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REGIONS OF VARIABILITY FOR BOUNDED UNIVALENT FUNCTIONS AND DISTORTION THEOREMS

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1. Notations

We will use the following notation:

$$K = \{z \in \boldsymbol{C};\ |z| < 1\}, \quad K^* = \{\xi \in \boldsymbol{C};\ |\xi| > 1\}.$$

S_M, $M > 1$, denotes the class of holomorphic and univalent functions f in K which have the form

(1) $$f(z) = z + a_2 z^2 + \ldots,$$

and satisfy the condition

(2) $$|f(z)| < M, \quad z \in K.$$

Σ_m, $0 \leqslant m < 1$, denotes the class of holomorphic and univalent functions F in K^* which have the form

(3) $$F(\xi) = \xi + \alpha_0 + \alpha_1/\xi + \ldots,$$

and satisfy the condition

(4) $$|F(\xi)| > m, \quad \xi \in K^*.$$

S_M^p (Σ_m^p), $p = 1, 2, \ldots, M > 1$, $0 \leqslant m < 1$, denote the classes of functions f_p (F_p) holomorphic and univalent in K (K^*) which have the form

(5) $$f_p(z) = z + a_{p+1}^{(p)} z^{p+1} + a_{2p+1}^{(p)} z^{2p+1} + \ldots$$
$$\left(F_p(\xi) = \xi + \frac{\alpha_{p-1}^{(p)}}{\xi^{p-1}} + \ldots\right),$$

and satisfy the condition (2) and (4), respectively.

We have $S_M^1 \equiv S_M$, $\Sigma_m^1 \equiv \Sigma_m$, $S_\infty^p \equiv S^p$, where S^p, $p = 1, 2, \ldots$, denotes the class of holomorphic and univalent function f_p in K which are p-symmetric and Σ_0^p, $p = 1, 2, \ldots$, denotes the class of holomorphic and univalent functions F_p in K^* which are p-symmetric and $F_p(\xi) \neq 0$, $\xi \in K^*$.

In the sequel we will use certain relations between the classes S^p and Σ^p (S_M^p and Σ_m^p).

Namely, if $f \in S$ (S_M), then

(6) $$f_p(z) = \sqrt[p]{f(z^p)} \in S^p \ \left(S^p_{\sqrt[p]{M}}\right), \quad p = 1, 2, \ldots, S^1 \equiv S \equiv S_\infty.$$

Moreover, if $f_p \in S_M^p$, then

(7) $$F_p(\xi) = [f_p(1/\xi)]^{-1} \in \Sigma^p_{1/M}, \quad p = 1, 2, \ldots$$

For a fixed $z = re^{i\varphi} \in K$, $0 < r < 1$, let us consider the functional

(8) $$J(f) = \log\left|\frac{f(z)}{z}\right| + i\log|f'(z)|,$$

where f is a function ranging over the class S_M.

The compactness of the class S_M and the continuity of the functional (8) imply that the set D of the values of (8) is a bounded closed connected set containing the origin.

Moreover, since the class S_M is rotation invariant, we may assume, with no loss of generality, that $z = r > 0$.

Let us write

(9) $$D_p = \{(U, V)\colon U + iV = J(f_p), f_p \in S_M^p\}, \quad p = 1, 2, \ldots,$$
$$D_1 = D.$$

2. Formulation of the results

In this note we determine the region D_p given by (9). We find explicit formulae for parametric equations of its boundary.

We also describe the region of variability

(10) $$G_p = \{(U, V)\colon U + iV = \hat{J}(F_p), F_p \in \Sigma_m^p\}, \quad p = 1, 2, \ldots,$$

where

(11) $$\hat{J}(F_p) = \log\left|\frac{F_p(\xi)}{\xi}\right| + i\log|F_p'(\xi)|, \quad F_p \in \Sigma_m^p,$$

and $\xi = Re^{i\varphi} \in K^*$, $R > 1$, is fixed.

To this end we will need the relation (7).

Using the regions D_p and G_p we will obtain the exact bounds for $|f_p'(z)|$, $f_p \in S_M^p$ and $|F_p'(\xi)|$, $F_p \in \Sigma_m^p$ in the terms of $|f_p(z)|$ and $|F_p(\xi)|$, respectively.

Moreover, the region D_p is also used in finding the exact estimates for the functional

(12) $$H(f_p) = \log\left|\frac{z^\alpha f_p'^\beta(z)}{f_p^\alpha(z)}\right|, \quad f_p \in S_M^p, \alpha, \beta \text{ real}.$$

For simplicity we restrict ourselves to giving the exact estimates (in terms of M and r) of the functionals

(13) $$H_1(f) = \log|f'(z)|, \quad f \in S_M,$$

(14) $$H_2(f) = \log\left|\frac{zf'(z)}{f(z)}\right|, \quad f \in S_M.$$

Analogous estimates can be given for functionals corresponding to (13) and (14) for the class Σ_m.

The main tool used in this paper is the method of Gutljanskiĭ [2], [3] basing on the general type of Löwner's equation, which shows how one can reduce the extremal problem on the class S_M to the corresponding problem on the class of functions with positive real part.

Our results extend the results of Aleksandrov and Kopanev [1].

Moreover, the estimates of (13) and (14) are obtained by a method completely different from that used in [5] and [7], [8].

Also, the results obtained make complete the corresponding results from [4] and [6].

Now we state our results.

THEOREM 1. *The set D of the values of the functional* (8) *on the class S_M is a closed convex set. The boundary of D, which we call Γ, has the form $\Gamma = \Gamma^+ \cup \Gamma^-$, where $\Gamma^+ = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.*

The arcs $\Gamma_1, \Gamma_2, \Gamma_3$ are given by the following parametric equations:

(15) $$\begin{cases} U = U_1(t) = \log(1+\varkappa_1)^2 + (t-1)\log r + \log\dfrac{1-t^2}{4} + t\log\dfrac{1+t}{1-t}, \\ V = V_1(t) = \log\dfrac{(1+\varkappa_1)^3}{1-\varkappa_1} + \dfrac{t^2-1}{t}\log r - \log(1-r^2) + \\ \qquad + \log t^2 + \dfrac{t^2-1}{t}\log\dfrac{1+t}{1-t}, \quad \dfrac{1-r}{1+r} \leqslant t \leqslant t_1, \end{cases}$$

(16) $$\begin{cases} U = U_2(t) = \dfrac{t-1}{t}\log M, \\ V = V_2(t) = \dfrac{t^2-1}{t^2}\log M + \log\left(\dfrac{1-r^2M^{-2/t}}{1-r^2}\right), \quad t_1 \leqslant t \leqslant t_2, \end{cases}$$

(17) $$\begin{cases} U = U_3(t) = \log(1-\varkappa_2)^2 + (t-1)\log r + \log\dfrac{t^2-1}{4} + t\log\dfrac{t+1}{t-1}, \\ V = V_3(t) = \log\dfrac{(1-\varkappa_2)^3}{1+\varkappa_2} + \dfrac{t^2-1}{t}\log r - \log(1-r^2) + \\ \qquad + \log t^2 + \dfrac{t^2-1}{t}\log\dfrac{t+1}{t-1}, \quad t_2 \leqslant t \leqslant \dfrac{1+r}{1-r}, \end{cases}$$

where $t_1 = t_1(r, M)$, $t_2 = t_2(r, M)$ *are the roots of the equations*

$$\left[r\left(\frac{1+t}{1-t}\right)\right]^t = M, \quad \left[r\left(\frac{t+1}{t-1}\right)\right]^t = M, \tag{18}$$

respectively.

The numbers $\varkappa_1 = \varkappa_1(r, M)$, $\varkappa_2 = \varkappa_2(r, M)$ *are given by the formulae*

$$\varkappa_1 = (2A-1)-2\sqrt{A^2-A}, \quad \varkappa_2 = (2B+1)-2\sqrt{B^2+B}, \tag{19}$$

where

$$A = M\left(\left[r\left(\frac{1+t}{1-t}\right)\right]^t(1-t^2)\right)^{-1}, \quad B = M\left(\left[r\left(\frac{t+1}{t-1}\right)\right]^t(t^2-1)\right)^{-1}. \tag{20}$$

The arc Γ^- *is given by the following equation*:

$$V = \log\frac{M^2e^{2U}(1-r^2)}{M^2-r^2e^{2U}}, \quad U_1\left(\frac{1-r}{1+r}\right) \leqslant U \leqslant U_3\left(\frac{1+r}{1-r}\right).$$

THEOREM 2. *The set* D_p, $p = 1, 2, \ldots$, *of the values of the functional* (8) *on the class* S_M^p *is a closed convex set. The boundary of* D_p, *which we call* Γ_p, *has the form*:

$$\Gamma_p = \Gamma_p^+ \cup \Gamma_p^- \quad \text{where} \quad \Gamma_p^+ = \Gamma_{1p}\cup\Gamma_{2p}\cup\Gamma_{3p}.$$

The arcs Γ_{1p}, Γ_{2p}, Γ_{3p} *are given by the following parametric equations*:

$$\begin{cases} U = U_{1p}(t) = \dfrac{1}{p}\left[\log(1+\varkappa_{1p})^2+(t-1)\log r^p+\log\dfrac{1-t^2}{4}+t\log\dfrac{1+t}{1-t}\right], \\ V = V_{1p}(t) = \log\dfrac{(1+\varkappa_{1p})^{(p+2)/p}}{1-\varkappa_{1p}}+\dfrac{(t-1)(p+t)}{pt}\log r^p-\log(1-r^{2p})+ \\ \qquad +\log t^2+\dfrac{1-p}{p}\log\dfrac{1-t^2}{4}+\dfrac{t^2-p}{pt}\log\dfrac{1+t}{1-t}, \\ \qquad\qquad \dfrac{1-r^p}{1+r^p} \leqslant t \leqslant t_{1p}, \end{cases} \tag{21}$$

$$\begin{cases} U = U_{2p}(t) = \dfrac{t-1}{t}\log M, \\ V = V_{2p}(t) = \dfrac{(t-1)(p+t)}{t^2}\operatorname{og} M+\log\left(\dfrac{1-r^{2p}M^{-2p/t}}{1-r^{2p}}\right), \quad t_{1p} \leqslant t \leqslant t_{2p}, \end{cases} \tag{22}$$

$$\begin{cases} U = U_{3p}(t) = \dfrac{1}{p}\left[\log(1-\varkappa_{2p})^2+(t-1)\log r^p+\log\dfrac{t^2-1}{4}+t\log\dfrac{t+1}{t-1}\right], \\ V = V_{3p}(t) = \log\dfrac{(1-\varkappa_{2p})^{(p+2)/p}}{1+\varkappa_{2p}}+\dfrac{(t-1)(t+p)}{pt}\log r^p-\log(1-r^{2p})+ \\ \qquad +\log t^2+\dfrac{1-p}{p}\log\dfrac{t^2-1}{4}+\dfrac{t^2-p}{pt}\log\dfrac{t+1}{t-1}, \\ \qquad\qquad t_{2p} \leqslant t \leqslant \dfrac{1+r^p}{1-r^p}, \end{cases} \tag{23}$$

where $t_{1p} = t_{1p}(r, M)$, $t_{2p} = t_{2p}(r, M)$ *are the roots of the equations*:

$$\left[r^p\left(\frac{1+t}{1-t}\right)\right]^t = M^p, \quad \left[r^p\left(\frac{t+1}{t-1}\right)\right]^t = M^p, \tag{24}$$

respectively.

The numbers $\varkappa_{1p} = \varkappa_{1p}(r, M)$, $\varkappa_{2p} = \varkappa_{2p}(r, M)$ *are given by the formulae*

$$\varkappa_{1p} = (2A_p-1)-2\sqrt{A_p^2-A_p}, \quad \varkappa_{2p} = (2B_p+1)-2\sqrt{B_p^2+B_p}, \tag{25}$$

where

$$A_p = M^p\left(\left[r^p\left(\frac{1+t}{1-t}\right)\right]^t(1-t^2)\right)^{-1}, \quad B_p = M^p\left(\left[r^p\left(\frac{t+1}{t-1}\right)\right]^t(t^2-1)\right)^{-1}. \tag{26}$$

The arc Γ_p^- *is given by the following equation*:

$$V = \log\frac{M^{2p}e^{(p+1)U}(1-r^{2p})}{M^{2p}-r^{2p}e^{2pU}}, \quad U_{1p}\left(\frac{1-r^p}{1+r^p}\right) \leqslant U \leqslant U_{3p}\left(\frac{1+r^p}{1-r^p}\right).$$

THEOREM 3. *The set* G_p, $p = 1, 2, \ldots$, *of the values of the functional* (11) *on the class* Σ_m^p *is a closed convex set. The boundary of* G_p, *which we call* $\hat{\Gamma}_p$, *has the form*:

$$\hat{\Gamma}_p = \hat{\Gamma}_p^+ \cup \hat{\Gamma}_p^- \quad \text{where} \quad \hat{\Gamma}_p^+ = \hat{\Gamma}_{1p}\cup\hat{\Gamma}_{2p}\cup\hat{\Gamma}_{3p}.$$

The arcs $\hat{\Gamma}_{1p}$, $\hat{\Gamma}_{2p}$, $\hat{\Gamma}_{3p}$ *are given by the following parametric equations*:

$$\begin{cases} U = U_{1p}(t) = -\dfrac{1}{p}\left[\log(1+\hat{\varkappa}_{1p})^2+(1-t)\log R^p+\log\dfrac{1-t^2}{4}+t\log\dfrac{1+t}{1-t}\right], \\ V = V_{1p}(t) = \log\dfrac{(1+\hat{\varkappa}_{1p})^{(p-2)/p}}{1-\hat{\varkappa}_{1p}}+\dfrac{(t-1)(t-p)}{pt}\log R^p- \\ \qquad -\log(1-R^{-2p})+\log t^2-\dfrac{p+1}{p}\log\dfrac{1-t^2}{4}- \\ \qquad -\dfrac{p+t^2}{pt}\log\dfrac{1+t}{1-t}, \quad \dfrac{R^p-1}{R^p+1} \leqslant t \leqslant \hat{t}_{1p}, \end{cases} \tag{27}$$

$$\begin{cases} U = U_{2p}(t) = \dfrac{t-1}{t}\log m, \\ V = V_{2p}(t) = \dfrac{(1-t)(p-t)}{t^2}\log m+\log\left(\dfrac{R^{2p}-m^{2p/t}}{R^{2p}-1}\right), \quad \hat{t}_{1p} \leqslant t \leqslant \hat{t}_{2p}, \end{cases} \tag{28}$$

$$\begin{cases} U = U_{3p}(t) = -\dfrac{1}{p}\left[\log(1-\hat{\varkappa}_{2p})^2+(1-t)\log R^p+\log\dfrac{t^2-1}{4}+t\log\dfrac{t+1}{t-1}\right], \\ V = V_{3p}(t) = \log\dfrac{(1-\hat{\varkappa}_{2p})^{(p-2)/p}}{1+\hat{\varkappa}_{2p}}+\dfrac{(t-1)(t-p)}{pt}\log R^p- \\ \qquad -\log(1-R^{-2p})+\log t^2-\dfrac{p+1}{p}\log\dfrac{t^2-1}{4}- \\ \qquad -\dfrac{p+t^2}{pt}\log\dfrac{t+1}{t-1}, \quad \hat{t}_{2p} \leqslant t \leqslant \dfrac{R^p+1}{R^p-1}, \end{cases} \tag{29}$$

where $t_{1p} = t_{1p}(R, m)$, $t_{2p} = t_{2p}(R, m)$ *are the roots of the equations*:

$$\left[R^p\left(\frac{1-t}{1+t}\right)\right]^t = m^p, \qquad \left[R^p\left(\frac{t-1}{t+1}\right)\right]^t = m^p, \tag{30}$$

respectively.

The numbers $\hat{\varkappa}_{1p} = \hat{\varkappa}_{1p}(R, m)$, $\hat{\varkappa}_{2p} = \hat{\varkappa}_{2p}(R, m)$ *are given by the formulae*

$$\hat{\varkappa}_{1p} = (2\hat{A}_p-1)-2\sqrt{\hat{A}_p^2-\hat{A}_p}, \qquad \hat{\varkappa}_{2p} = (2\hat{B}_p+1)-2\sqrt{\hat{B}_p^2+\hat{B}_p}, \tag{31}$$

where

$$\hat{A}_p = m^{-p}\left(\left[R^{-p}\left(\frac{1+t}{1-t}\right)\right]^t(1-t^2)\right)^{-1}, \qquad \hat{B}_p = m^{-p}\left(\left[R^{-p}\left(\frac{t+1}{t-1}\right)\right]^t(t^2-1)\right)^{-1}. \tag{32}$$

The arc $\hat{\Gamma}_p^-$ *is given by the following equation*:

$$V = \log\frac{m^{-2p}e^{(1-p)U}(1-R^{-2p})}{m^{-2p}-R^{-2p}e^{-2pU}}, \qquad U_{1p}\left(\frac{R^p-1}{R^p+1}\right) \leqslant U \leqslant U_{3p}\left(\frac{R^p+1}{R^p-1}\right).$$

THEOREM 4. *Let* $f \in S_M$, *then for fixed* $|f(z)|$ *and* $z = re^{i\varphi} \in K$ *the following sharp estimates hold*:

$$|f'(z)| \leqslant \begin{cases} \dfrac{M^2}{(M^2-|f(z)|^2)}\cdot\dfrac{(1-x_1^2)^2}{(1-r^2)}\cdot\left|\dfrac{f(z)}{z}\right|^2\left(\dfrac{r}{x_1}\right)^{-4x_1^2/(1-x_1^2)} \\ \qquad\qquad \text{for } |f(z)| \leqslant rM^{1-1/t_1}, \\ \dfrac{M}{(1-r^2)}\cdot\dfrac{1-r^2M^{-2\left(\log\frac{Mr}{|f(z)|}\right)/\log M}}{M^{\left(\log\frac{Mr}{|f(z)|}/\log M\right)^2}} \\ \qquad\qquad \text{for } rM^{1-1/t_1} \leqslant |f(z)| \leqslant rM^{1-1/t_2}, \\ \dfrac{M^2}{(M^2-|f(z)|^2)}\cdot\dfrac{(1-x_2^2)^2}{(1-r^2)}\left|\dfrac{f(z)}{z}\right|^2\left(\dfrac{x_2}{r}\right)^{4x_2^2/(1-x_2^2)} \\ \qquad\qquad \text{for } |f(z)| \geqslant rM^{1-1/t_2}, \end{cases} \tag{33}$$

where t_1, t_2 *are the roots of the equations* (18).

The numbers x_1, x_2 *are the unique roots of the equations*:

$$\frac{M^2}{(M+|f(z)|)^2}(1+x)^2\left|\frac{f(z)}{z}\right|\left(\frac{r}{x}\right)^{2x/(1+x)} = 1, \tag{34}$$

$$\frac{M^2}{(M-|f(z)|)^2}(1-x)^2\left|\frac{f(z)}{z}\right|\left(\frac{x}{r}\right)^{2x/(1-x)} = 1, \tag{35}$$

respectively.

We have the sharp estimate from below

$$|f'(z)| \geqslant \frac{M^2(1-r^2)}{M^2-|f(z)|^2}\left|\frac{f(z)}{z}\right|^2 \geqslant \frac{M^2(1-r^2)}{M^2-(E^-)^2}\left(\frac{E^-}{r}\right)^2, \tag{36}$$

where

$$E^+\ (E^-) = \max_{\substack{|z|=r\\ f\in S_M}}\ (\min)|f(z)|.$$

THEOREM 5. *Let* $f_p \in S_M^p$, $p = 1, 2, \ldots$, *then for fixed* $|f_p(z)|$ *and* $z = re^{i\varphi} \in K$ *the following sharp estimates hold*:

$$|f_p'(z)| \leqslant \begin{cases} \dfrac{M^{2p}}{(M^{2p}-|f_p(z)|^{2p})}\cdot\dfrac{(1-x_{1p}^2)^2}{(1-r^{2p})}\left|\dfrac{f_p(z)}{z}\right|^{p+1}\left(\dfrac{r^p}{x_{1p}}\right)^{-4x_{1p}^2/(1-x_{1p}^2)} \\ \qquad\qquad \text{for } |f_p(z)| \leqslant rM^{1-1/t_{1p}}, \\ \dfrac{M^p}{(1-r^{2p})}\cdot\dfrac{1-r^{2p}M^{-2p\left(\log\frac{Mr}{|f_p(z)|}\right)/\log M}}{M^{p\left(\log\frac{Mr}{|f_p(z)|}/\log M\right)^2}} \\ \qquad\qquad \text{for } rM^{1-1/t_{1p}} \leqslant |f_p(z)| \leqslant rM^{1-1/t_{2p}}, \\ \dfrac{M^{2p}}{(M^{2p}-|f_p(z)|^{2p})}\cdot\dfrac{(1-x_{2p}^2)^2}{(1-r^{2p})}\left|\dfrac{f_p(z)}{z}\right|^{p+1}\left(\dfrac{x_{2p}}{r^p}\right)^{4x_{2p}^2/(1-x_{2p}^2)} \\ \qquad\qquad \text{for } |f_p(z)| \geqslant rM^{1-1/t_{2p}}, \end{cases} \tag{37}$$

where t_{1p}, t_{2p} *are the roots of the equations* (24).

The numbers x_{1p}, x_{2p} *are the unique roots of the equations*:

$$\frac{M^{2p}}{(M^p+|f_p(z)|^p)^2}(1+x)^2\left|\frac{f_p(z)}{z}\right|^p\left(\frac{r^p}{x}\right)^{2x/(1+x)} = 1, \tag{38}$$

$$\frac{M^{2p}}{(M^p-|f_p(z)|^p)^2}(1-x)^2\left|\frac{f_p(z)}{z}\right|^p\left(\frac{x}{r^p}\right)^{2x/(1-x)} = 1, \tag{39}$$

respectively.

We have the following sharp estimate from below:

$$|f_p'(z)| \geqslant \frac{M^{2p}(1-r^{2p})}{M^{2p}-|f_p(z)|^{2p}}\left|\frac{f_p(z)}{z}\right|^{p+1} \geqslant \frac{M^{2p}(1-r^{2p})}{M^{2p}-(E_p^-)^{2p}}\left(\frac{E_p^-}{r}\right)^{p+1}, \tag{40}$$

where

$$E_p^+\ (E_p^-) = \max_{\substack{|z|=r\\ f_p\in S_M^p}}\ (\min)|f_p(z)|$$

(see Lemma 4 below).

THEOREM 6. *Let* $F_p \in \Sigma_m^p$, $p = 1, 2, \ldots$, *then for fixed* $|F_p(\xi)|$, $\xi = Re^{i\varphi} \in K^*$, *the following sharp estimates hold*:

$$
(41)\qquad |F_p'(\xi)| \leqslant \begin{cases} \dfrac{(1-\hat{x}_{1p}^2)^2}{(|F_p(\xi)|^{2p}-m^{2p})(R^{2p}-1)}\left|\dfrac{F_p(\xi)}{\xi}\right|^{p+1}(R^p\hat{x}_{1p})^{4\hat{x}_{1p}^2/(1-\hat{x}_{1p}^2)} \\ \qquad\qquad \text{for } |F_p(\xi)| \geqslant Rm^{(1/\hat{t}_{1p})-1}, \\ \dfrac{m^{p\left(\log \frac{|F_p(\xi)|}{mR}/\log m\right)^2}}{m^p(R^{2p}-1)}\left[R^{2p}-m^{-2p\left(\log\frac{|F_p(\xi)|}{Rm}/\log m\right)}\right] \\ \qquad\qquad \text{for } Rm^{(1/\hat{t}_{2p})-1} \leqslant |F_p(\xi)| \leqslant Rm^{(1/\hat{t}_{1p})-1}, \\ \dfrac{(1-\hat{x}_{2p}^2)^2}{(|F_p(\xi)|^{2p}-m^{2p})(R^{2p}-1)}\left|\dfrac{F_p(\xi)}{\xi}\right|^{p+1}(R^p\hat{x}_{2p})^{4\hat{x}_{2p}^2/(1-\hat{x}_{2p}^2)} \\ \qquad\qquad \text{for } |F_p(\xi)| \leqslant Rm^{(1/\hat{t}_{2p})-1}, \end{cases}
$$

where t_{1p}, t_{2p} *are roots of the equations* (30).

The numbers x_{1p}, x_{2p} *are the unique roots of the equations*:

$$
(42)\qquad \frac{|F_p(\xi)|^{2p}}{(|F_p(\xi)|^p+m^p)^2}(1+x)^2\left|\frac{F_p(\xi)}{\xi}\right|^{-p}(R^px)^{-2x/(1+x)} = 1,
$$

$$
(43)\qquad \frac{|F_p(\xi)|^{2p}}{(|F_p(\xi)|^p-m^p)^2}(1-x)^2\left|\frac{F_p(\xi)}{\xi}\right|^{-p}(R^px)^{2x/(1-x)} = 1,
$$

respectively.

We have the following sharp estimate from below:

$$
(44)\qquad |F_p'(\xi)| \geqslant \frac{R^{2p}-1}{|F_p(\xi)|^{2p}-m^{2p}}\left|\frac{F_p(\xi)}{\xi}\right|^{p+1} \geqslant \frac{R^{2p}-1}{(\hat{E}_p^+)^{2p}-m^{2p}}\left(\frac{\hat{E}_p^+}{R}\right)^{p+1},
$$

where

$$
\hat{E}_p^+ \ (\hat{E}_p^-) = \max_{\substack{|\xi|=R \\ F_p\in\Sigma_m^p}} (\min) |F_p(\xi)|.
$$

THEOREM 7. *For every function* $f \in S_M$ *the following sharp estimates hold:*

$$
(45)\qquad A(r, M) \leqslant |f'(z)| \leqslant B(r, M), \qquad z = re^{i\varphi} \in K,
$$

where for every $r \in (0, 1)$ *and* $M \in (1, +\infty)$ *we have*

$$
(46)\qquad A(r, M) = \frac{x(1-(x/M))}{r(1+(x/M))}\cdot\frac{1-r}{1+r},
$$

and $x \in (0, M)$ *is the root of the equation*

$$
(47)\qquad \frac{((x/M)+1)^2}{x} = \frac{(1+r)^2}{r}.
$$

The quantity $B(r, M)$ is defined in a more complicated way, namely:

1° for every $r \in (0, 1)$ and $M \in (M_1(r), +\infty)$, where

$$
M_1(r) = \max\left(1, \frac{r}{(1-r)^2}\cdot\frac{[1-h(r)]^2}{h(r)}\right), \qquad h(r) = -1+\frac{1+r}{\sqrt{2r}},
$$

we have

$$
(48)\qquad B(r, M) = \frac{y}{r}\cdot\frac{(1-(y/M))}{(1+(y/M))}\frac{1+r}{1-r},
$$

where $y \in (0, Mh(r))$ is the root of the equation

$$
(49)\qquad \frac{((y/M)-1)^2}{y} = \frac{(1-r)^2}{r},
$$

2° for every $r \in (1/2, 1)$ and $M \in (M_2(r), M_1(r)]$, where $M_2(r) = 8r^3$, we have

$$
(50)\qquad \log B(r, M) = \log\frac{2u[1-(u/M)^2]}{1-r^2} - \frac{\sqrt{\left(\frac{u}{M}\right)^2+2\left(\frac{u}{M}\right)-1}}{1+\frac{u}{M}}\log r\,\frac{\left(1+\frac{u}{M}\right)+\sqrt{\left(\frac{u}{M}\right)^2+2\left(\frac{u}{M}\right)-1}}{\left(1+\frac{u}{M}\right)-\sqrt{\left(\frac{u}{M}\right)^2+2\left(\frac{u}{M}\right)-1}},
$$

where $u \in [Mh(r), \frac{1}{2}M)$ is the root of the equation

$$
(51)\qquad \log\frac{\left(\frac{u}{M}-1\right)^2}{2u\left[\left(\frac{u}{M}\right)^2+2\left(\frac{u}{M}\right)-1\right]} + \frac{\left(1+\frac{u}{M}\right)}{\sqrt{\left(\frac{u}{M}\right)^2+2\left(\frac{u}{M}\right)-1}}\log r\,\frac{\left(1+\frac{u}{M}\right)+\sqrt{\left(\frac{u}{M}\right)^2+2\left(\frac{u}{M}\right)-1}}{\left(1+\frac{u}{M}\right)-\sqrt{\left(\frac{u}{M}\right)^2+2\left(\frac{u}{M}\right)-1}} = 0;
$$

3° for every $r \in (1/2, 1)$ and $M \in (1, M_2(r)]$ we have

$$
(52)\qquad \log B(r, M) = \log M\frac{1-(v/M)^2}{1-r^2} + \frac{v^2}{M^2-v^2}\log\frac{v}{Mr},
$$

where $v \in [\frac{1}{2}M, M]$ is the root of the equation

$$
(53)\qquad \log M + \frac{M^2-v^2}{v^2}\log\frac{v}{Mr} = 0.
$$

THEOREM 8. *For every function* $f \in S_M$ *the following sharp estimates hold*:

$$
(54)\qquad \hat{A}(r, M) \leqslant \left|\frac{zf'(z)}{f(z)}\right| \leqslant \hat{B}(r, M), \qquad z = re^{i\varphi} \in K,
$$

where for every $r \in (0, 1)$ *and* $M \in (1, +\infty)$ *we have*

$$\hat{A}(r, M) = \frac{1+(x/M)}{1-(x/M)} \cdot \frac{1-r}{r(1+r)} \tag{55}$$

and $x \in (0, M)$ *is the root of the equation* (47).

The quantity $\hat{B}(r, M)$ is defined in the following way:

1° for every $r \in (0, 2-\sqrt{3})$ and $M \in (1, +\infty)$, as well as for $r \in [2-\sqrt{3}, 1)$ and $M \in (\hat{M}_1(r), +\infty)$, where

$$\hat{M}_1(r) = \max\left(1, \frac{2-\hat{h}(r)}{\hat{h}^2(r)}\right), \quad \hat{h}(r) = \left(\frac{1-r}{\sqrt{2r}}\right)^2,$$

we have

$$\hat{B}(r, M) = \frac{1-(y/M)}{1+(y/M)} \cdot \frac{1+r}{r(1-r)}, \tag{56}$$

where y is the root of the equation (49).

2° for every $r \in [2-\sqrt{3}, 1)$, $M \in (\hat{M}_2(r), \hat{M}_1(r)]$, where $\hat{M}_2(r) = [(2+\sqrt{3})r]^{\sqrt{3}}$, we have

$$\log \hat{B}(r, M) = \frac{\sqrt{2\frac{u}{M}}}{1+\frac{u}{M}} \log \frac{\left(1+\frac{u}{M}\right)-\sqrt{2\frac{u}{M}}}{\left(1+\frac{u}{M}\right)+\sqrt{2\frac{u}{M}}} \frac{1}{r} + \log\left[\frac{4r}{1-r^2} \frac{1-\left(\frac{u}{M}\right)^2}{1+\left(\frac{u}{M}\right)^2}\right], \tag{57}$$

where $u \in (M\tau_1(r), (2-\sqrt{3})M)$, is the root of the equation

$$\log \frac{1}{8M} \frac{\left(\frac{u}{M}-1\right)^2\left[\left(\frac{u}{M}\right)^2+1\right]}{\left(\frac{u}{M}\right)^2} + \frac{1+\frac{u}{M}}{\sqrt{2\frac{u}{M}}} \log r \frac{\left(1+\frac{u}{M}\right)+\sqrt{2\frac{u}{M}}}{\left(1+\frac{u}{M}\right)-\sqrt{2\frac{u}{M}}} = 0, \tag{58}$$

and $\tau_1(r)$ is the root of the equation

$$\frac{(\tau_1+1)^2}{2\tau_1} = \frac{\hat{h}(r)+2}{\hat{h}(r)}.$$

3° for every $r \in [2-\sqrt{3}, 1)$, $M \in (1, \hat{M}_2(r)]$ we have

$$\log \hat{B}(r, M) = \frac{1+\left(\frac{v}{M}\right)^2}{2\left[1-\left(\frac{v}{M}\right)^2\right]} \log \frac{\frac{v}{M}}{r} + \log r \frac{1-\left(\frac{v}{M}\right)^2}{(1-r^2)\frac{v}{M}}, \tag{59}$$

where $v \in ((2-\sqrt{3})M, M)$ is the root of the equation

$$2\frac{1-(v/M)^2}{1+(v/M)^2} \log \frac{r}{(v/M)} - \log M = 0. \tag{60}$$

3. Lemmas

Let $\mathfrak{P}$ denote the class of holomorphic functions p in K which have the form

$$p(z) = 1+c_1 z+ \ldots, \tag{61}$$

and satisfy the condition $\mathrm{Re}\, p(z) > 0$, $z \in K$.

Further, by $\mathfrak{P}^{[\alpha,\beta]}$ we denote the class of functions $p\colon K \times [\alpha, \beta] \Rightarrow \boldsymbol{C}$ such that for fixed $w \in K$ the function p is measurable viewed as a function of $x \in [\alpha, \beta]$ and for almost all $x \in [\alpha, \beta]$ the function p regarded as a function of $w \in K$ belongs to the class $\mathfrak{P}$.

Let us consider the differential equation

$$\frac{dw}{dx} = -wp(w, x), \quad w = w(z, x), \quad p \in \mathfrak{P}^{[0,+\infty)} \tag{62}$$

with the initial condition $w(z, 0) = z$, $z \in K$.

We have

LEMMA 1 [2]. *A function f belongs to S_M iff it can be represented in the form*

$$f(z) = \lim_{x \to \log M} e^x w(z, x; p), \quad p \in \mathfrak{P}^{[0,+\infty)}, \tag{63}$$

where $w = w(z, x; p)$ *is the solution of* (62) *for almost all* $x \in [0, +\infty)$ *and satisfies the initial condition* $w(z, 0; p) = z$, $z \in K$.

LEMMA 2 [2]. *Let* $0 < t = t(x) < 1$, $\Theta = \Theta(x)$ *be continuous real valued functions and let* $w_0(x) = t(x)e^{i\Theta(x)}$, $x \in [\alpha, \beta]$. *Then for any function* $p = p(w, x) \in \mathfrak{P}^{[\alpha,\beta]}$ *we have*

$$H(w, x) = \frac{p\left(\frac{t-w}{1-tw}e^{i\Theta}, x\right) - i\,\mathrm{Im}\, p(w_0, x)}{\mathrm{Re}\, p(w_0, x)} \in \mathfrak{P}^{[\alpha,\beta]}, \tag{64}$$

and the formula for the "inverse transformation" has the form

$$p(w, x) = \frac{H\left(\frac{t-we^{-i\Theta}}{1-twe^{-i\Theta}}, x\right) - i\,\mathrm{Im}\, H(t, x)}{\mathrm{Re}\, H(t, x)}. \tag{65}$$

The next lemma is the key lemma; it is a consequence of (62) and (63) and it shows how to reduce the extremal problem from the class S_M to the class $\mathfrak{P}^{[0,1]}$ and then to the class $\mathfrak{P}$.

The lemma in its form presented here is an immediate consequence of Theorem 1 from [2].

LEMMA 3 [2]. *The functional* $\hat{I}(F)$ *mapping the class* $\mathfrak{P}^{[0,1]}$ *onto* D, *has the following form*

$$\hat{I}(F) = \int_{\varrho/M}^{r} \mathrm{Re}\big(F(t, t)-1\big)\frac{dt}{t} + i\int_{\varrho/M}^{r} \mathrm{Re}\left(F(t, t)-1+\frac{t}{1-t^2}F'_w(0, t)\right)\frac{dt}{t}, \tag{66}$$

where $\varrho = \varrho(F)$ is the unique root of the equation

$$\log M = \int_{\varrho/M}^{r} \operatorname{Re} F(t, t) \frac{dt}{t}. \tag{67}$$

The function $f \in S_M$ for which $J(f) = \hat{I}(F)$ can be found from (63) and (62), where p has the form (65) and $H(w, x) = F(w, t(x))$.

The functions $t(x)$ and $\Theta(x)$ are solutions of the system of equations

$$\begin{aligned} &\operatorname{Re} F(t, t) d\log t + dx = 0, \\ &\operatorname{Re} F(t, t) d\Theta - \operatorname{Im} F(t, t) dx = 0 \end{aligned} \tag{68}$$

with the initial conditions: $t(0) = r$, $\Theta(0) = \varphi$.

LEMMA 4. *If $f_p \in S_M^p$, then for arbitrary $z = re^{i\varphi} \in K$ the following sharp estimates hold*

$$E_p^- = \left[\frac{2}{(1+r^p)+\sqrt{(1+r^p)^2-4r^p/M^p}}\right]^{2/p} r \leqslant |f_p(z)| \tag{69}$$

$$\leqslant r\left[\frac{2}{(1-r^p)+\sqrt{(1-r^p)^2+4r^p/M^p}}\right]^{2/p} = E_p^+.$$

This result can be obtained from the corresponding result for the class S_M [2] and relation (6).

LEMMA 5 [9]. *Let $\Phi(\xi, \omega)$ be a real valued function defined in the domain $\{(\xi, \omega)\colon \operatorname{Re}\xi > 0, |\omega| < +\infty\}$ and assume that, in every disk $|\omega - \omega_0| \leqslant R$, Φ attains its proper supremum and infimum in the boundary of the disk.*

Let $z = re^{i\varphi} \in K$, $0 < r < 1$, and let us write

$$I = \operatorname*{extr}_{p \in \mathfrak{P}} \operatorname*{extr}_{|z|=r} \Phi(p(z), zp'(z)). \tag{70}$$

Then I is attained within the family of functions which have the form

$$p(z) = \lambda_1 \frac{1+ze^{-i\theta_1}}{1-ze^{-i\theta_1}} + \lambda_2 \frac{1+ze^{-i\theta_2}}{1-ze^{-i\theta_2}},$$

where θ_1, θ_2 are real numbers and $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 0$, $\lambda_1 + \lambda_2 = 1$.

Moreover, we have the equality

$$I = \operatorname*{extr}_{\xi} \operatorname*{extr}_{\omega} \Phi(\xi, \omega),$$

where ξ is ranging over the disk $|\xi - a| \leqslant b$ and ω over the circle (ξ being fixed)

$$\{\omega\colon |\omega - \tfrac{1}{2}(\xi^2-1)| = \tfrac{1}{2}(b^2 - |\xi - a|^2)\}, \tag{71}$$

where

$$a = \frac{1+r^2}{1-r^2}, \quad b = \frac{2r}{1-r^2}. \tag{72}$$

LEMMA 6. *Let $t \in (0, r]$, $r \in (0, 1)$ and $\lambda \in (-\infty, +\infty)$ are fixed numbers and assume that q is a function from the class $\mathfrak{P}$.*

Then the values of

$$\max_{q \in \mathfrak{P}}(\min) \operatorname{Re}\left[(\lambda+1)q(t) + \frac{t}{1-t^2} q'(0) - 1\right] \tag{73}$$

are attained for functions $q_0 \in \mathfrak{P}$ such that:

(a) *in the case of maximum:*

$$\operatorname{Re} q_0(t) = \begin{cases} \dfrac{1-t}{1+t} & \text{for} \quad 0 < \tau \leqslant \dfrac{1-t}{1+t}, \\ \tau & \text{for} \quad \dfrac{1-t}{1+t} \leqslant \tau \leqslant \dfrac{1+t}{1-t}, \\ \dfrac{1+t}{1-t} & \text{for} \quad \tau \geqslant \dfrac{1+t}{1-t}, \end{cases} \quad \text{and } \lambda \geqslant -1, \tag{74}$$

(b) *in the case of minimum:*

$$\operatorname{Re} q_0(t) = \begin{cases} \dfrac{1+t}{1-t} & \text{for} \quad 0 < \tau \leqslant 1, \\ \dfrac{1-t}{1+t} & \text{for} \quad \tau \geqslant 1, \end{cases} \quad \text{and } \lambda \geqslant -1, \tag{75}$$

where $\tau = (-\lambda-1)^{-1/2}$, $\lambda < -1$,

Moreover, the extremal function satisfies the equality

$$\frac{t}{1-t^2} q_0'(0) = a - \frac{1}{\operatorname{Re} q_0(t)}, \quad a = \frac{1+t^2}{1-t^2}. \tag{76}$$

Proof. Let the assumptions of the lemma be satisfied. Then for any $z_0 \in K$ and $\Theta \in [-\pi, \pi)$ we have

$$p(z) = \frac{q\left(\dfrac{z_0+ze^{i\Theta}}{1+e^{i\Theta}\bar{z}_0 z}\right) - i\operatorname{Im} q(z_0)}{\operatorname{Re} q(z_0)} \in \mathfrak{P}. \tag{77}$$

From (77) we find

$$q(z) = \frac{p\left(e^{-i\Theta}\dfrac{z-z_0}{1-\bar{z}_0 z}\right) - i\operatorname{Im} p(-z_0 e^{-i\Theta})}{\operatorname{Re} p(-z_0 e^{-i\Theta})}. \tag{78}$$

Choosing z_0 and Θ appropriately we get

$$q(t) = \frac{1 - i\operatorname{Im} p(t)}{\operatorname{Re} p(t)}, \quad q'(0) = \frac{(t^2-1)p'(t)}{\operatorname{Re} p(t)}.$$

Now we can apply Lemma 5. Writing

$$\xi' = \frac{1 - i\operatorname{Im}\xi}{\operatorname{Re}\xi}, \quad \omega' = \frac{(t^2-1)\omega}{t\operatorname{Re}\xi},$$

we see that the disk $|\xi-a|\leqslant b$ goes onto the disk $|\xi'-a|\leqslant b$ and the circle (71) goes onto the circle

$$\left\{\omega' : \left|\omega'-\frac{(\xi'-1)(\overline{\xi'}+1)}{b\operatorname{Re}\xi'}\right| = \frac{b^2-|\xi'-a|^2}{b\operatorname{Re}\xi'}\right\}. \tag{79}$$

Now we see that in order to find (73) it is enough to evaluate

$$\operatorname*{extr}_{\xi'}\operatorname*{extr}_{\omega'}\left[(\lambda+1)\xi'+\frac{t}{1-t^2}\omega'-1\right],$$

where ω' ranges over the circle (79) (ξ' being fixed) and ξ' ranges over the circle $|\xi'-a| = b$.

It can be shown, by a not too long calculation, that the extrema in (73) are attained on the boundaries of the corresponding circles and that equalities (74)–(76) hold.

4. Proofs of the theorems

Proof of Theorem 1. From (8), (9) and Lemma 3 we infer that every point $W = U+iV$ of D has the representation

$$\begin{cases} U = \log\dfrac{\varrho}{r}, \\ V = \displaystyle\int_{\varrho/M}^{r} \operatorname{Re}\left[F(t,t)-1+\frac{t}{1-t^2}F'_w(0,t)\right]\frac{dt}{t}, \quad F = F(w,t)\in\mathfrak{P}^{[0,1]}, \end{cases} \tag{80}$$

where $\varrho = \varrho(F)$ is the unique root of the equation

$$\log M = \int_{\varrho/M}^{r} \operatorname{Re}F(t,t)\frac{dt}{t}, \quad \varrho = |f(z)|, \tag{81}$$

and F is an arbitrary function of the class $\mathfrak{P}^{[0,1]}$.

It is clear that if F_1, F_2 are in $\mathfrak{P}^{[0,1]}$ then also for every $\gamma\in[0,1]$ we have $F_\gamma = \gamma F_1+(1-\gamma)F_2\in\mathfrak{P}^{[0,1]}$, which implies that if $W_1, W_2\in D$, then by (66) also $W_\gamma = \gamma W_1+(1-\gamma)W_2\in D$. So the set D is closed and convex, and in order to describe it fully it is sufficient to find its boundary.

Let Γ be the boundary of D. For every fixed ϱ, $\varrho\in[E^-, E^+]$, the intersection $D(\varrho)$ of the set D with the straight line $U = \log(\varrho/r)$ is a segment or point. Thus in order to find the boundary of D it is enough to find the ends of the segments $D(\varrho)$. This in turn is equivalent to determining the values of

$$\max_{F\in\mathfrak{P}_\varrho^{[0,1]}}(\min)\,V(F) = \max_{F\in\mathfrak{P}_\varrho^{[0,1]}}(\min)\int_{\varrho/M}^{r}\operatorname{Re}\left[F(t,t)-1+\frac{t}{1-t^2}F'_w(0,t)\right]\frac{dt}{t}, \tag{82}$$

where $\mathfrak{P}_\varrho^{[0,1)}$ consists of functions F from $\mathfrak{P}^{[0,1]}$ which satisfy the equality (81).

The extremum (82) under condition (81) is equivalent to evaluating the extremum of

$$\psi(F) = \lambda\int_{\varrho/M}^{r}\operatorname{Re}F(t,t)\frac{dt}{t}+\int_{\varrho/M}^{r}\operatorname{Re}\left[F(t,t)-1+\frac{t}{1-t^2}F'_w(0,t)\right]\frac{dt}{t}, \tag{83}$$

where λ is a real number and F is an arbitrary function from the class $\mathfrak{P}^{[0,1]}$.

From Lemma 3 it follows that a point $W_0\in\Gamma$ is the image of a function $F_0\in\mathfrak{P}^{[0,1]}$ iff

$$\begin{cases} U_0 = \log\dfrac{\varrho}{r} \quad \left(= \displaystyle\int_{\varrho/M}^{r}\operatorname{Re}\big(F_0(t,t)-1\big)\frac{dt}{t}\right), \\ V_0 = \operatorname*{extr}_{F\in\mathfrak{P}_\varrho^{[0,1]}}\displaystyle\int_{\varrho/M}^{r}\operatorname{Re}\left[F(t,t)-1+\frac{t}{1-t^2}F'_w(0,t)\right]\frac{dt}{t}, \quad E^-\leqslant\varrho\leqslant E^+, \end{cases} \tag{84}$$

i.e. iff for arbitrary fixed real λ the function F_0 realizes the extremum of the functional

$$\left\{\lambda\int_{\varrho_0/M}^{r}\operatorname{Re}F(t,t)\frac{dt}{t}+\int_{\varrho_0/M}^{r}\operatorname{Re}\left[F(t,t)-1+\frac{t}{1-t^2}F'_w(0,t)\right]\frac{dt}{t}\right\}, \tag{85}$$

where $\varrho_0 = \varrho(F_0)$ is the unique root of the equation

$$\log M = \int_{\varrho/M}^{r}\operatorname{Re}F_0(t,t)\frac{dt}{t}. \tag{86}$$

For arbitrary fixed $t\in(0,r)$ and real λ consider the functional

$$\Omega(q,t,\lambda) = \lambda\operatorname{Re}q(t)+\operatorname{Re}\left[q(t)-1+\frac{t}{1-t^2}q'(0)\right] \tag{87}$$

defined on the class $\mathfrak{P}$.

Now we select from the class $\mathfrak{P}^{[0,1]}$ all functions $F = F(w,t)$ which for every fixed $t\in(0,r)$ realize the maximum of (87) within the class $\mathfrak{P}$. The collection of such functions is denoted by $\mathfrak{P}[\lambda]$.

We remark that if $q_0\in\mathfrak{P}$ realizes the extremum of (87), then for every fixed $t\in(0,r)$ there exists a function $F_0(w,t)\in\mathfrak{P}^{[0,1]}$ such that $q_0(w) = F_0(w,t)$ and, moreover, $F_0(w,t)\in\mathfrak{P}[\lambda]$. Thus for an arbitrary function $F(w,t)$ we have (handling the case of maximum, the case of minimum being similar):

$$\int_{\varrho_0/M}^{r}\left(\lambda\operatorname{Re}F(t,t)+\operatorname{Re}\left[F(t,t)-1+\frac{t}{1-t^2}F'_w(0,t)\right]\right)\frac{dt}{t}$$
$$\leqslant\int_{\varrho_0/M}^{r}\max_{F\in\mathfrak{P}^{[0,1]}}\left(\lambda\operatorname{Re}F(t,t)+\left[\operatorname{Re}F(t,t)-1+\frac{t}{1-t^2}F'_w(0,t)\right]\right)\frac{dt}{t}$$

$$= \int_{\varrho/M}^{r} \max_{q\in\mathfrak{P}} \left(\lambda \operatorname{Re} q(t) + \operatorname{Re}\left[q(t)-1+\frac{t}{1-t^2}q'(0)\right]\right)\frac{dt}{t}$$

$$= \int_{\varrho_0/M}^{r} \left(\lambda \operatorname{Re} q_0(t) + \operatorname{Re}\left[q_0(t)-1+\frac{t}{1-t^2}q_0'(0)\right]\right)\frac{dt}{t}$$

$$= \int_{\varrho_0/M}^{r} \left(\lambda \operatorname{Re} F_0(t,t) + \operatorname{Re}\left[F_0(t,t)-1+\frac{t}{1-t^2}F'_{0_w}(0,t)\right]\right)\frac{dt}{t},$$

which proves the equality

$$\max_{F\in\mathfrak{P}^{[0,1]}} \int_{\varrho_0/M}^{r} \left(\lambda \operatorname{Re} F(t,t) + \operatorname{Re}\left[F(t,t)-1+\frac{t}{1-t^2}F'_{w}(0,t)\right]\right)\frac{dt}{t}$$

$$= \int_{\varrho/M}^{r} \left(\lambda \operatorname{Re} F_0(t,t) + \operatorname{Re}\left[F_0(t,t)-1+\frac{t}{1-t^2}F'_{0_w}(0,t)\right]\right)\frac{dt}{t}.$$

Now we see that to a function F_0 which realizes the maximum of (83) corresponds a function q_0 which realizes the maximum of (87). Hence in order to find the function $F_0 = F_0(w,t) \in \mathfrak{P}^{[0,1]}$ which defines, *via* the transformation (80), the desired point of Γ, it is equivalent to determine the function $q_0(w) \in \mathfrak{P}$ which gives the extremum of functional (87).

Thus the boundary Γ of D is the image of the set $\bigcup_{-\infty<\lambda<+\infty} \mathfrak{P}[\lambda]$ under transformation (80).

Now we will find the equations of the boundary of D. To this effect we integrate the extremal function q_0 and use the formulae (84) and the results of Lemma 6.

First we will determine the part of the boundary which corresponds to the case of maximum in (82).

According to the result of Lemma 6 we will consider the following cases:

(a) $0 < \tau \leqslant (1-r)/(1+r)$. In this case we have $\operatorname{Re} q_0(t) = (1-t)/(1+t)$ and then by (84) we have (writing $\varkappa = \varrho/M$);

$$U_0 = \int_{\varkappa}^{r}\left(\frac{1-t}{1+t}-1\right)\frac{dt}{t} = \log\left(\frac{1+\varkappa}{1+r}\right)^2,$$

$$V_0 = U_0 + \int_{\varkappa}^{r}\left(\frac{1+t^2}{1-t^2}-\frac{1+t}{1-t}\right)\frac{dt}{t} = \log\left(\frac{1-r}{1-\varkappa}\right)\left(\frac{1+\varkappa}{1+r}\right)^3,$$

where $\varkappa$ is found from the condition (86), i.e.

$$\int_{\varkappa}^{r}\frac{1-t}{1+t}\,\frac{dt}{t} = \log M,$$

which implies

$$(88) \qquad \varkappa = \left[M\frac{(1+r)^2}{2r}-1\right] - \sqrt{\left[M\frac{(1+r)^2}{2r}-1\right]^2-1}.$$

Finally, in this case, we get one point on Γ^+, namely the point

$$(89) \qquad \begin{cases} U_0 = \log\left(\dfrac{1+\varkappa}{1+r}\right)^2, \\ V_0 = \log\left(\dfrac{1-r}{1-\varkappa}\right)\left(\dfrac{1+\varkappa}{1+r}\right)^3, \end{cases}$$

where $\varkappa$ is given by (88).

(b) $\tau \geqslant (1+r)/(1-r)$. In this case we have $\operatorname{Re} q_0(t) = (1+t)/(1-t)$ and then by (84) we find:

$$U_0 = \int_{\varkappa}^{r}\left(\frac{1+t}{1-t}-1\right)\frac{dt}{\iota} = \mathrm{og}\left(\frac{1-\varkappa}{1-r}\right)^2,$$

$$V_0 = U_0 + \int_{\varkappa}^{r}\left(\frac{1+t^2}{1-t^2}-\frac{1-t}{1+t}\right)\frac{dt}{t} = \log\left(\frac{1+r}{1+\varkappa}\right)\left(\frac{1-\varkappa}{1-r}\right)^3.$$

Condition (86) gives

$$\int_{\varkappa}^{r}\frac{1+t}{1-t}\,\frac{dt}{t} = \log M,$$

which implies

$$(90) \qquad \varkappa = \left[M\frac{(1-r)^2}{2r}+1\right] - \sqrt{\left[M\frac{(1-r)^2}{2r}+1\right]^2-1}.$$

In this case we also get one point on Γ^+, namely the point

$$(91) \qquad \begin{cases} U_0 = \log\left(\dfrac{1-\varkappa}{1-r}\right)^2, \\ V_0 = \log\left(\dfrac{1+r}{1+\varkappa}\right)\left(\dfrac{1-\varkappa}{1-r}\right)^3, \end{cases}$$

where $\varkappa$ is given by (90).

Now we consider the third case, a more complicated one:

(c) $(1-r)/(1+r) < \tau < (1+r)/(1-r)$ (then we have $\operatorname{Re} q_0(t) = \tau$).

1° First assume that $(1-r)/(1+r) < \tau < 1$ and $\varkappa \geqslant (1-\tau)/(1+\tau)$. Then the condition (86) has the form

$$\int_{\varkappa}^{r}\tau\frac{dt}{t} = \log M,$$

which implies

$$\varkappa = rM^{-1/\tau}. \tag{92}$$

Then by (84) we obtain (using also (92)):

$$\begin{cases} U_0 = U_0(\tau) = \int\limits_{\varkappa}^{r} (\tau-1)\frac{dt}{t} = \frac{\tau-1}{\tau}\log M, \\ V_0 = U_0(\tau) + \int\limits_{\varkappa}^{r}\left(\frac{1+t^2}{1-t^2}-\frac{1}{\tau}\right)\frac{dt}{t} = \frac{\tau^2-1}{\tau^2}\log M + \log\left(\frac{1-r^2M^{-2/\tau}}{1-r^2}\right). \end{cases} \tag{93}$$

If $\tau = 1$, we get one point on the V axis

$$\begin{cases} U_0 = 0, \\ V_0 = \log\frac{1-r^2M^{-2}}{1-r^2}. \end{cases} \tag{94}$$

The condition $\varkappa \geqslant (1-\tau)/(1+\tau)$ is equivalent (in view to (92)) to the condition

$$\left[r\left(\frac{1+\tau}{1-\tau}\right)\right]^{\tau} \geqslant M. \tag{95}$$

It is easy to see that the equation $\left[r\left(\frac{1+\tau}{1-\tau}\right)\right]^{\tau} = M$ has a unique root $\tau_1 = \tau_1(r, M) \in ((1-r)/(1+r), 1)$.

Now we see that in the case of $(1-r)/(1+r) < \tau \leqslant 1$ and $\varkappa \geqslant (1-\tau)/(1+\tau)$, the function q_0 such that $\mathrm{Re}\, q_0(t) = \tau$ gives rise to the part of Γ^+ which has equations (93), the parameter τ changing in the interval $[\tau_1(r, M), 1]$.

2° Secondly, assume that $(1-r)/(1+r) < \tau < 1$ and $\varkappa \leqslant (1-\tau)/(1+\tau)$. Then we have:

$$\mathrm{Re}\, q_0(t) = \begin{cases} \frac{1-t}{1+t} & \text{for} \quad \varkappa \leqslant t \leqslant \frac{1-\tau}{1+\tau}, \\ \tau & \text{for} \quad \frac{1-\tau}{1+\tau} \leqslant t \leqslant r. \end{cases}$$

Using (84) we get:

$$U_0 = U_0(\tau) = \int\limits_{\varkappa}^{r} (\mathrm{Re}\, q_0(t)-1)\frac{dt}{t} = \int\limits_{\varkappa}^{(1-\tau)/(1+\tau)} \left(\frac{1-t}{1+t}-1\right)\frac{dt}{t} + \int\limits_{(1-\tau)/(1+\tau)}^{r} (\tau-1)\frac{dt}{t}$$

$$= \log\frac{(1+\varkappa)^2(1+\tau)^2}{4} + (\tau-1)\log r\left(\frac{1+\tau}{1-\tau}\right),$$

$$V_0 = V_0(\tau) = U_0(\tau) + \int\limits_{\varkappa}^{r}\left(a - \frac{1}{\mathrm{Re}\, q_0(t)}\right)\frac{dt}{t}$$

$$= U_0(\tau) + \int\limits_{\varkappa}^{(1-\tau)/(1+\tau)} \left(\frac{1+t^2}{1-t^2} - \frac{1+t}{1-t}\right)\frac{dt}{t} + \int\limits_{(1-\tau)/(1+\tau)}^{r} \left(\frac{1+t^2}{1-t^2} - \frac{1}{\tau}\right)\frac{dt}{t}$$

$$= \log\frac{(1+\varkappa)^2(1+\tau)^2}{4} + (\tau-1)\log r\left(\frac{1+\tau}{1-\tau}\right) + \log\tau\left(\frac{1+\varkappa}{1-\varkappa}\right) +$$

$$+\log\frac{r}{1-r^2} - \log\frac{1-\tau^2}{4\tau} - \frac{1}{\tau}\log r\left(\frac{1+\tau}{1-\tau}\right).$$

In order to find $\varkappa = \varkappa(r, M)$ we use condition (86) which in the case considered takes on the form

$$\int\limits_{\varkappa}^{(1-\tau)/(1+\tau)} \frac{1-t}{1+t}\frac{dt}{t} + \int\limits_{(1-\tau)/(1+\tau)}^{r} \tau\frac{dt}{t} = \log M.$$

After simplification we get

$$\varkappa = \varkappa(r, M, \tau) = (2A-1) - \sqrt{A^2-A}, \tag{96}$$

where $A = Mr^{-\tau}(1+\tau)^{-(1+\tau)}(1-\tau)^{-(1-\tau)}$.

Finally, in the case of $(1-r)/(1+r) < \tau < 1$ and $\varkappa \leqslant (1-\tau)/(1+\tau)$ we obtain the partion of the boundary Γ^+ with parametric equations

$$\begin{cases} U_0(\tau) = \log\frac{(1+\varkappa)^2(1+\tau)^2}{4} + (\tau-1)\log r\left(\frac{1+\tau}{1-\tau}\right), \\ V_0(\tau) = \log\frac{(1+\varkappa)^2(1+\tau)^2}{4} + (\tau-1)\log r\left(\frac{1+\tau}{1-\tau}\right) + \\ \qquad +\log\tau\left(\frac{1+\varkappa}{1-\varkappa}\right) + \log\frac{r}{1-r^2} - \log\frac{1-\tau^2}{4\tau} - \frac{1}{\tau}\log r\left(\frac{1+\tau}{1-\tau}\right), \end{cases} \tag{97}$$

where $\varkappa$ is given by (96) and the parameter τ is changing in the interval $\left[\frac{1-r}{1+r}, \tau_1(r, M)\right]$.

To complete the proof we consider the case of $1 < \tau \leqslant (1+r)/(1-r)$ and $\varkappa \geqslant (\tau-1)/(\tau+1)$ or $\varkappa \leqslant (\tau-1)/(\tau+1)$; but this can be done in a way quite analogous to what was done above.

It is also worthwhile mentioning that all the three arcs which form the curve Γ^+ meet tangentially.

The equation of Γ^- has been obtained by the integration of the function q_0 given by (75).

After suitable rearrangements, simplification and change of the notation we get the final form of Theorem 1 (formulae (15)–(20)).

Proof of Theorem 2. We get the results from Theorem 2 if we apply relation (6) in Theorem 1.

Proof of Theorem 3. The results of Theorem 3 are obtained from Theorem 2 by applying relation (7).

Proof of Theorem 4. Using the convexity of D we find the maximum value of V when U is fixed on all the three arcs $\Gamma_1, \Gamma_2, \Gamma_3$. For example, taking into account the equations of Γ_1, we determine from the first equation of (15) the parameter t. Using the formula for $\varkappa_1$ and putting the value of t just calculated into the second equation of (15) we get (33).

Proof of Theorem 5. The result of Theorem 5 follows from the results of Theorem 4 by using relation (6).

Proof of Theorem 6. The result of Theorem 6 follows from the result of Theorem 5 by using relation (7).

Proof of Theorem 7 *and* 8. As an application of Theorem 1 we get distortion theorems for the class S_M, namely the sharp bounds for $|f'(z)|$ and $\left|\frac{zf'(z)}{f(z)}\right|$.

These estimates have a rather complicated form; they were found in [5] and [8], respectively, by the variational method of Charzyński and Janowski.

Here these results are simple corollaries following from the shape of the region D for the class S_M (Theorem 1).

In order to prove the statemants we have to extremize V and $(V-U)$ respectively, as $(U, V)\in D$. But this is a linear case and so the extremum is attained on the boundary. As far as we have not too bad parametric equations of Γ^+, we obtain the results of Theorems 7 and 8 by straightforward calculations, which are rather long.

The fact that Γ^+ consists of three arcs implies that in both upper estimates we have three formulae, but in the lower estimates we have only one.

All results and estimates contained in the paper are sharp, which follows from Lemma 6 in which the corresponding estimates for the class $\mathfrak{P}$ are sharp.

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