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EXPANSION OF ANALYTIC FUNCTIONS IN SERIES OF CLASSICAL ORTHOGONAL POLYNOMIALS

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The problem of representation of a function by series in a given function system is essential in analysis. The significance of power series, Dirichlet series and orthogonal function series is well known. In particular, the so-called classical orthogonal polynomials connected with the names of Jacobi, Legendre, Chebyshev, Laguerre and Hermite, play an important role in the theory of functions. Moreover, there are many applications in which these system of polynomials appear in a very natural way.

The purpose of this note is to state certain results on the representation of analytic functions by series in classical orthogonal polynomials. In Section 1 we recall the definitions of Jacobi, Laguerre and Hermite polynomials. Further, with the aid of asymptotic formulas for classical orthogonal polynomials we give a description of the regions of convergence of series in these polynomials. Section 2 is devoted to the main subject of this note, namely, the expansion of analytic functions into series of classical orthogonal polynomials. The final Section 3 contains another approach to the problem of representation of analytic functions by means of series in Laguerre, resp. Hermite polynomials.

1. Classical orthogonal polynomials

For convenience, in formulating the definitions of Jacobi, Laguerre and Hermite polynomials we follow G. Szegö [12], resp. H. Bateman and A. Erdelyi [2].

1.1. If $\alpha, \beta > -1$, the system of Jacobi polynomials $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ is (uniquely) determined by the orthogonality property

$$\int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta} P_m^{(\alpha,\beta)}(t)\, P_n^{(\alpha,\beta)}(t)\,dt = I_n^{(\alpha,\beta)}\delta_{mn},$$

where

$$I_n^{(\alpha,\beta)} = \begin{cases} \dfrac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)} & \text{if} \quad n \geqslant 1, \\ \dfrac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} & \text{if} \quad n = 0 \end{cases}$$

and the condition that the coefficient of z^n in $P_n^{(\alpha,\beta)}(z)$ is positive.

If $\alpha = \beta$, Jacobi polynomials are called ultraspherical or Gegenbauer polynomials. Their particular cases are Legendre polynomials ($\alpha = \beta = 0$) and Chebyshev polynomials of the first ($\alpha = \beta = -1/2$) resp. second kind ($\alpha = \beta = 1/2$).

If $\alpha > -1$, the system of Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ is determined by the property

$$\int_0^{\infty} t^{\alpha}\exp(-t)L_m^{(\alpha)}(t)L_n^{(\alpha)}(t)\,dt = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\,\delta_{mn}$$

and the assumption that the coefficient of z^n in $(-1)^n L_n^{(\alpha)}(z)$ is positive.

Finally, Hermite polynomials $\{H_n(z)\}_{n=0}^{\infty}$ are determined by the property

$$\int_{-\infty}^{\infty} \exp(-t^2)H_m(t)H_n(t)\,dt = \sqrt{\pi}\, 2^n\Gamma(n+1)\,\delta_{mn}$$

provided that the coefficient of z^n in $H_n(z)$ is positive.

1.2. In order to describe the region and also the mode of convergence of a series in Jacobi polynomials

$$\sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(z) \tag{1.1}$$

one needs an asymptotic formula for $P_n^{(\alpha,\beta)}(z)$ as $n \to +\infty$ and z belongs to an arbitrary compact subset of the region $C-[-1, 1]$. The corresponding formula was given at the end of the 19-th century by G. Darboux [3], [12, (8.21.9)] and has the form

$$P_n^{(\alpha,\beta)}(z) = P^{(\alpha,\beta)}(z)\,n^{-1/2}\{\omega(z)\}^n\{1+p_n^{(\alpha,\beta)}(z)\}. \tag{1.2}$$

Here $\omega(z)$ is the inverse of Zukowski transformation $z = \frac{1}{2}(\omega+\omega^{-1})$ for which $\omega(\infty) = \infty$, $P^{(\alpha,\beta)}(z) \neq 0$ and $\{p_n^{(\alpha,\beta)}(z)\}_{n=1}^{\infty}$ are analytic functions holomorphic in the region $C-[-1, 1]$ and such that $\lim_{n\to+\infty} p_n^{(\alpha,\beta)}(z) = 0$ uniformly on every compact subset of this region.

Further, if $1 < r < +\infty$, we denote by $E(r)$ the interior of the ellipse $\gamma(r)$: $|\omega(z)| = r$ and we assume by definition that $E(+\infty) = C$. Then, as usual, an application of the asymptotic formula (1.2) leads to the following

PROPOSITION 1.1. *Let* $\lambda = \overline{\lim\limits_{n\to+\infty}} \sqrt[n]{|a_n|}$ *and* $R = \lambda^{-1}$. *Then*:

(i) *if* $R \leqslant 1$, *the series* (1.1) *is divergent at every point of the region* $C-[-1, 1]$;

(ii) *if* $1 < R \leqslant +\infty$, *the series* (1.1) *is absolutely uniformly convergent on every compact subset of the region* $E(R)$ *and diverges at every point of the region* $C-\overline{E(R)}$.

O. Perron [5] has studied in details the asymptotic properties of the confluent hypergeometric function $\Phi(a, c; z)$ as z or one of the parameters a, c tends to infinity. Using his general results and also the relation

$$L_n^{(\alpha)}(z) = \binom{n+\alpha}{n}\Phi(-n, \alpha+1; z) \tag{1.3}$$

one can derive asymptotic formulas for Laguerre polynomials in the region $C-[0, +\infty)$ and on the ray $[0, +\infty)$ [12, (8.22.3), (8.22.2)]. These formulas are sufficient to describe the region of convergence of a series of the kind

$$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z). \tag{1.4}$$

More precisely, if $\lambda_0 = -\overline{\lim\limits_{n\to+\infty}}(2\sqrt{n})^{-1}\ln|a_n| > 0$, then the series (1.4) is absolutely convergent at every point of the region $\Delta(\lambda_0)$: $\mathrm{Re}(-z)^{1/2} < \lambda_0$. But basing only on Perron's formulas one cannot answer the question what is the mode of convergence of the series (1.4) on an arbitrary compact subset of the region $\Delta(\lambda_0)$.

In the author's paper [7] it was shown, by means of the integral representation

$$L_n^{(\alpha)}(z) = \frac{z^{-\alpha/2}\exp z}{\Gamma(n+1)}\int_0^{\infty} t^{n+\alpha/2}\exp(-t)\,J_\alpha(2\sqrt{tz})\,dt, \tag{1.5}$$

where J_α is the Bessel function of the first kind with index α, that if $n \to +\infty$ and $z \to \infty$ in the region $\overline{\Delta(\lambda)}$: $\mathrm{Re}(-z)^{1/2} \leqslant \lambda$ then ($z = x+iy$)

$$L_n^{(\alpha)}(z) = O\left(|z|^{-\alpha/2-1/4}n^{\alpha/2-1/4}\exp\left(x+2\lambda\sqrt{n}\right)\right).$$

Further, it was shown how the above inequality can be used to get statements analogous to the classical Abel's lemma and Cauchy–Hadamard's theorem. We shall formulate the corresponding result as

PROPOSITION 1.2. *Let* $\lambda_0 = -\overline{\lim\limits_{n\to+\infty}}(2\sqrt{n})^{-1}\ln|a_n|$, *then*:

(i) *if* $\lambda_0 \leqslant 0$, *the series* (1.4) *is divergent at every point of the region* $C-[0, +\infty)$;

(ii) *if* $0 < \lambda_0 \leqslant +\infty$, *the series* (1.4) *is absolutely uniformly convergent on every compact subset of the region* $\Delta(\lambda_0)$ *and diverges at every point of the region* $C-\overline{\Delta(\lambda_0)}$.

For series in Hermite polynomials, i.e. series of the type

$$\sum_{n=0}^{\infty} a_n H_n(z) \tag{1.6}$$

the following holds:

PROPOSITION 1.3. *Let* $\tau_0 = -\overline{\lim_{n\to+\infty}}(2n+1)^{-1/2}\ln|(2n/e)^{n/2}a_n|$, *then:*

(i) *if* $\tau_0 \leqslant 0$, *the series* (1.6) *is divergent at every point of the open set* $C-(-\infty, +\infty)$;

(ii) *if* $0 < \tau_0 \leqslant +\infty$, *the series* (1.6) *is absolutely uniformly convergent on every compact subset of the region* $S(\tau_0)$: $|\mathrm{Im}\,z| < \tau_0$ *and diverges at every point of the open set* $C-\overline{S(\tau_0)}$.

The proof is based on the asymptotic formula for Hermite polynomials given by Szegö [12, (8.22.7)].

2. Representation of analytic functions by series in classical orthogonal polynomials

Classical orthogonal polynomials are one of the oldest and most powerful tools of analysis. Nevertheless, some of the (main) results concerning the expansion of analytic functions into series of these polynomials were obtained quite recently. Maybe one of the reasons is that in the case of a system of polynomials orthogonal over an infinite interval of the real axis (e.g. Laguerre or Hermite polynomials), the classical tool of complex analysis, namely, the Cauchy integral formula cannot be applied in general. As regards Jacobi polynomials, such a difficulty does not exist and the representation problem can be solved by means of the asymptotic formulas for these polynomials and for the corresponding functions of second kind $\{Q_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$.

The last system is defined in the region $C-[-1,1]$ by the equalities $(\alpha, \beta > -1)$

$$Q_n^{(\alpha,\beta)}(z) = -\int_{-1}^{1}\frac{(1-t)^{\alpha}(1+t)^{\beta}P_n^{(\alpha,\beta)}(t)}{t-z}\,dt, \quad n = 0, 1, 2, \ldots \tag{2.1}$$

For the system (2.1) the following asymptotic formula holds:

$$Q_n^{(\alpha,\beta)}(z) = Q^{(\alpha,\beta)}(z)n^{-1/2}\{\omega(z)\}^{-n-1}\{1+q_n^{(\alpha,\beta)}(z)\}, \quad n \geqslant 1, \tag{2.2}$$

where $Q^{(\alpha,\beta)}(z) \neq 0$ such that $\{q_n^{(\alpha,\beta)}(z)\}_{n=1}^{\infty}$ are holomorphic functions in the region $C-[-1,1]$ and $\lim_{n\to+\infty} q_n^{(\alpha,\beta)}(z) = 0$ uniformly on every compact subset of this region [12, (8.71.19)].

2.1. Basing on Christoffel–Darboux formula for Jacobi polynomials and functions of second kind [12, (4.62.19)] and on the asymptotic formulas (1.2) and (2.2), it is easy to prove the following

LEMMA 2.1. *Let* $\alpha, \beta > -1$, $1 < r < +\infty$ *and let* ψ *be a function L-integrable on the ellipse* $\gamma(r)$. *Then the function*

$$\int_{\gamma(r)}\frac{\psi(\zeta)}{\zeta-z}\,d\zeta, \quad \zeta \in C-\gamma(r)$$

can be expanded in the region $E(r)$ *into a series of type* (1.1) *with the coefficients*

$$a_n = \frac{1}{J_n^{(\alpha,\beta)}}\int_{\gamma(r)}\psi(\zeta)Q_n^{(\alpha,\beta)}(\zeta)\,d\zeta, \quad n = 0, 1, 2, \ldots$$

A trivial application of the Cauchy integral formula and the above lemma lead to the first one of the main results of the present note:

THEOREM I. *Let* $\alpha, \beta > -1$, $1 < R \leqslant +\infty$ *and let* f *be a complex function, holomorphic in the region* $E(R)$. *Then,* f *can be represented in this region by a series of type* (1.1) *with the coefficients* $(1 < r < R)$

$$a_n = \frac{1}{2\pi i J_n^{(\alpha,\beta)}}\int_{\gamma(r)}f(\zeta)Q_n^{(\alpha,\beta)}(\zeta)\,d\zeta, \quad n = 0, 1, 2, \ldots$$

Jacobi polynomials are particular cases of Gauss' hypergeometric function $F(a, b, c; z)$. More precisely,

$$\begin{aligned}P_n^{(\alpha,\beta)}(z) &= \binom{n+\alpha}{n}F(-n, n+\alpha+\beta+1, \alpha+1; (1-z)/2)\\ &= (-1)^n\binom{n+\beta}{n}F(-n, n+\alpha+\beta+1, \beta+1; (1+z)/2)\end{aligned}$$

and if α, β are arbitrary complex numbers such that $\alpha, \beta, \alpha+\beta \neq -1, -2, \ldots$, the above representations give a very natural generalization of Jacobi polynomials. Without going into all details, we just remark that Theorem I is valid also for the generalized Jacobi polynomials [1], [11].

2.2. The problem of expansion of analytic functions into series of Hermite polynomials found a solution in 1940 by E. Hille [4]. In order to formulate his result, we shall introduce suitable spaces of holomorphic functions which in fact have been described in Hille's paper.

If $0 < \tau_0 \leqslant +\infty$, we denote by $H(\tau_0)$ the space of all complex functions $f(z)$ holomorphic in the stripe $S(\tau_0)$: $|\mathrm{Im}\,z| < \tau_0$ and having the following property: for every $0 \leqslant \tau < \tau_0$ there exists a constant $B(\tau) \geqslant 0$ such that if $z = x+iy \in \overline{S}(\tau)$: $|\mathrm{Im}\,z| \leqslant \tau$, the inequality holds:

$$|f(z)| \leqslant B(\tau)\exp\{x^2/2-|x|(\tau^2-y^2)^{1/2}\}. \tag{2.3}$$

By means of the definition just given, Hille's result can be formulated as follows:

THEOREM II. *A complex function* f *holomorphic in the region* $S(\tau_0)$ $(0 < \tau_0 \leqslant +\infty)$ *can be represented in this region by a series of type* (1.6) *if and only if* $f \in H(\tau_0)$.

Let us mention that the proof of Theorem II, given in [4], is not easy and especially in proving the sufficiency of the condition (2.3) the author had to overcome considerable difficulties.

2.3. In 1947 H. Pollard's paper [6] appear in which, basing on Hille's work [4], the author solved the problem of representation of analytic functions by series in the polynomials $\{L_n(z^2)\}_{n=0}^{\infty} = \{L_n^{(0)}(z^2)\}_{n=0}^{\infty}$, i.e. by series of Laguerre polynomials in the case of $\alpha = 0$. Pollard's result is the following:

THEOREM III. *A complex function f holomorphic and even in the region $S(\tau_0)$ $(0 < \tau_0 \leqslant +\infty)$ can be represented in this region by a series in the polynomials $\{L_n(z^2)\}_{n=0}^{\infty}$ if and only if $f \in H(\tau_0)$.*

Remark. It is evident that by the mapping $\zeta = z^2$ the stripe $S(\tau_0)$ is transformed onto the domain $\Delta(\tau_0)$.

In the case of $\alpha \neq 0$, instead of Laguerre polynomials one can operate with the polynomials $\{L_n^{(\alpha)}(z^2)\}_{n=0}^{\infty}$. But the most striking thing is that the method used by Pollard cannot be generalized to the case of arbitrary $\alpha > -1$. Moreover, as far as we know, the problem of expansion of analytic functions into series of Laguerre polynomials in the general case, has remained open till the end of 1978.

A solution of the above problem, which is based also on Hille's theorem but uses a quite different idea, is given in the author's paper [8]. As usual, the main result is proved by means of certain auxiliary statements, which we now formulate as the "key lemma":

LEMMA 2.2. *Let $0 < \tau_0 \leqslant +\infty$ and $f \in H(\tau_0)$, then:*

(a) $zf(z) \in H(\tau_0)$;

(b) $\int_0^1 \varphi(t) f(zt)\,dt \in H(\tau_0)$ *if* $\int_0^1 |\varphi(t)|\,dt < +\infty$;

(c) $f'(z) \in H(\tau_0)$;

(d) *if $-1/2 < \alpha < 1/2$, the integral transformation*

$$f(z) = P^{(\alpha)}(F; z) = \frac{1}{\Gamma(\alpha+1/2)} \int_0^1 (1-t^2)^{\alpha-1/2} F(zt)\,dt \tag{2.4}$$

is an isomorphism of the space $H(\tau_0)$ and its inverse is

$$F(z) = Q^{(\alpha)}(f; z) = \frac{d}{dz}\left\{\frac{2z}{\Gamma(-\alpha+1/2)} \int_0^1 (1-t^2)^{-\alpha-1/2} t^{2\alpha+1} f(zt)\,dt\right\}.$$

Proof. The properties (a) and (b) have been established by Pollard [6, p. 362–363]. The property (c) is a consequence of Theorem II. Indeed, $f(z)$ can be represented in the region $S(\tau_0)$ by a series in Hermite polynomials, uniformly convergent on every compact subset of this region. In view of the relation $H_n'(z) = 2nH_{n-1}(z)$ $(n = 1, 2, 3, \ldots)$, $f'(z)$ is also representable in $S(\tau_0)$ by a series in Hermite polynomials, and therefore $f'(z) \in H(\tau_0)$.

If $F(z) \in H(\tau_0)$, then (b) implies that $f(z) = P^{(\alpha)}(F; z) \in H(\tau_0)$. Furthermore, the equalities

$$f^{(k)}(0) = \frac{\Gamma(k/2+1/2)}{2\Gamma(\alpha+k/2+1/2)} F^{(k)}(0), \quad k = 0, 1, 2, \ldots$$

lead to the conclusion that the mapping $P^{(\alpha)}$ is injective.

Let us note that in view of the properties (a), (b) and (c), the integral transformation $F(z) = Q^{(\alpha)}(f; z)$ is "well defined" and $F(z) \in H(\tau_0)$ if $f(z) \in H(\tau_0)$. Moreover, by using power series with centre at the origin for the functions F and f, it is easy to show that $P^{(\alpha)}(F; z) = f(z)$. In other words, the mapping (2.4) is surjective.

Now we are going to prove that the mapping (2.4) is continuous with respect to the topology of the space $H(\tau_0)$. Let us mention that if $G \subset C$ is a region, we assume (as usual) that the space $A(G)$ of all complex functions holomorphic in G is endowed with the topology of uniform convergence on compact subset of G. It is well known that this topology is metrizable and that $A(G)$ is a Fréchet space. In what follows the (linear) subspaces of $A(G)$ are considered with the induced topology.

Let a sequence $\{F_n(z)\}_{n=1}^{\infty} \subset H(\tau_0)$ be convergent in the space $H(\tau_0)$ to a function $F(z) \in H(\tau_0)$. For arbitrary $T > 0$ and $0 \leqslant \tau < \tau_0$ the convergence is uniform on the set $K(T, \tau) = \{z = x+iy\colon |x| \leqslant T,\ |y| \leqslant \tau\}$ and (2.4) implies that the sequence $\{P^{(\alpha)}(F_n; z)\}_{n=1}^{\infty}$ is also uniformly convergent to the function $P^{(\alpha)}(F; z)$ on the set $K(T, \tau)$. Since every compact subset of the region $S(\tau_0)$ is contained in a set of type $K(T, \tau)$, one can conclude that $\lim_{n\to+\infty} P^{(\alpha)}(F_n; z) = P^{(\alpha)}(F; z)$ in the topology of the space $H(\tau_0)$. Therefore, the operator $P^{(\alpha)}$ is continuous and in a similar way it can be proved that the same holds for the operator $Q^{(\alpha)}$.

Remark. The integral transformation $P^{(\alpha)}$ is in fact an integro-differential operator of Riemann–Liouville's type and $Q^{(\alpha)}$ is its inverse. Let us also note that the operator $P^{(0)}$ has been used in [6, p. 362, 364] but without saying anything about the property (d).

Now we are able to prove a statement which is a generalization of Theorem III. At first we observe that Lemma 2.2 holds if one replaces $H(\tau_0)$ by the space $\tilde{H}(\tau_0)$ of all even functions belonging to $H(\tau_0)$.

Let us note that a natural generalization of Laguerre polynomials is given by the relation (1.3) under the only assumption that the parameter α is an arbitrary (complex) number different from $-1, -2, \ldots$ From now on we deal with Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ assuming that α is real and not a negative integer.

THEOREM IV. *Let $\alpha \neq -1, -2, \ldots$ be real and $0 < \tau_0 \leqslant +\infty$. A complex function f analytic in the region $S(\tau_0)$ can be represented in this region by a series in the polynomials $\{L_n^{(\alpha)}(z^2)\}_{n=0}^{\infty}$ if and only if $f \in \tilde{H}(\tau_0)$.*

Proof. It is sufficient to prove Theorem IV in the case of $-1/2 < \alpha \leqslant 1/2$ since by means of elementary relations for Laguerre polynomials [2, v. II, 10.12, (23), (24)] the statement can be carried over to the case of any real $\alpha \neq -1, -2, \ldots$ The case of $\alpha = 1/2$ also can be omitted, since

$$L_n^{(1/2)}(z^2) = (-1)^n 2^{-2n-1} (n!)^{-1} z^{-1} H_{2n+1}(z), \quad n = 0, 1, 2, \ldots$$

Let us put

$$\tilde{L}^{(\alpha)} = \{\tilde{L}_n^{(\alpha)}(z)\}_{n=0}^{\infty} = \{(-1)^n(\Gamma(n+\alpha+1))^{-1}L_n^{(\alpha)}(z^2)\}_{n=0}^{\infty},$$
$$\tilde{H} = \{\tilde{H}_n(z)\}_{n=0}^{\infty} = \{2\sqrt{\pi}\,((2n)!)^{-1}H_{2n}(z)\}_{n=0}^{\infty}.$$

Then, in view of the relation between Hermite and Laguerre polynomials due to Uspensky [13], [12, (5.6.5)], we have $\tilde{L}_n^{(\alpha)}(z) = P^{(\alpha)}(\tilde{H}_n; z)$ $(n = 0, 1, 2, \ldots)$. Further, by Theorem II the system $\tilde{H}$ is a basis for the space $\tilde{H}(\tau_0)$ and Lemma 2.2(d) yields immediately that the system $\tilde{L}^{(\alpha)}$ is also a basis in that space. In other words, every function $f \in \tilde{H}(\tau_0)$ can be expanded in the region $S(\tau_0)$ into a series of the polynomials $\{L_n^{(\alpha)}(z^2)\}_{n=0}^{\infty}$.

Let us assume that a complex function f is represented in the region $S(\tau_0)$ by a series in the polynomials $\{L_n^{(\alpha)}(z^2)\}_{n=0}^{\infty}$ and put $F(z) = Q^{(\alpha)}(f; z)$. By Lemma 2.2(d), $\tilde{H}_n(z) = Q^{(\alpha)}(\tilde{L}_n^{(\alpha)}; z)$ $(n = 0, 1, 2, \ldots)$ and since the operator $Q^{(\alpha)}$ is continuous and the function F is even, one can assert that F is representable in the region $S(\tau_0)$ by a series in the polynomials $\{H_{2n}(z)\}_{n=0}^{\infty}$. According to Theorem II, the function F belongs to $\tilde{H}(\tau_0)$ and, therefore, $f(z) = P^{(\alpha)}(F; z) \in \tilde{H}(\tau_0)$.

3. Hankel's transform and the representation of analytic functions by series in Laguerre polynomials

The integral representation (1.5) shows that the complex function $\Gamma(n+1)z^{\alpha/2} \times \exp(-z)L_n^{(\alpha)}(z)$, which is analytic in the region $C-(-\infty, 0]$, is the image of the function $t^{n+\alpha/2}\exp(-t)$ under a transformation of Hankel's type. Therefore, one might suppose that if an analytic function $f(z)$ is represented by a series of type (1.4), the function $z^{\alpha/2}\exp(-z)f(z)$ must be also Hankel's transform of a suitable complex function. A result in this connection is announced in the short communications [9], [10]. Here we shall give in some details the proof of the corresponding statement basing on the properties of Laguerre functions of second kind $\{M_n^{(\alpha)}(z)\}_{n=0}^{\infty}$. The last system is defined in the region $C-[0, +\infty)$ by the equalities

$$M_n^{(\alpha)}(z) = -\int_0^{\infty} \frac{t^{\alpha}\exp(-t)L_n^{(\alpha)}(t)}{t-z}\,dt, \quad n = 0, 1, 2, \ldots \tag{3.1}$$

Using the Rodrigues formula for Laguerre polynomials, from (3.1) we get easily

$$M_n^{(\alpha)}(z) = -\int_0^{\infty} \frac{t^{n+\alpha}\exp(-t)}{(t-z)^{n+1}}\,dt, \quad n = 0, 1, 2, \ldots \tag{3.2}$$

If $\mathrm{Re}\,z < 0$ and $l(z)$ is the ray $\{\zeta = (-z)\cdot t, 0 \leqslant t < +\infty\}$, the Cauchy integral theorem gives

$$M_n^{(\alpha)}(z) = -\int_{l(z)} \frac{\zeta^{n+\alpha}\exp(-\zeta)}{(\zeta-z)^{n+1}}\,d\zeta = -(-z)^{\alpha}\int_0^{\infty} \frac{t^{n+\alpha}\exp(zt)}{(1+t)^{n+1}}\,dt.$$

The last integral representation leads to a generating function for the system (3.1), namely the function ($\mathrm{Re}\,z < 0$, $w \in C$)

$$M^{(\alpha)}(z, w) = \sum_{n=0}^{\infty} \frac{M_n^{(\alpha)}(z)}{n!}\,w^n = -(-z)^{\alpha}\int_0^{\infty} \frac{t^{\alpha}}{1+t}\exp\left\{\frac{wt}{1+t}+zt\right\}dt. \tag{3.3}$$

Let $G(\sigma)$ $(0 < \sigma \leqslant +\infty)$ be the class of all entire functions $g(w)$ having the property

$$\overline{\lim_{|w|\to+\infty}}\,(2\sqrt{|w|})^{-1}(\ln|g(w)|-|w|) \leqslant -\sigma. \tag{3.4}$$

The class $G(\sigma)$ can be described by means of the following

LEMMA. *An (entire) function*

$$g(w) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\,w^n \tag{3.5}$$

is in the class $G(\sigma)$ if and only if

$$\overline{\lim_{n\to+\infty}}\,(2\sqrt{n})^{-1}\ln|a_n| \leqslant -\sigma. \tag{3.6}$$

Proof. In the case $0 < \sigma < +\infty$ the necessity of condition (3.6) can be established directly by means of the Cauchy inequalities for the Taylor coefficients of an analytic function. Namely, for every $0 < \delta < \sigma$ there exist $D(\delta) > 0$ and $N(\delta) > 0$ such that if $n > N(\delta)$ then

$$|a_n| \leqslant n!\,n^{-n}\max_{|w|=n}|g(w)| \leqslant D(\delta)n!\,n^{-n}\exp[n-2(\sigma-\delta)\sqrt{n}]$$

and Stirling's formula yields $\overline{\lim\limits_{n\to+\infty}}\,(2\sqrt{n})^{-1}\ln|a_n| \leqslant -\sigma+\delta$.

In proving that (3.6) is sufficient for a function (3.5) to be in the class $G(\sigma)$ we shall use an asymptotic formula for the Laguerre functions of second kind:

$$M_n^{(\alpha)}(z) = -\sqrt{\pi}\exp(z/2)(-z)^{\alpha/2-1/4}n^{\alpha/2-1/4}\exp\{-2\sqrt{n}(-z)^{1/2}\}\{1+\mu_n^{(\alpha)}(z)\}. \tag{3.7}$$

Here $\{\mu_n^{(\alpha)}(z)\}_{n=1}^{\infty}$ are holomorphic functions in the region $C-[0, +\infty)$ and such that $\lim\limits_{n\to+\infty}\mu_n^{(\alpha)}(z) = 0$ uniformly on every compact subset of this region. The above formula can be derived from the relation

$$M_n^{(\alpha)}(z) = -\Gamma(n+\alpha+1)(-z)^{\alpha}\Psi(n+\alpha+1, \alpha+1; -z).$$

where $\Psi(a, c; z)$ is Tricomi's confluent hypergeometric function.

Having in view the asymptotic formula (3.7) we can assert that if the sequence $\{a_n\}_{n=0}^{\infty}$ satisfies (3.6), then for every $0 < \delta < \sigma$ there exists $C(\delta)$ such that $|a_n| \leqslant C(\delta)\{-M_n^{(0)}[-(\sigma-\delta)^2]\}$ $(n = 0, 1, 2, \ldots)$ and, therefore, $|g(w)| \leqslant -C(\delta)\times M^{(0)}[-(\sigma-\delta)^2, |w|]$. Further, from (3.3) it follows that

$$|g(w)| = O\Big\{\int\limits_0^\infty \exp[|w|t(1+t|^{-1}-(\sigma-\delta)^2t]dt\Big\}$$

$$= O\Big\{\int\limits_1^\infty \exp[|w|-(\sigma-\delta)^2t-|w|t^{-1}]dt\Big\}$$

and after some computation we get that if $|w|\to+\infty$ then

$$|g(w)| = O\{\sqrt{|w|}\exp[|w|-2(\sigma-\delta)\sqrt{|w|}]\}.$$

Since $0<\delta<\sigma$ is arbitrary, it follows that the entire function g belongs to $G(\sigma)$.

Now, by the aid of the Lemma just proved we establish the following

THEOREM V. *Let $0<\lambda_0\leqslant+\infty$ and $\alpha>-1$. A complex function f analytic in the region $\Delta(\lambda_0)$ can be represented in this region as a series of Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^\infty$ if and only if in the region $\tilde{\Delta}(\lambda_0)=\Delta(\lambda_0)-(-\lambda_0^2,0]$ the representation holds*:

$$f(z) = z^{-\alpha/2}\exp z\int\limits_0^\infty t^{\alpha/2}\exp(-t)g(t)J_\alpha(2\sqrt{zt})\,dt, \tag{3.8}$$

with a function $g\in G(\lambda_0)$.

Proof. If $g\in G(\lambda_0)$, then the integral in (3.8) is absolutely uniformly convergent on every compact subset of the region $\tilde{\Delta}(\lambda_0)$. We shall prove this in the case $\lambda_0<+\infty$. If $K\subset\tilde{\Delta}(\lambda_0)$ is compact, let $\lambda=\sup\limits_{z\in K}\mathrm{Re}(-z)^{1/2}$ and $\delta=(\lambda_0-\lambda)/2$. Then the asymptotic formula [2, v. II, 7.13, (3)] and inequality (3.4) yield that if $t\to+\infty$, then

$$t^{\alpha/2}\exp(-t)|g(t)J_\alpha(2\sqrt{zt})| = O\{t^{\alpha/2}\exp(-2\delta\sqrt{t})\} \tag{3.9}$$

uniformly on $z\in K$.

Further, if g is represented by the power series (3.5), the inequality (3.6) holds for $\sigma=\lambda_0$ and, therefore (Proposition 1.2), the series (1.4) is convergent in the region $\Delta(\lambda_0)$.

If we define for $\nu=0,1,2,\ldots$ and $z\in\tilde{\Delta}(\lambda_0)$

$$R_\nu(z) = f(z)-\sum_{n=0}^{\nu}a_nL_n^{(\alpha)}(z),$$

from (1.5) and (3.8) it follows that

$$R_\nu(z) = z^{-\alpha/2}\exp z\int\limits_0^\infty t^{\alpha/2}\exp(-t)\Big\{\sum_{n=\nu+1}^{\infty}\frac{a_nt^n}{n!}\Big\}J_\alpha(2\sqrt{zt})dt. \tag{3.10}$$

The function

$$g^*(w) = \sum_{n=0}^{\infty}\frac{|a_n|}{n!}w^n$$

is also in the class $G(\lambda_0)$, therefore, if we replace g by g^* in (3.9), we can assert that for every $\varepsilon>0$ and every $z\in\tilde{\Delta}(\lambda_0)$ there exists $T=T(\varepsilon)>0$ such that

$$\int\limits_T^\infty t^{\alpha/2}\exp(-t)g^*(t)|J_\alpha(2\sqrt{zt})|dt<\varepsilon.$$

Thus for every $\nu=0,1,2,\ldots$ we have

$$\Big|\int\limits_T^\infty t^{\alpha/2}\exp(-t)\Big\{\sum_{n=\nu+1}^{\infty}\frac{a_nt^n}{n!}\Big\}J_\alpha(2\sqrt{zt})\,dt\Big| \tag{3.11}$$

$$\leqslant\int\limits_T^\infty t^{\alpha/2}\exp(-t)g^*(t)\,|J_\alpha(2\sqrt{zt})|dt<\varepsilon.$$

There exists $N=N(\varepsilon)>0$ with the property that if $\nu>N$ and $0\leqslant t\leqslant T$, then $\Big|\sum\limits_{n=\nu+1}^{\infty}(n!)^{-1}a_nt^n\Big|<\varepsilon$ and, therefore,

$$\Big|\int\limits_0^T t^{\alpha/2}\exp(-T)\Big\{\sum_{n=\nu+1}^{\infty}\frac{a_nt^n}{n!}\Big\}J_\alpha(2\sqrt{zt})\,dt\Big| \tag{3.12}$$

$$= O\Big\{\varepsilon\int\limits_0^\infty t^{\alpha/2}\exp(-t)|J_\alpha(2\sqrt{zt})|dt\Big\} = O(\varepsilon).$$

From (3.10), (3.11) and (3.12) it follows that $R_\nu(z)=O(\varepsilon)$, $\nu>N$, i.e. the series (1.4) represents the function f in the region $\tilde{\Delta}(\lambda_0)$ and, therefore, in the region $\Delta(\lambda_0)$ as well.

Now suppose that a complex function f is represented in the region $\Delta(\lambda_0)$ by the series (1.4). Then it follows from Proposition 1.2 and the Lemma that the (entire) function (3.5) is in the class $G(\lambda_0)$. By means of the integral transformation (3.8) this function defines a complex function $\tilde{f}$, analytic in the region $\tilde{\Delta}(\lambda_0)$. But as we have just seen, the function $\tilde{f}$ is represented in this region by the series (1.4) and, hence, $f=\tilde{f}$.

As a consequence of Theorem V and the relations between Hermite and Laguerre polynomials [12, (5.6.1)], one can easily get a statement which gives a necessary and sufficient condition for an analytic function to be represented by a series in Hermite polynomials, namely

THEOREM VI. *A complex function f holomorphic in the region $S(\tau_0)$ ($0<\tau_0\leqslant+\infty$) can be expanded there into a series of Hermite polynomials if and only if in $S(\tau_0)$ the integral representation holds*:

$$f(z) = \exp z^2\int\limits_0^\infty \exp(-t^2)\{g(t^2)\cos(2zt)+h(t^2)\sin(2zt)\}dt$$

where $g,h\in G(\tau_0)$.

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ON THE EQUIVALENCE BETWEEN LOCALLY POLAR AND GLOBALLY POLAR SETS IN C^n

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B. Josefson [3] has recently proved that every locally C^n-polar set is a globally C^n-polar set. Using the method of the proof developed by Josefson we prove that every locally C^n-polar set E is an L-polar set (i.e. there exists a function W plurisubharmonic in C^n such that $W = -\infty$ on E and $W(x) \leqslant \beta+\log^+|x|$ for all $x \in C^n$, where β is a real constant).

1. Introduction

Given an open subset D of C^n we denote by PSH(D) the family of all functions plurisubharmonic in D. We denote by L the class of all functions U plurisubharmonic in C^n such that

$$U(x) \leqslant \beta+\log^+|x|, \quad x \in C^n,$$

where β is a real constant depending on U and $|x| := \max\limits_{1\leqslant i\leqslant n} |x_i|$.

The aim of this paper is to prove the following

THEOREM. *Given any subset E of C^n the following conditions are equivalent*:

(a) *E is locally C^n-polar, i.e. for every point $a \in E$ there exist a neighbourhood U_a of a and a function $W \in \mathrm{PSH}(U_a)$ such that $W = -\infty$ on $E\cap U_a$*;

(b) *E is L-polar, i.e. there exists a function W of the class L such that $W = -\infty$ on E*;

(c) *E is globally C^n-polar, i.e. there exists $W \in \mathrm{PSH}(C^n)$ such that $W = -\infty$ on E.*

The implication (a) $\Rightarrow$ (c) was a question posed by P. Lelong [4] which has been recently solved by B. Josefson [3]. The main tool of the proof given by Josefson is an "elementary" Lemma on systems of homogeneous linear equations. The same lemma will be basic for the proof of our theorem.

The equivalence of locally polar and globally polar sets in R^n with respect to