$, we see also that v is upper semicontinuous. This completes the proof that v is plurisubharmonic.

Now let $\tau \geqslant 0$ be given and let σ be any real number greater than τ. If $x \in P(u_\tau)$ we know that $v_\sigma(x) = -\infty$ so, if $\mathrm{Re}\, t \geqslant 0$,

$$v(x, t) = \sup_{\sigma\leqslant\tau}\big(v_\sigma(x)+\mathrm{Re}\, t\sigma\big) \leqslant \sup_{\sigma\leqslant\tau} v_\sigma(x)+\mathrm{Re}\, t\tau \leqslant v(x, 0)+\mathrm{Re}\, t\tau.$$

Since we have proved that $v(x, 0) < +\infty$ we see that $M(x) \leqslant \tau$; in other words we have proved that $P(u_\tau) \subset E_\tau'$ for all $\tau \geqslant 0$. The inclusion $P(u_{-\infty}) \subset E_{-\infty}'$ holds trivially.

Conversely, to any positive τ and any point $x \notin P(u_\tau)$ we can choose a dyadic number $\sigma \leqslant \tau$ but arbitrarily close to τ. We must have $v_\sigma(x) > -\infty$ and so the estimate

$$v(x, t) \geqslant v_\sigma(x)+\mathrm{Re}\, t\sigma$$

shows that $M(x) \geqslant \sigma$, thus $M(x) \geqslant \tau$. This proves that $E_\tau \subset P(u_\tau)$ if $\tau > 0$ and it is easily checked that $E_0 = P(u_{-\infty}) \subset P(u_0)$. The proof is now completed by taking unions and intersections in the relation $E_\tau \subset P(u_\tau) \subset E_\tau'$.

References

[1] U. Cegrell, *Removable singularities for plurisubharmonic functions and related problems*, Proc. London Math. Soc. 36 (1978), 310–336.

[2] A. D. Ioffe and V. M. Tihomirov, *Theory of extremal problems*, North-Holland, 1979.

[3] C. O. Kiselman, *The partial Legendre transformation for plurisubharmonic functions*, Invent. Math. 49 (1978), 137–148.

[4] —, *Densité des fonctions plurisousharmoniques*, Bull. Soc. Math. France 107 (1979), 295–304.

[5] P. Lelong, *Fonctionnelles analytiques et fonctions entières (n variables)*, Cours d'été de Montréal 1967. Les Presses de l'Université de Montréal 1968.

[6] P. Lelong, *Un théorème de fonctions inverses dans les espaces vectoriels topologiques complexes et ses applications à des problèmes de croissance en analyse complexe*, Sém. P. Lelong, H. Skoda, 1976/77, Lecture Notes in Mathematics 694, 172–195, Springer-Verlag, 1978.
[7] R. T. Rockafellar, *Convex analysis*, Princeton University Press, 1970.
[8] M. Tsuji, *Theory of meromorphic functions in a neighborhood of a closed set of capacity zero*, Japan. J. Math. 19 (1944), 139–154.

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REGIONS OF VARIABILITY FOR BOUNDED UNIVALENT FUNCTIONS AND DISTORTION THEOREMS

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1. Notations

We will use the following notation:

$$K = \{z \in \boldsymbol{C};\ |z| < 1\}, \quad K^* = \{\xi \in \boldsymbol{C};\ |\xi| > 1\}.$$

S_M, $M > 1$, denotes the class of holomorphic and univalent functions f in K which have the form

(1) $$f(z) = z + a_2 z^2 + \ldots,$$

and satisfy the condition

(2) $$|f(z)| < M, \quad z \in K.$$

Σ_m, $0 \leqslant m < 1$, denotes the class of holomorphic and univalent functions F in K^* which have the form

(3) $$F(\xi) = \xi + \alpha_0 + \alpha_1/\xi + \ldots,$$

and satisfy the condition

(4) $$|F(\xi)| > m, \quad \xi \in K^*.$$

S_M^p (Σ_m^p), $p = 1, 2, \ldots$, $M > 1$, $0 \leqslant m < 1$, denote the classes of functions f_p (F_p) holomorphic and univalent in K (K^*) which have the form

(5) $$f_p(z) = z + a_{p+1}^{(p)} z^{p+1} + a_{2p+1}^{(p)} z^{2p+1} + \ldots$$
$$\left(F_p(\xi) = \xi + \frac{\alpha_{p-1}^{(p)}}{\xi^{p-1}} + \ldots\right),$$

and satisfy the condition (2) and (4), respectively.

We have $S_M^1 \equiv S_M$, $\Sigma_m^1 \equiv \Sigma_m$, $S_\infty^p \equiv S^p$, where S^p, $p = 1, 2, \ldots$, denotes the class of holomorphic and univalent function f_p in K which are p-symmetric and Σ_0^p, $p = 1, 2, \ldots$, denotes the class of holomorphic and univalent functions F_p in K^* which are p-symmetric and $F_p(\xi) \neq 0$, $\xi \in K^*$.