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SOME APPLICATIONS OF THE BERGMAN KERNEL TO GEOMETRICAL THEORY OF FUNCTIONS

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In this note we discuss certain aspects of the problem, very general in its nature, of applications of the Bergman kernel [2] to the geometrical theory of functions of complex variables.

1. Basic notations and notions

As usual, by C^n we will denote the space of n complex variables $z_j = x_j + iy_j$, $j = 1, 2, \ldots, n$. B will denote balls in C^n, i.e.

$$B = B(z^0, \varepsilon) = \Big\{z = (z_1, z_2, \ldots, z_n) \in C^n \colon \sum_{\nu=1}^{n} |z_\nu - z_\nu^0|^2 < \varepsilon^2\Big\},$$

while U will denote polydiscs, i.e.

$$U = U(z^0, \varepsilon) = \{z \in C^n \colon |z_1 - z_1^0| < \varepsilon, \ldots, |z_n - z_n^0| < \varepsilon\}.$$

The usual symbols

$$\partial = \sum_{\nu=1}^{n} \frac{\partial}{\partial z_\nu} dz_\nu, \qquad \bar\partial = \sum_{\nu=1}^{n} \frac{\partial}{\partial \bar z_\nu} d\bar z_\nu$$

will be used, where

$$\frac{\partial}{\partial z_\nu} = \frac{1}{2}\left(\frac{\partial}{\partial x_\nu} - i\frac{\partial}{\partial y_\nu}\right), \qquad \frac{\partial}{\partial \bar z_\nu} = \frac{1}{2}\left(\frac{\partial}{\partial x_\nu} + i\frac{\partial}{\partial y_\nu}\right), \qquad \nu = 1, 2, \ldots, n.$$

A map f of a domain $D \subset C^n$ will be called *biholomorphic* if it is holomorphic in D and possesses an inverse map f^{-1} which is holomorphic in the domain $G = f(D)$.

In a domain $D \subset C^n$ we consider the Hibert space of holomorphic functions

$$L^2H(D) = \Big\{f \in H(D) \colon ||f||_D^2 = \int_D |f|^2 d\omega < +\infty\Big\}$$

with scalar product

$$\langle \varphi, \psi \rangle_D = \int_D \varphi \overline{\psi}\, d\omega$$

($d\omega$ is the volume element in C^n). We will be concerned only with domains for which this space is non-trivial. They will be called *domains of a bounded type* (such are, for example, all bounded domains, while the space C^n is not of that type).

There are various ways of introducing the Bergman kernel of a domain. The most classical one is as follows: $\{\varphi_\nu\}_{\nu=0}^{\infty}$ denotes an orthonormal basis of $L^2H(D)$. A *Bergman kernel* (of the system of functions $\{\varphi_\nu\}_{\nu=0}^{\infty}$) will be called the sum of the series

$$K(z, \overline{\zeta}) = \sum_{\nu=0}^{\infty} \varphi_\nu(z)\overline{\varphi_\nu(\zeta)}, \qquad (z, \zeta \in D). \tag{1}$$

This is in fact the most natural way of calculating explicitly the expression $K(z, \overline{\zeta})$ for a given domain. For instance, let $U = \{z \in C^n\colon |z_1| < 1, \ldots, |z_n| < 1\}$ be a polydisc, and B be the ball $\{z \in C^n\colon |z_1|^2 + \ldots + |z_n|^2 < 1\}$. Then

$$K_U(z, \overline{\zeta}) = \frac{1}{\pi^n} \prod_{\nu=1}^{n} \frac{1}{(1 - z_\nu \overline{\zeta}_\nu)^2}; \quad K_B(z, \overline{\zeta}) = \frac{n!}{\pi^n \left(1 - \sum_{\nu=1}^{n} z_\nu \overline{\zeta}_\nu\right)^{n+1}}.$$

Practically, however, the cases when a similar calculation is possible, are rather difficult to meet. Regarding some explicit expressions of the Bergman kernel in certain domains, see, for example, [2], [5], [4].

The simplest basic properties of the Bergman kernel, which follow directly from the definition, are

(i) $K(z, \overline{\zeta})$ is holomorphic in z and antiholomorphic in ζ. We say that $K(z, \overline{\zeta})$ is a holomorphic function in $(z, \overline{\zeta})$ in $D \times D^*$, where $D^* = \{z = (z_1, \ldots, z_n) \in C^n\colon (\overline{z}_1, \ldots, \overline{z}_n) \in D\}$;

(ii) antisymmetry: $\overline{K(z, \overline{\zeta})} = K(\zeta, \overline{z})$;

(iii) a "reproducing property": for each function $f \in L^2H(D)$ and for any point $\zeta \in D$

$$f(\zeta) = \int_D K(\zeta, \overline{z}) f(z)\, d\omega_z.$$

Remark. The Bergman kernel, defined as above, seems to depend on the choice of the basis $\{\varphi_\nu\}_{\nu=0}^{\infty}$. But this is not the case. Actually, there is exactly one Bergman kernel, which will be denoted by $K_D(z, \overline{\zeta})$, assigned to a given domain D of a bounded type in C^n. This follows from the fact that the following extremal problem has a unique solution:

"for a fixed $\zeta \in D$ to find a function f_0 in the class $E = \{f \in L^2H(D)\colon f(\zeta) = 1\}$ which minimizes $\|f\|_D$."

Only one function $f_0 = f_0(z; \zeta)$ exists for which

$$\|f_0\|_D^2 = \inf_{f \in E} \|f\|_D^2.$$

It is usually denoted by $M_D(z; \zeta)$ and is called a *minimal function for the domain D with respect to the point ζ*. Its relation with the sum of the series (1) is simple:

$$K(z, \overline{\zeta}) = \frac{M_D(z; \zeta)}{\|M_D\|_D^2}. \tag{2}$$

Thus with every domain $D \subset C^n$ of a bounded type there is associated exactly one Bergman kernel $K_D(z, \overline{\zeta})$ $(z, \zeta \in D)$.

DEFINITION 1. $K_D(z) = K_D(z, \overline{z}) = \sum_{\nu=0}^{\infty} |\varphi_\nu(z)|^2$ will be called the *Bergman function of the domain D*, [2].

In domains of a bounded type this function assumes positive values since it is the reciprocal value of $\inf \|f\|_D^2$ in the class E.

We will also mention one way of characterizing the Bergman kernel due to M. Skwarczyński [18]:

BASIC LEMMA. *For an arbitrary domain D and $t \in D$ the function $K_D(\cdot, \overline{t})$ is uniquely characterized as an element $\varphi \in L^2H(D)$ which satisfies the conditions*

(a) $\varphi(t) \geqslant \|\varphi\|_D^2$,

(b) *if* $f \in L^2H(D)$ *and* $f(t) \geqslant \|f\|_D^2$, *then* $f(t) \leqslant \varphi(t)$.

The following important property of the Bergman function is the reason for its frequent use and application in a number of problems from the theory of functions of complex variables.

BASIC PROPOSITION [2]. *Let $f\colon D \to G = f(D)$ be a biholomorphic map. Then*

$$K_D(z) = (K_G \circ f)(z) \cdot |I_f(z)|^2. \tag{3}$$

$I_f(z)$ denotes the Jacobian of the map f at the point z. From (3) *we obtain* $\ln K_G(w) = \ln K_D(z) - \ln I_f(z) - \ln \overline{I_f(z)}$, *where* $w = f(z)$.

Taking into account that $\overline{\partial} \ln I_f = \partial \ln \overline{I_f} = 0$, in view of being holomorphic, we find that the differential form

$$\partial\overline{\partial} \ln K_D = \sum_{\mu,\nu=1}^{n} \frac{\partial^2 \ln K_D}{\partial z_\mu \partial \overline{z}_\nu} dz_\mu \wedge d\overline{z}_\nu \tag{4}$$

is invariant under biholomorphic maps of this domain. The bilinear form, associated to (4)

$$ds_D^2 = \sum_{\mu,\nu=1}^{n} \frac{\partial^2 \ln K_D}{\partial z_\mu \partial \overline{z}_\nu} dz_\mu d\overline{z}_\nu, \tag{5}$$

is called the *Bergman form of the domain D*.

It is easy to prove that in bounded domains the Bergman form is Hermitian and positive definite. This fact implies the strict plurisubharmonicity of the function $\ln K_D(z)$, as well as the plurisubharmonicity of $K_D(z)$. So, if D is a domain for which its Bergman function increases to infinity when approaching its boundary, then D is a domain of holomorphy.

DEFINITION 2. The Hermitian metric defined in the domain $D \subset C^n$ by means of the fundamental form (5) is called the *Bergman metric.*

Every domain which is complete with regard to the Bergman metric, is a domain of holomorphy. The converse implication is, in general not true.

We recall that in this metric biholomorphic maps are isometries. If $b_D(z_1, z_2)$ is the distance between two points z_1 and z_2 in this metric and $f\colon D \to G = f(D)$ is a biholomorphic map, then

$$b_D(z_1, z_2) = b_G\big(f(z_1), f(z_2)\big),$$

where b_G is the distance in the Bergman metric in the domain G.

Finally, we mention a proposition due to Bremerman stating that the Bergman kernel of the product of two domains $D \subset C^n(z)$ and $G \subset C^m(w)$ is equal to the product of the Bergman kernels of these domains, i.e.

$$K_{D\times G}(z, w, \bar{\zeta}, \bar{\omega}) = K_D(z, \bar{\zeta}) K_G(w, \bar{\omega}). \tag{6}$$

Hence follows the relation between the elements of the Bergman metric in the domains $D\times G$, D and G:

$$ds^2_{D\times G} = ds^2_D + ds^2_G. \tag{7}$$

2. Sequences of domains and their Bergman kernels

A. Consider a bounded domain $D \subset C^n$ and a sequence of subdomains of D such that

$$D_m \subset D_{m+1}, \quad D = \bigcup_{m=1}^{\infty} D_m. \tag{8}$$

For arbitrary $z, t \in D$ and for sufficiently large indices m one can consider the functions $K_{D_m}(z, \bar{t})$ and $K_D(z, \bar{t})$. The convergence of the sequence $\{K_{D_m}(z, \bar{t})\}_{m=1}^{\infty}$ is the object of the subsequent considerations.

In 1967 I proved the following theorem [12]:

THEOREM 1. *Let a bounded domain $D \subset C$ be the union of an increasing sequence of domains D_m, $m = 1, 2, \ldots$ Then the sequence of the Bergman kernels $K_{D_m}(z, \bar{t})$ converges to $K_D(z, \bar{t})$ locally uniformly in z in D.*

The idea of the proof is quite simple. It is easy to see that the sequence of the minimal functions $M_m = M_{D_m}(z; t)$ for the domains D_m (regarding t as fixed) is a uniformly bounded family of analytic functions and one can apply the compactness principle of Montel families. Then it is sufficient to employ relation (2).

In fact, the proof of this proposition, given in [12], works under the more general assumptions that D is an arbitrary domain of a bounded type in $C^n (n \geqslant 1)$; moreover it can be proved that the sequence $K_{D_m}(z, \bar{t})$ tends to $K_D(z, \bar{t})$ locally uniformly in $D \times D^*$, i.e. without assuming that t is a fixed point. This might be done by merging the idea of the original proof of Theorem 1 with an argument of Skwarczyński [19]. Although I have been aware of this fact since 1968, a proof for the general case has not been published yet, unless we count my Ph. D. thesis [Sofia, 1974]. Let us now prove

THEOREM 2. *Under the assumptions* (8) *on the domain D and subdomains D_m, the sequence $K_{D_m}(z, \bar{t})$ converges to $K_D(z, \bar{t})$ as $m \to \infty$ locally uniformly in $(z, \bar{t}) \in D \times D^*$.*

Proof. First, let us fix an arbitrary $t \in D$. Denote by F an arbitrary closed subdomain of D. An index $m_0 = m_0(F)$ exists such that $F \subset D_m$ for any $m \geqslant m_0$. Then

$$||M_m||_F \leqslant ||M_m||_{D_m} \leqslant ||M||_D$$

in view of a well-known monotony property. The sequence $M_m = M_{D_m}(z; t)$ consists of holomorphic functions whose norms are uniformly bounded and it is possible to extract a subsequence which is uniformly convergent on F. Denote by $g(z)$ its limit. This is a function holomorphic in D and $g(t) = 1$. As regards its norm, we get $||g||_F = ||\lim\limits_{\alpha} M_{m_\alpha}||_F = \lim\limits_{\alpha} ||M_{m_\alpha}||_F \leqslant ||M||_D$, i.e. $||g||_D \leqslant ||M||_D$. By the uniqueness of solution of the extremal problem in the class E it follows that $g \equiv M$ or

$$M(z; t) = \lim_{\alpha\to\infty} M_{m_\alpha}(z; t), \tag{9}$$

where the convergence is uniform in z on $F \subset D$.

Applying once more the monotony property we find that $K_{D_m}(t, \bar{t})$ form a monotonely decreasing sequence of real numbers bounded from below by $K_D(t, \bar{t})$. In view of (9) and (2), it contains a subsequence converging exactly to $K_D(t, \bar{t})$, i.e. for the sequence itself we have

$$K_D(t, \bar{t}) = \lim_{m\to\infty} K_{D_m}(t, \bar{t}). \tag{10}$$

Finally, consider $K_{D_m}(s, \bar{s})$ for an arbitrary $s \in F$. It is easily verified that $m_1 = m_1(F)$ exists, as well as a constant A_F, so that $K_{D_m}(s, \bar{s}) \leqslant A_F$ for any $s \in F$ and $m \geqslant m_1$. Examine the absolute value $|K_{D_m}(z, \bar{t})|$ on any compact subset L of $D \times D^*$, which can obviously be included in a compact set of the form $F \times F^*$. Then

$$|K_{D_m}(z, \bar{t})| \leqslant [K_{D_m}(z, \bar{z})]^{1/2} [K_{D_m}(t, \bar{t})]^{1/2} \leqslant A_F$$

for $m \geqslant m_1$ and $(z, \bar{t}) \in D \times D^*$. Thus the sequence $\{K_{D_m}(z, \bar{t})\}_{m=1}^{\infty}$ is uniformly bounded and a convergent subsequence $\{K_{D_{m\beta}}(z, \bar{t})\}_\beta$ can be extracted which tends uniformly on L to a function denoted by $k(z,t)$. Let us fix $t \in L$ and form the integral

$$\int_L |k(\cdot,t)|^2 = \lim_\beta \int_L |K_{D_{m\beta}}(\cdot,\bar{t})|^2 \leqslant \lim_\beta \int_{D_{m\beta}} |K_{D_{m\beta}}(\cdot,\bar{t})|^2 = \lim_\beta K_{D_{m\beta}}(t,\bar{t}) = k(t,t) = K_D(t,\bar{t}).$$

Therefore,

$$\|k(\cdot,t)\|_D^2 \leqslant k(t,t) = K_D(t,\bar{t}).$$

The above inequality shows that $k(z,t)$ satisfies the condition (a) of the basic lemma from § 1. Each function $f \in L^2H(D)$ satisfying (a) satisfies also $f(t) \leqslant K_D(t,\bar{t})$, and the right-hand side of the inequality is exactly $k(t,t)$, i.e. K_D and k satisfy the conditions of the lemma and hence $K_D(z,\bar{t}) \equiv k(z,t)$. Since the sequence $K_{D_{m\beta}}$ was chosen arbitrarily, the sequence $\{K_{D_m}(z,\bar{t})\}_{m=1}^\infty$ itself is locally uniformly convergent in $D \times D^*$ and tends to $K_D(z,\bar{t})$.

This theorem gives grounds to conclude that if I denotes some biholomorphic invariant related to the Bergman function, then $I(D) = \lim_{m\to\infty} I(D_m)$.

Theorems 1 and 2 found applications in certain problems concerning the so-called *Lu Qi-Keng conjecture* (see, for instance [17], [15], [13]). The conjecture says that when D is a bounded domain in C^n, the Bergman kernel does not vanish in $D \times D^*$. Following Skwarczyński [17], we introduce the following

DEFINITION 3. A domain $D \subset C^n$ will be called a *Lu Qi-Keng domain* if the equation $K_D(z,\bar{t}) = 0$ has no solution in $D \times D^*$.

It turns out that the Lu Qi-Keng conjecture does not hold, for example, in the case of multiply connected domains in the plane. In [17] it is shown that for a suitable choise of the number r, the annulus $R = \{z \in C: 0 < r < |z| < 1\}$ is not a Lu Qi-Keng domain. Moreover, Rosenthal [15] proved that "for any $p \geqslant 3$, there is a p-connected domain G_p in C which is not a Lu Qi-Keng domain".

In the case of simply-connected domains in C^n $(n \geqslant 2)$, the problem of validity of the conjecture seems to remain open. There are some results concerning this subject (see [1], [3], [11]).

B. Let us now turn our attention to the analogous situation when a given domain is approximated from "outside", that is, when we are concerned with a decreasing sequence of domains $D_m \supset D_{m+1} \supset D$. Certain interesting results regarding the problem were obtained by Skwarczyński and Iwiński [19], [20].

The matters are now much more complicated, even in the case of domains in the plane. For instance, if U_0 is the unit disc, $D = U_0 \setminus [0,1)$, $D_m = \{z \in C: |z| < 1+1/m\}$, then $K_{D_m} \to K_{U_0} \neq K_D$. Therefore one has to exclude from considerations slit domains and assume that $D = \operatorname{int} \bar{D}$. Another natural assumption to make is that $|\partial D| = 0$. With these assumptions on D, it is not difficult to prove

THEOREM 3 [20]. *Let* $\bar{D} = \bigcap_{m=1}^\infty D_m$, $D \subset D_{m+1} \subset D_m$, $D = \operatorname{int}\bar{D}$ *and* $|\partial D| = 0$. *Then* $K_{D_m}(z,\bar{t}) \to K_D(z,\bar{t})$ *as* $m \to \infty$ *locally uniformly in* $D \times D^*$ *if and only if the space* $H(\bar{D})$ *of all functions holomorphic in some neighbourhood of* $\bar{D}$ *is dense in* $L^2H(D)$.

At present there exist no general characterizations of domains in C^n $(n \geqslant 2)$, for which $H(\bar{D})$ is dense in $L^2H(D)$. In the case $n = 1$ such a characterization was given by Havin [6]. From Havin's result we obtain

THEOREM 4 [20]. *Let* $\bar{D} = \bigcap_{m=1}^\infty D_m$, $D \subset D_{m+1} \subset D_m$, $D = \operatorname{int}\bar{D}$ *and* $|\partial D| = 0$. *Then* $K_{D_m}(z,\bar{t}) \to K_D(z,\bar{t})$ *as* $m \to \infty$ *locally uniformly in* $(z,t) \in D \times D$ *if and only if the set of all* $z \in \partial D$ *which do not belong to the fine closure of the complement of* $\bar{D}$ *has a zero logarithmic capacity.*

If $C \setminus \bar{D}$ has a finite number of components none of which is reduced to a single point, then the conditions of Theorem 4 are fulfilled. On the other hand, there are domains which do not satisfy conditions of that kind.

3. Kobayashi and Carathéodory pseudometrics related to sequences of complex manifolds

A natural question that arises is to generalize Theorems 2, 3 and 4 from Section 2 to the case of complex manifolds. In this section we present the results obtained by V. Hristov [7], [8].

Let M be a connected complex manifold. The Carathéodory and Kobayashi pseudometrics can be introduced in the following way [9]. Let $\varrho(a,b)$ be the *Poincaré–Bergman distance* between two points a and b from the unit disc U, defined by

$$\varrho(a,b) = \frac{1}{2}\log\frac{|1-\bar{a}b|+|b-a|}{|1-\bar{a}b|-|b-a|}.$$

DEFINITION 4. By the *Carathéodory pseudodistance* between points p and q from M we mean the number

$$c_M(p,q) = \sup_f \varrho\big(f(p), f(q)\big),$$

where the supremum is taken over all holomorphic maps $f\colon M \to U$. $O(M,U)$ denotes the family of all these maps.

DEFINITION 5 (following Royden [16]). Let M be a complex manifold, $p \in M$ and let $v \in T_p(M)$ be a tangent vector. Consider the holomorphic maps $f\colon U_R = \{|z| < R\} \to M$, $f(0) = p$, $f'(0) = v$. Write

$$\Phi(p,v) = 1/[\sup\{R: \exists f\colon U_R \to M, f(0) = p, f'(0) = v\}].$$

The *Kobayashi pseudodistance* in M between the points p and q is defined as the number

$$k_M(p,q) = \inf\int_0^1 \Phi\big(\gamma(t),\gamma'(t)\big)\,dt,$$

where $\gamma\colon [0,1] \to M$ is a piecewise-smooth arc with origin $p = \gamma(0)$ and end $q = \gamma(1)$.

These two pseudodistances possess the following important property, which can be viewed as a generalization of the Schwarz lemma: holomorphic maps between complex manifolds are distance-decreasing in each of these pseudometrics, that is, for every holomorphic map $f\colon M \to N$ and any two points $p, q \in M$, we have

$$c_M(p, q) \geqslant c_N\big(f(p), f(q)\big); \qquad k_M(p, q) \geqslant k_N\big(f(p), f(q)\big).$$

Besides, the inequality

$$c_M(p, q) \leqslant k_M(p, q)$$

holds for p, $q \in M$, and turns into the equality

$$c_M(p, q) = k_M(p, q) = \varrho_M(p, q)$$

when $M \equiv U$. The above inequality shows that the Kobayashi (Carathéodory) pseudodistance is the "greatest" (the "smallest") among those distances which do not increase under holomorphic transformations of U (M) into M (U).

To obtain his results Hristov uses the notion of "*kernel of a sequence of manifolds*" and "*convergence of a sequence of manifolds to its kernel*". He considers a sequence of complex manifolds $\{M_j\}_{j=1}^{\infty}$ with the same dimension, for which the intersection $\bigcap\limits_{j=1}^{\infty} M_j$ (as an intersection of sets) is not empty and has an open interior in all M_j. Moreover, let every pair of manifolds of the sequence have compatible structures in their common parts.

DEFINITION 6. A manifold M is called a *kernel of the sequence* $\{M_j\}_j$ if M is a maximal subset of $\bigcup\limits_{j=1}^{\infty} M_j$ whose every point is contained in M_j with an open neighbourhood for all sufficiently large indices j. We will say that the sequence $\{M_j\}_j$ *tends to its kernel* M if M is a kernel for every subsequence of $\{M_j\}_j$.

The analogues of Theorem 2 hold when a given complex manifold is approximated "from inside". More precisely, we have

THEOREM 5. *Let M be a complex manifold and $\{M_j\}_j$ be a sequence of submanifolds having M as a kernel and converging to M. Then the limit $\lim\limits_{j\to\infty} c_{M_j}(p, q)$ exists for any $p, q \in M$ (which, for sufficiently large indices, belong to M_j) and this limit equals $c_M(p, q)$.*

THEOREM 6. *Under the same assumptions, the limit $\lim\limits_{j\to\infty} k_{M_j}(p, q)$ exists and equals $k_M(p, q)$.*

The case of approximation from "outside", as in item B of § 2, seems to be more complicated and together with the natural restrictions imposed on the sequence of manifolds, we need additional "density" on the families of holomorphic maps carried by the manifolds.

THEOREM 7. *Let $\{M_j\}_j$ be a sequence of complex manifolds containing M, having M as a kernel and converging to M, and suppose that every holomorphic map $f\colon M \to U$ can be represented as a limit of a sequence of holomorphic maps $f_j\colon M_j \to U$. Then, if the limit $\lim\limits_{j\to\infty} c_{M_j}(p, q)$ exists for two points $p, q \in M$, it equals $c_M(p, q)$.*

THEOREM 8. *Under the same assumptions on the sequence $\{M_j\}_j$, assume that there is a sequence of holomorphic maps $\varphi_j\colon M_j \to M$ which converges to id_M. Then, if for two points $p, q \in M$, the limit $\lim\limits_{j\to\infty} k_{M_j}(p, q)$ exists, this limit equals $k_M(p, q)$.*

4. The Bremermann theorem and its inverses

In this section we discuss a problem posed by me in 1975 in [14]. In § 1 we referred to the following proposition:

THEOREM (H. J. Bremermann, *Lectures on functions of a complex variable*, Univ. of Michigan Press, Ann Arbor, Mich., 1955). *If G is the Cartesian product of domains $B \subset C^n(z)$ and $D \subset C^m(w)$, then*

$$K_G(z, w, \bar\zeta, \bar\omega) = K_B(z, \bar\zeta) K_D(w, \bar\omega). \tag{11}$$

The problem is to find some kind of an inverse to this theorem. For the time being two ways of approaching the subject have been found. In [14] I suggested to consider domains satisfying conditions as follows: let G be a bounded domain from $C^n \times C^m$ containing the origin of this space and such that

(i) for each point $z \in B = \pi_1(G)$ the open set $S_z = \pi_2\big(G \cap \pi_1^{-1}(z)\big) = \{w \in C^m\colon \exists (z, w) \in G,\ z = \pi_1(z, w),\ w = \pi_2(z, w)\}$ is a domain in C^m_z containing the origin of C^m;

(ii) for any point $z \in B$, there is a biholomorphic mapping $\varphi_z\colon R \to S_z$ with $\varphi_z(0) = 0$, where $R = \pi_2(G)$.

Examples of such domains can be given: such are, for example (when $n = m = 1$), all Reinhardt domains, all Hartogs domains with symmetry plane $w = 0$ and so on.

Remark. In the original definition, given in [14], condition (ii) is missing. I owe this change to a remark of K. Diederich in 1976.

Let (z, w) be an arbitrary point from such a domain G. Obviously, we can form $K_G(z, w)$, $K_B(z)$ and $K_{S_z}(w)$. The direct Bremermann theorem states that, if $S_z \equiv R$, then $K_G = K_B \cdot K_{S_z}$.

Now, consider a domain of the above type, for which

$$K_G(z, w) = K_B(z) K_{S_z}(w) \tag{12}$$

at every point. Is it true that the "fibres" S_z do not differ, that is, is it true that $G = B \times S_0$?

The answer of this question is not so simple. Without loss of generality, the case $n = m = 1$ can be considered. Moreover, we can consider a Reinhardt domain;

for example, let $G = \{|z| < 1,\ |w| < g(|z|)\}$. Then (Theorem 2.3 of [14]), if at a point from G, say, at the origin, we have $K_G(0,0) = K_B(0)K_{S_0}(0)$, then $G = \{|z| < 1\} \times \{|w| < g(0)\}$.

Things do not look like that, as far as Hartogs domains are concerned. Even if we take a Hartogs domain which is complete, that is, if $G = \{(z, w) \in C^2: z \in B \subset C,\ |w| < R(z)\}$, it does not always follow that (12) implies $G = B \times S_0$. Let us point out a counter-example: let the unit disc be the base of the domain and let $R(z) = \frac{1}{2}\exp(z+\bar{z})$. Here the condition (12) is fulfilled, but the domain is not a cartesian product. The least we can say in this case is that the given domain is biholomorphically equivalent to the bidisc U^2 (cf. [14]).

In this way we come up to the conjecture that the domains for which the corresponding equality between the Bergman functions (12) holds are biholomorphically equivalent to cartesian products. But it again turns out that this is not always true. The reason is: as we showed in § 1, the direct Bremermann theorem asserts that the equality (6) holds. But the latter also yields (7), whence we find that

$$ds^2_{B\times D}|_B = ds^2_B,$$

i.e. the Bergman metric element for the domain $B \times D$ restricted to the base B coincides with Bergman metric element for the base. When considering domains of our class satisfying only (12), we find that $ds^2_G|_B = ds^2_B +$ a certain expression depending on $\partial\bar{\partial}\ln K_{S_z}(w)|_{w=0}$. The vanishing of the above expression actually involves imposing new restrictions of geometric nature on the domain G.

It is only under those restrictions that one can prove theorems (analogous to Theorems 4.2, 4.3, 4.4 from [14]) asserting that such a domain is biholomorphically equivalent to a polydisc or to a ball. For example, a Hartogs domain of the form $G = \{(z, w) \in C^2:\ z \in B,\ |w| < R(z)\}$, where B is simply-connected can be mapped onto the bidisc U^2 by a transformation which leaves the symmetry plane $\{w = 0\}$ ground, if and only if $-\ln R(z)$ is a function harmonic in B.

I will end with a short remark on another way of approach to the problem. Here I mean the results obtained by E. Ligocka and published in [10]. For this purpose, recall that a domain $D \subset R^n$ is called a domain of existence of a function f (assumed to be analytic in D) if and only if: whenever U is a domain in R^n and g is an analytic function on U such that $U \cap D \neq \emptyset$ and $f = g$ on some nonempty open set W, $W \subset U \cap D$, then $U \subset D$ and $f = g$ on U.

Ligocka proved a general proposition concerning the domains of existence of real-analytic functions, whence follows another inverse of the Bremermann theorem.

THEOREM 9. *Let $D \subset R^n \times R^m$ be a domain of existence of an analytic function f. Suppose that there exists a point $z_0 = (x_0, y_0)$ such that $f(x, y) = g(x)h(y)$ on some open neighbourhood of z_0, $g(x)$ and $h(y)$ are real-analytic functions defined on neighbourhoods of x_0 and y_0, respectively. Then:*

1. $D = \pi_1(D) \times \pi_2(D)$, where π_1 denotes the projection of $R^n \times R^m$ onto R^n and π_2 denotes the projection of $R^n \times R^m$ onto R^m.

2. $f(x, y) = G(x)H(y)$ on D, $G(x) = g(x)$ on a neighbourhood of x_0, $H(y) = h(y)$ on a neighbourhood of y_0.

3. $G(x)$ and $H(y)$ are real-analytic, $\pi_1(D)$ is the domain of existence for $G(x)$ and $\pi_2(D)$ is a domain of existence for $H(y)$.

An immediate consequence of this theorem is

THEOREM 10. *Let D be a domain in $C^n \times C^m$ which is a domain of existence for its Bergman function $K_D(z)$. Suppose that there exists $z_0 \in D$ such that*

$$K_D(z) = K_1(z_1)K_2(z_2), \quad z = (z_1, z_2),\ z_1 \in C^n,\ z_2 \in C^m$$

on some neighbourhood of z_0. Then

1. $D = \pi_1(D) \times \pi_2(D)$.

2. $K_D(z, \bar{w}) = K_{\pi_1(D)}(z_1, \bar{w}_1)K_{\pi_2(D)}(z_2, \bar{w}_2)$ on D, where $K_{\pi_1(D)}$ denotes the Bergman kernel of the domain $\pi_1(D) \subset C^n$, $K_{\pi_2(D)}$ denotes the Bergman kernel of $\pi_2(D) \subset C^m$.

3. There exists a constant $c \neq 0$ such that

$$K_1(z_1) = cK_{\pi_1(D)}(z_1, \bar{z}_1), \quad K_2(z_2) = \frac{1}{c}K_{\pi_2(D)}(z_2, \bar{z}_2)$$

on some neighbourhood of z.

Up to the present we do not have a satisfactory characterization of domains in C^n which are domains of existence of their Bergman functions in the general case. We can only say that every bounded domain which is complete with respect to the Bergman metric is also a domain of existence for its Bergman function.

References

[1] D. Bell, Compositio Math. 21 (4) (1969), 329–330.
[2] S. Bergman, *The kernel function and conformal mapping*, Math. Surveys V, A. M. S. (1970).
[3] I. Davidov, C. R. Acad. Bulg. Sci. 30 (1) (1977), 13–16.
[4] Egorycev, Izv. Vyšš. Učebn. Zav; Matematika 1 (1975), 101–103.
[5] B. A. Fuks, *Special chapters on theory of analytic functions of several complex variables*, Moscow 1963.
[6] V. Havin, Dokl. Akad. Nauk SSSR 178 (1968), 1025–1029.
[7] V. Hristov, C. R. Acad. Bulg. Sci. 29 (7) (1976), 951–954.
[8] —, Ph. D. thesis, autorefferat, Sofia (1977).
[9] S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings*, M. Dekker (1970).
[10] E. Ligocka, Bull. Acad. Pol. Sci., Sér. Math. 26 (6) (1978), 495–499.
[11] S. Matsuura, Pacif. J. Math. 49 (2) (1973), 407–416.
[12] I. Ramadanov, C. R. Acad. Bulg. Sci. 20 (8) (1967), 759–762.
[13] —, Izv. Mat. Inst. BAN, 10 (1969), 231–237.
[14] —, Annales de l'Institut Fourier (Grenoble) 25 (2) (1975), 193–211.

[15] P. Rosenthal, Proc. Amer. Math. Soc. 21 (1969), 33–35.
[16] H.L. Royden, Proc. Maryland Conf. Lecture notes N° 185.
[17] M. Skwarczyński, Proc. Amer. Math. Soc. 22 (1969), 305–310.
[18] —, Thesis, Warsaw University 1969.
[19] M. Skwarczyński and T. Iwiński, Bull. Warsaw University 1972.
[20] —, —, in: *Approximation theory*, Proc. Conf. Poznań 1972, Warsaw 1975, 117–120.

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EXPANSION OF ANALYTIC FUNCTIONS IN SERIES OF CLASSICAL ORTHOGONAL POLYNOMIALS

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The problem of representation of a function by series in a given function system is essential in analysis. The significance of power series, Dirichlet series and orthogonal function series is well known. In particular, the so-called classical orthogonal polynomials connected with the names of Jacobi, Legendre, Chebyshev, Laguerre and Hermite, play an important role in the theory of functions. Moreover, there are many applications in which these system of polynomials appear in a very natural way.

The purpose of this note is to state certain results on the representation of analytic functions by series in classical orthogonal polynomials. In Section 1 we recall the definitions of Jacobi, Laguerre and Hermite polynomials. Further, with the aid of asymptotic formulas for classical orthogonal polynomials we give a description of the regions of convergence of series in these polynomials. Section 2 is devoted to the main subject of this note, namely, the expansion of analytic functions into series of classical orthogonal polynomials. The final Section 3 contains another approach to the problem of representation of analytic functions by means of series in Laguerre, resp. Hermite polynomials.

1. Classical orthogonal polynomials

For convenience, in formulating the definitions of Jacobi, Laguerre and Hermite polynomials we follow G. Szegö [12], resp. H. Bateman and A. Erdelyi [2].

1.1. If $\alpha, \beta > -1$, the system of Jacobi polynomials $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ is (uniquely) determined by the orthogonality property

$$\int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta} P_m^{(\alpha,\beta)}(t) P_n^{(\alpha,\beta)}(t)\,dt = I_n^{(\alpha,\beta)}\delta_{mn},$$