K becomes

$$K' = \frac{(j_1+1)(j_2-1)}{j_1 j_2} K = \frac{j_1 j_2 + (j_2 - j_1) - 1}{j_1 j_2} K > K.$$

This proves our statement.

In our case we have (see 5) and 3)):

$$m + \frac{n-1}{2} \leqslant j = n + r \leqslant m + n - 1$$

and so we find for the number e of vertices of $\Pi^{(r)}$ the estimates in 6).

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CLASSICAL EXTREMAL PROBLEMS FOR UNIVALENT FUNCTIONS

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1

Denote by S the class of functions

$$(S) \qquad f(z) = c_0 + c_1 z + c_2 z^2 + \ldots,$$

regular and univalent in the unit disc D: $|z| < 1$.

Let $L(z_1, z_2)$ be the curve $z = z(s)$, $0 \leqslant s \leqslant \bar{s}$, $z_1 = z(0)$, $z_2 = z(\bar{s})$, $|z_1| < |z_2|$, for which $z'(s)$ and $r'(s) = |z(s)|'$ exist and are continuous except for a finite number of values of s. The parameter s denotes the length of the arc.

By $\mathscr{L}(z_1, z_2, f)$ denote the image of $L(z_1, z_2)$ by means of $f(z) \in S$. $\bar{L}(z_1, z_2)$ and $\bar{\mathscr{L}}(z_1, z_2, f)$ denote the lengths of $L(z_1, z_2)$ and $\mathscr{L}(z_1, z_2, f)$, respectively.

THEOREM I. *If $f(z) \in S$ and $|z_1| < |z_2| < 1$, then*

$$(1) \qquad \frac{1-|z_1|\,|z_2|}{(1+|z_1|)^2(1+|z_2|)^2} \leqslant \frac{\bar{\mathscr{L}}(z_1, z_2, f)}{\bar{L}(z_1, z_2)} \leqslant \frac{1-|z_1|\,|z_2|}{(1-|z_1|)^2(1-|z_2|)^2},$$

where the upper estimate holds true if $r'(s) \geqslant 0$.

For $|z| \leqslant r < 1$, one obtains

THEOREM I*. *If $f(z) \in S$ and $|z_1| < |z_2| \leqslant r < 1$, then*

$$(1^*) \qquad \frac{1-r}{(1+r)^3} \leqslant \frac{\bar{\mathscr{L}}(z_1, z_2, f)}{\bar{L}(z_1, z_2)} \leqslant \frac{1+r}{(1-r)^3},$$

where the upper estimate holds true if $r'(s) \geqslant 0$.

As a corollary we get:

THEOREM $\bar{\text{I}}$. *If $f(z) \in S$ and $|z_1| < |z_2| \leqslant r < 1$, then*

$$(2) \qquad \frac{1-|z_1|\,|z_2|}{(1+|z_1|)^2(1+|z_2|)^2} \leqslant \left|\frac{f(z_1)-f(z_2)}{z_1-z_2}\right| \leqslant \frac{1-|z_1|\,|z_2|}{(1-|z_1|)^2(1-|z_2|)^2},$$

where the left inequality holds if the segment joining the points $f(z_1)$ and $f(z_2)$ lies entirely in the image $f(D)$ of the unit disc by means of $f(z)$, while the right inequality holds if, on the segment joining z_1 with z_2, $|z|$ only increases or only decreases.

Under the same conditions the following inequalities hold:

$$\frac{1-r}{(1+r)^3} \leqslant \left| \frac{f(z_1)-f(z_2)}{z_1-z_2} \right| \leqslant \frac{1+r}{(1-r)^3}. \tag{2*}$$

These theorems comprise (generalize) the classical Koebe theorem [1]:

THEOREM K. *If $f(z) \in S$ and $|z| \leqslant r < 1$, then*

$$\frac{1-r}{(1+r)^3} \leqslant |f'(z)| \leqslant \frac{1+r}{(1-r)^3}. \tag{3*}$$

The bounds in (3*) *are reached by the functions $f(z) = z/(1-z)^2$.*

The inequalities (2), for $z_1 = 0$, $z_2 = z$, yield the Bieberbach theorem:

THEOREM B. *If $f(z) \in S$ and $|z| \leqslant r < 1$, then*

$$\frac{|z|}{(1+|z|)^2} \leqslant |f(z)| \leqslant \frac{|z|}{(1-|z|)^2} \tag{4}$$

and

$$\frac{1}{(1+r)^2} \leqslant \left| \frac{f(z)}{z} \right| \leqslant \frac{1}{(1-r)^2}. \tag{4*}$$

The bounds in (4) *and* (4*) *are reached by the function $f(z) = z/(1-z)^2$.*

In the proof of Theorems I and I* the Koebe theorem and the integral method of Bieberbach [2] are used.

Proof of Theorems I *and* I*. Let $L(z_1, z_2)$ be a curve $z = z(s)$, $0 \leqslant s \leqslant \bar{s}$, in the unit disc D, $z(0) = z_1$, $z(\bar{s}) = z_2$, joining the points z_1 and z_2, $|z_1| < |z_2|$. It can be assumed that $z'(s)$ and $|z(s)|'$, $0 \leqslant s \leqslant \bar{s}$, exist and are continuous except for a finite number of values of s. Here s is the length of the arc along the curve.

Let $\mathscr{L}(z_1, z_2, f)$ be the image of $L(z_1, z_2)$ by means of the function $f(z) \in S$. The lengths of these curves are denoted by $\bar{L}(z_1, z_2)$ and $\bar{\mathscr{L}}(z_1, z_2, f)$, respectively.

Since $L(z_1, z_2)$ is rectifiable, there exists a positive integer $p \geqslant 1$, such that $(p-1)(|z_2|-|z_1|) < \bar{L}(z_1, z_2) \leqslant p(|z_2|-|z_1|)$.

Then

$$\bar{\mathscr{L}}(z_1, z_2, f) = \int_0^{\bar{s}} |f'(z)|\,|z'(s)|\,ds = \int_0^{\bar{s}} |f'(z)|\,|dz|. \tag{5}$$

(A) Let $\varphi(z) = \varphi(|z|) = (1+|z|)/(1-|z|)^3$. In view of the Koebe theorem

$$\int_0^{\bar{s}} |f'(z)|\,|dz| \leqslant \int_0^{\bar{s}} \varphi(|z|)\,|dz|.$$

Let $\zeta_1 = z_1, \zeta_2, \ldots, \zeta_{pn}, z_2$ be $pn+1$ points on $L(z_1, z_2)$ dividing its arc into pn equal parts. Then

$$\int_0^{\bar{s}} \varphi(|z|)\,|dz| = \lim_{n\to\infty} \sum_{k=1}^{pn} \varphi(|\zeta_k|)\,|\zeta_{k+1}-\zeta_k| = \lim_{n\to\infty} \sum_{k=1}^{pn} \varphi(|\zeta_k|)\,\frac{\bar{L}(z_1, z_2)}{pn}.$$

Assume that $r'(s) = |z(s)|' \geqslant 0$. Then the numbers $|\zeta_1| = |z_1| \leqslant |\zeta_2| \leqslant \ldots \leqslant |\zeta_{pn}| \leqslant |\zeta_{pn+1}| = |z_2|$ and they are in the interval $[|z_1|, |z_2|]$ of length $|z_2|-|z_1| > 0$.

Divide the interval $[|z_1|, |z_2|]$ into pn parts, all equal to $(|z_2|-|z_1|)/pn$. Since $|\zeta_{k+1}-\zeta_k| = \bar{L}(z_1, z_2)/pn$ is p times greater than $(|z_2|-|z_1|)/pn = \alpha_n^p$ at most, then in every interval consisting of p consecutive intervals of length α_n^p there is at least one of the numbers $|\zeta_k|$. Now divide the interval $[|z_1|, |z_2|]$ into n equal intervals. Each of them represents a group of p intervals of length $(|z_2|-|z_1|)/pn$ each. All these n groups will be given consecutive numbers. Among the numbers $|\zeta_\nu|$ lying in the νth of the groups consisting of p consecutive intervals of length $(|z_2|-|z_1|)/pn$ each, by $|\zeta_\nu^*|$ denote the number for which $\varphi(|\zeta_k|)$ is the greatest.

Then,

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{pn} \varphi(|\zeta_k|)\,\frac{\bar{L}(z_1, z_2)}{pn} &= \frac{\bar{L}(z_1, z_2)}{|z_2|-|z_1|} \lim_{n\to\infty} \sum_{k=1}^{pn} \varphi(|\zeta_k|)\,\frac{|z_2|-|z_1|}{pn} \\
&\leqslant \frac{\bar{L}(z_1, z_2)}{|z_2|-|z_1|} \lim_{n\to\infty} \sum_{\nu=1}^{n} \varphi(|\zeta_\nu^*|)\,\frac{|z_2|-|z_1|}{n} \\
&= \frac{\bar{L}(z_1, z_2)}{|z_2|-|z_1|} \int_0^{\bar{s}} \varphi(|z|)\,d|z|,
\end{aligned}$$

i.e.

$$\bar{\mathscr{L}}(z_1, z_2, f) \leqslant \frac{\bar{L}(z_1, z_2)}{|z_2|-|z_1|} \int_{|z_1|}^{|z_2|} \frac{(1+|z|)\,d|z|}{(1-|z|)^3} = \bar{L}(z_1, z_2)\,\frac{1-|z_1|\,|z_2|}{(1-|z_1|)^2(1-|z_2|)^2}.$$

Therefore

$$\frac{\bar{\mathscr{L}}(z_1, z_2, f)}{\bar{L}(z_1, z_2)} \leqslant \frac{1-|z_1|\,|z_2|}{(1-|z_1|)^2(1-|z_2|)^2} \leqslant \frac{1+r}{(1-r)^3},$$

under the condition that $r'(s) \geqslant 0$ in the interval from $|z_1|$ to $|z_2|$ and $|z_2| \leqslant r < 1$.

(B) Let $\psi(z) = \psi(|z|) = (1-|z|)/(1+|z|)^3$. In view of the Koebe theorem

$$\int_0^{\bar{s}} |f'(z)|\,|dz| \geqslant \int_0^{\bar{s}} \psi(|z|)\,|dz|.$$

First assume that $r'(s) \geqslant 0$ in the interval $|z_1| \leqslant s \leqslant |z|$. Repeating the considerations and notations from item (A), among the numbers $|\zeta_k|$, lying in the νth of the groups consisting of p consecutive intervals of length $(|z_2|-|z_1|)/pn$ each, by $|\zeta^*|$ denote the number for which $\psi(|z_k|)$ is the smallest. Then by analogy with the mentioned above

$$\bar{\mathscr{L}}(z_1, z_2, f) \geqslant \frac{\bar{L}(z_1, z_2)}{|z_2|-|z_1|} \int_{|z_1|}^{|z_2|} \frac{(1-|z|)\,d|z|}{(1+|z|)^3} = \bar{L}(z_1, z_2)\,\frac{1-|z_1|\,|z_2|}{(1+|z_1|)^2(1+|z_2|)^2},$$

i.e.

$$\frac{\mathscr{L}(z_1, z_2, f)}{\bar{L}(z_1, z_2)} \geqslant \frac{1-|z_1||z_2|}{(1+|z_1|)^2(1+|z_2|)^2} \geqslant \frac{1-r}{(1+r)^3},$$

where $|z_2| \leqslant r < 1$.

In this case, the condition $r'(s) \geqslant 0$ can be easily discharged. Namely, $L = L(z_1, z_2)$ can be represented in the form $L(z_1, z_2) = L_1(z_1, z_2) + L_2(z)$, where $L_1 = L_1(z_1, z_2)$ is a curve $z = z(s)$ in D joining z_1 with z_2 and for which $r'(s) \geqslant 0$, while $L_2 = L_2(z)$, is a sum of linear sets.

Then,

$$(L)\int_{|z_1|}^{|z_2|} \psi(|z|)d|z| = (L_1)\int_{|z_1|}^{|z_2|} \psi(|z|)d|z| + (L_2)\int \psi(|z|)d|z| \geqslant (L_1)\int_{|z_1|}^{|z_2|} \psi(|z|)d|z|.$$

Thus, Theorems I and I* are proved.

Note that in view of (5) the inequalities (1) might be written in the form:

$$(1') \qquad \frac{1-|z_1||z_2|}{(1+|z_1|)^2(1+|z_2|)^2} \leqslant \frac{1}{\bar{L}(z_1, z_2)} \int_{L(z_1, z_2)} |f'(z)||dz| \leqslant \frac{1-|z_1||z_2|}{(1-|z_1|)^2(1-|z_2|)^2}.$$

If we choose the curve $L(z_1, z_2)$ in such a way that its image $\mathscr{L}(z_1, z_2, f)$, through $f(z) \in S$, be the segment joining $f(z_1)$ with $f(z_2)$, then

$$\frac{\mathscr{L}(z_1, z_2, f)}{\bar{L}(z_1, z_2)} = \frac{|f(z_1)-f(z_2)|}{\bar{L}(z_1, z_2)} \leqslant \left|\frac{f(z_1)-f(z_2)}{z_1-z_2}\right|$$

and thus the left-hand side of (2) is established.

If the curve $L(z_1, z_2)$ is the segment joining z_1 with z_2, then under the conditions of Theorem $\bar{\mathrm{I}}$:

$$\frac{\mathscr{L}(z_1, z_2, f)}{\bar{L}(z_1, z_2)} = \frac{\mathscr{L}(z_1, z_2, f)}{|z_1-z_2|} \geqslant \left|\frac{f(z_1)-f(z_2)}{z_1-z_2}\right|.$$

In this way Theorem I is proved.

Let $f(z) \in S$ map the unit disc D convexly. In this case (see [2], p. 83), if $|z| \leqslant r < 1$:

$$(3^*) \qquad \frac{1}{(1+|z|)^2} \leqslant |f'(z)| \leqslant \frac{1}{(1-|z|)^2}.$$

Using (5) and the approach stated in (A) and (B) we obtain:

THEOREM I_1. *If $f(z) \in S$ is a convex function and $|z_1| < |z_2| < 1$ then:*

$$(6) \qquad \frac{1}{(1+|z_1|)(1+|z_2|)} \leqslant \frac{\mathscr{L}(z_1, z_2, f)}{\bar{L}(z_1, z_2)} \leqslant \frac{1}{(1-|z_1|)(1-|z_2|)},$$

$$(7) \qquad \frac{1}{(1+|z_1|)(1+|z_2|)} \leqslant \left|\frac{f(z_1)-f(z_2)}{z_1-z_2}\right| \leqslant \frac{1}{(1-|z_1|)(1-|z_2|)},$$

where the upper estimate in (6) *holds true if $r'(s) \geqslant 0$, while in* (7) *it is true provided $|z|$ only increases or only decreases on the segment joining $|z_1|$ with $|z_2|$.*

Let the function

$$(8) \qquad f_k(z) = z + c_1^{(k)} z^{k+1} + c_2^{(k)} z^{2k+1} + \ldots, \qquad k = 1, 2, \ldots$$

be k-symmetric and univalent in the disc $|z| < 1$.

Analogously, from (5), (A) and (B) we obtain the theorems:

THEOREM I_2. *If $f_k(z) \in S_k$, then under the conditions of Theorem* I, *we have*:

$$(9) \qquad \frac{1}{|z_2|-|z_1|} \int_{|z_1|}^{|z_2|} \left(\frac{1-r^k}{1+r^k}\right)^3 \frac{dr}{(1-r^k)^{2/k}} \leqslant \frac{\mathscr{L}(z_1, z_2, f_k)}{\bar{L}(z_1, z_2)} \leqslant \frac{1}{|z_2|-|z_1|} \int_{|z_1|}^{|z_2|} \left(\frac{1+r^k}{1-r^k}\right)^3 \frac{dr}{(1+r^k)^{2/k}}$$

and for the conditions of Theorem $\bar{\mathrm{I}}$:

$$(10) \qquad \frac{1}{|z_2|-|z_1|} \int_{|z_1|}^{|z_2|} \left(\frac{1-r^k}{1+r^k}\right)^3 \frac{dr}{(1-r^k)^{2/k}} \leqslant \left|\frac{f_k(z_1)-f_k(z_2)}{z_1-z_2}\right| \leqslant \frac{1}{|z_2|-|z_1|} \int_{|z_1|}^{|z_2|} \left(\frac{1+r^k}{1-r^k}\right)^3 \frac{dr}{(1+r^k)^{2/k}}.$$

THEOREM I_3. *If $f_k(z) \in S_k$ is a convex function, then under the conditions of Theorem* I_1 *we have*:

$$(11) \qquad \frac{1}{|z_2|-|z_1|} \int_{|z_1|}^{|z_2|} \frac{dr}{(1+r^k)^{2/k}} \leqslant \frac{\mathscr{L}(z_1, z_2, f_k)}{\bar{L}(z_1, z_2)} \leqslant \frac{1}{|z_2|-|z_1|} \int_{|z_1|}^{|z_2|} \frac{dr}{(1-r^k)^{2/k}}$$

and

$$(12) \qquad \frac{1}{|z_2|-|z_1|} \int_{|z_1|}^{|z_2|} \frac{dr}{(1+r^k)^{2/k}} \leqslant \left|\frac{f_k(z_1)-f_k(z_2)}{z_1-z_2}\right| \leqslant \frac{1}{|z_2|-|z_1|} \int_{|z_1|}^{|z_2|} \frac{dr}{(1-r^k)^{2/k}}.$$

2

By S denote the class of functions

$$(S) \qquad f(z) = z + c_2 z^2 + c_3 z^3 + \ldots,$$

regular and univalent in the unit disc $|z| < 1$.

By S_k, $k = 1, 2, \ldots$ ($S_1 \equiv S$) denote the subset of S of functions

$$(S_k) \qquad f_k(z) = z + c_{k+1}z^{k+1} + c_{2k+1}z^{2k+1} + \ldots = z + c_1^{(k)}z^{k+1} + c_2^{(k)}z^{2k+1} + \ldots$$

k-symmetric, univalent and regular in the disc $|z| < 1$.

Note that if $f_1(z) \in S_1$, then $f_k(z) = \sqrt[k]{f_1(z^k)} \in S_k$.

By Σ denote the class of functions

$$(\Sigma) \qquad F(\zeta) = \zeta + \gamma_0 + \gamma_1/\zeta + \ldots$$

univalent and holomorphic in the domain $|\zeta| > 1$, except for the simple pole at the improper point.

If $f(z) \in S$, then $F(\zeta) = 1/f(1/\zeta) \in \Sigma$.

The classical theorem of Koebe [1] states:

THEOREM K (Verzerrungssatz). *If $f(z) \in S$ and $|z| \leqslant r < 1$, then*

$$(1) \qquad \frac{1-r}{(1+r)^3} \leqslant |f'(z)| \leqslant \frac{1+r}{(1-r)^3}.$$

The bounds in (1) *are reached by the function $f(z) = z/(1-z)^2$.*

From this theorem Bieberbach [2] obtained:

THEOREM B. *If $f(z) \in S$ and $|z| \leqslant r < 1$, then*

$$(2) \qquad \frac{|z|}{(1+|z|)^2} \leqslant |f(z)| \leqslant \frac{|z|}{(1-|z|)^2};$$

if $f_k(z) \in S_k$, $k = 1, 2, \ldots$, $|z| \leqslant r < 1$, then

$$(3) \qquad \frac{|z|}{(1+|z|^k)^{2/k}} \leqslant |f_k(z)| \leqslant \frac{|z|}{(1-|z|^k)^{2/k}}.$$

The bounds in (2) *and* (3) *are reached by the functions $f(z) = z/(1-z^k)^{2/k}$.*

From (1) *and* (3) *follows: If $f_k(z) \in S_k$, $k = 1, 2, \ldots$, $|z| \leqslant r < 1$, then*

$$(1') \qquad \left(\frac{1-r^k}{1+r^k}\right)^3 \frac{1}{(1-r^k)^{2/k}} \leqslant |f_k'(z)| \leqslant \left(\frac{1+r^k}{1-r^k}\right)^3 \frac{1}{(1+r^k)^{2/k}}.$$

The bounds in (1') *are reached by the functions $f(z) = z/(1-z^k)^{2/k}$.*

G. M. Goluzin [3] found:

THEOREM G. *If $\zeta_1, \zeta_2, \ldots, \zeta_n$ ($n \geqslant 1$) belong to the domain $|\zeta| > 1$ and $F(\zeta) \in \Sigma$, then for $|\zeta| > 1$*

$$(4) \qquad \left|\prod_{\nu,\nu'=1}^{n}\left(1-\frac{1}{\zeta_\nu\bar{\zeta}_{\nu'}}\right)\right| \leqslant \left|\prod_{\nu,\nu'=1}^{n}\frac{F(\zeta_\nu)-F(\zeta_{\nu'})}{\zeta_\nu-\zeta_{\nu'}}\right| \leqslant 1\Big/\left|\prod_{\nu,\nu'=1}^{n}\left(1-\frac{1}{\zeta_\nu\bar{\zeta}_{\nu'}}\right)\right|,$$

where in the product in the middle, in the case of $\nu = \nu'$, the corresponding factor should be interpreted as $F'(\zeta_\nu)$.

The estimates (4) *are exact for $\zeta_\nu = \zeta e^{2\pi\nu i/n}$, $\nu = 1, 2, \ldots, n$, $|\zeta| > 1$.*

The left estimate is exact always when $n = 2$.

1. In [4] Iliev, using Theorem G, established:

THEOREM I. *If $f_k(z) \in S_k$ and $|z| \leqslant r < 1$, $|z_1| \leqslant r$, $|z_2| \leqslant r$, $z_1 \neq z_2$, then*

$$(1.1) \qquad \frac{1-r^2}{(1+r^k)^{4/k}} \leqslant \left|\frac{f_k(z_1)-f_k(z_2)}{z_1-z_2}\right| \leqslant \frac{1}{(1-r^2)(1-r^k)^{4/k}}.$$

The left inequality is exact for $k = 1$ and $k = 2$.

The above theorem generalizes the Koebe theorem.

Proof. As Goluzin [3] remarked, if $F(\zeta) \in \Sigma$ and $|\zeta_1| > 1$, then

$$(1.2) \qquad G(\zeta) = \frac{(1-|\zeta_1|^2)F'(\zeta_1)}{F\left(\dfrac{1+\zeta_1\zeta}{\zeta_1+\zeta}\right)-F(\zeta_1)} = \zeta+\beta_0+\frac{\beta_1}{\zeta}+\ldots$$

belongs to the class Σ, so that (4), $n = 1$, in $|\zeta| > 1$ yields

$$(1.3) \qquad 1-1/|\zeta|^2 \leqslant |G'(\zeta)| \leqslant \frac{1}{1-1/|\zeta|^2}.$$

Following Goluzin's considerations, after setting $\zeta = \dfrac{1-\bar{\zeta}_1\zeta_2}{\zeta_2-\zeta_1}$, $|\zeta_2| > 1$, (1.3) yields

$$(1.4) \qquad \frac{(1-|\zeta_1|^2)(1-|\zeta_2|^2)}{|1-\zeta_1\bar{\zeta}_2|^2} \leqslant \left|F'(\zeta_1)F'(\zeta_2)\left(\frac{\zeta_2-\zeta_1}{F(\zeta_2)-F(\zeta_1)}\right)^2\right| \leqslant \frac{|1-\zeta_1\bar{\zeta}_2|^2}{(1-|\zeta_1|^2)(1-|\zeta_2|^2)}.$$

Besides, from (4) for $n = 2$, we obtain

$$(1.5) \qquad \left(1-\frac{1}{|\zeta_1|^2}\right)\left(1-\frac{1}{|\zeta_2|^2}\right)\left|1-\frac{1}{\zeta_1\bar{\zeta}_2}\right|^2 \leqslant \left|F'(\zeta_1)F'(\zeta_2)\left(\frac{F(\zeta_1)-F(\zeta_2)}{\zeta_1-\zeta_2}\right)^2\right| \leqslant \frac{1}{\left(1-\dfrac{1}{|\zeta_1|^2}\right)\left(1-\dfrac{1}{|\zeta_2|^2}\right)\left|1-\dfrac{1}{\zeta_1\bar{\zeta}_2}\right|^2}.$$

From (1.4) and (1.5) we find

$$(1.6) \qquad \left(1-\frac{1}{|\zeta_1|^2}\right)^{1/2}\left(1-\frac{1}{|\zeta_2|^2}\right)^{1/2} \leqslant \left|\frac{F(\zeta_1)-F(\zeta_2)}{\zeta_1-\zeta_2}\right| \leqslant \frac{1}{\left(1-\dfrac{1}{|\zeta_1|^2}\right)^{1/2}\left(1-\dfrac{1}{|\zeta_2|^2}\right)^{1/2}}.$$

Let $f(z) \in S$. Then $f(\zeta) = 1/f(1/\zeta) \in \Sigma$. If $|z_1| < 1$, $|z_2| < 1$ and $\zeta_1 = 1/z_1$, $\zeta_2 = 1/z_2$, (1.6) yields

$$(1.7) \qquad \left|\frac{f(z_1)f(z_2)}{z_1z_2}\right|(1-|z_1|^2)^{1/2}(1-|z_2|)^{1/2} \leqslant \left|\frac{f(z_1)-f(z_2)}{z_1-z_2}\right| \leqslant \left|\frac{f(z_1)f(z_2)}{z_1z_2}\right|\frac{1}{(1-|z_1|^2)^{1/2}(1-|z_2|^2)^{1/2}}.$$

If $f(z) = f_k(z) \in S_k$ from (1.7) and (3), for $|z_1| = r_1$, $|z_2| = r_2$, follow the inequalities

$$(1.8)\qquad \frac{(1-r_1^2)^{1/2}(1-r_2^2)^{1/2}}{(1+r_1^k)^{2/k}(1+r_2^k)^{2/k}} \leqslant \left|\frac{f_k(z_1)-f_k(z_2)}{z_1-z_2}\right| \leqslant \frac{1}{(1-r_1^k)^{2/k}(1-r_2^k)^{2/k}(1-r_1^2)^{1/2}(1-r_2^2)^{1/2}}$$

whence, for $r_1 \leqslant r$, $r_2 \leqslant r$, follows (1.1).

2. In [5] G. Szegö found a method with the help of which one might study the partial sums of univalent functions. He stated the following

THEOREM S I. *If the function*

$$(2.1)\qquad f(z) = z+c_2z^2+\ldots$$

belongs to the class S, then its partial sums

$$(2.2)\qquad \sigma_n(z) = z+c_2z^2+\ldots+c_nz^n, \quad n = 1, 2, \ldots$$

are univalent in the disc $|z| < 1/4$. The constant $1/4$ cannot be substituted by a greater one.

In the proof Szegö applies his following theorem:

THEOREM S II. *If $f(z) \in S$, $|z_1| < 1$, $|z_2| < 1$, $z_1 \neq z_2$, then*

$$(2.3)\qquad \left(\frac{1-|z_2|}{1+|z_2|}\right)^2 \frac{|1-\bar{z}_2z_1|}{(|z_1-z_2|+|1-\bar{z}_2z_1|)^2} \leqslant \left|\frac{f(z_1)-f(z_2)}{z_2-z_1}\right| \leqslant \left(\frac{1+|z_2|}{1-|z_2|}\right)^2 \frac{|1-\bar{z}_2z_1|}{(|z_1-z_2|-|1-\bar{z}_2z_1|)^2}.$$

The inequalities (2.3) are not exact. Applying Theorem I Iliev proves, using the Szegö method, the theorems that follow in this point and simplifies the proof of Theorem S II, given by Szegö.

THEOREM 1. *Let the function*

$$(2.4)\qquad f_2(z) = z+c_3z^3+\ldots$$

belong to the class S_2. Then its partial sums

$$(2.5)\qquad \sigma_n^{(2)}(z) = z+c_3z^3+\ldots+c_{2n+1}z^{2n+1}, \quad n = 1, 2, \ldots$$

are univalent in the disc $|z| < 1/\sqrt{3}$. The constant $1/\sqrt{3}$ cannot be substituted by a greater one.

S. Takahachi [6] established the above theorem in the following particular cases: 1. if the domain, where the variable $\omega = f_2(z)$, $|z| < 1$ is defined, is convex and 2. if this domain is starlike with respect to $\omega = 0$. K. Joh [7] gave one proof for the general case which turned out to be wrong (see [8], p. 1154 and [9], pp. 980–981). However, his direct proof for $n = 1$ and $n = 2$, i.e. for $\sigma_3^{(2)}(z)$ and $\sigma_5^{(2)}(z)$ is true, so that the theorem can be considered as established in these cases. He later proved [10] the theorem for the case when all the coefficients $c_{2\nu+1}$ in (2.4) are real. In one of his next papers [11], he achieved new results in the general case without being able to prove completely the theorem. In [4] and [12] the theorem was established completely and was proved for every integer $n \geqslant 3$.

In [13] and [14] the following theorem is established:

THEOREM 2. *Let the function*

$$(2.6)\qquad f_3(z) = z+c_1^{(3)}z^4+c_2^{(3)}z^7+\ldots$$

belong to the class S_3. Then its partial sums

$$(2.7)\qquad \sigma_n^{(3)}(z) = z+c_1^{(3)}z^4+\ldots+c_n^{(3)}z^{3n+1}, \quad n = 1, 2, \ldots$$

are univalent in the disc $|z| < \sqrt[3]{3}/2$. The constant $\sqrt[3]{3}/2$ cannot be replaced by a greater one.

Proof of Theorem 1. Let $f_2(z) \in S_2$. For $r = 1/\sqrt{3}$, if $|z_1| < 1/\sqrt{3}$, $|z_2| < 1/\sqrt{3}$, $z_1 \neq z_2$ (1.1) yields ([12]; [4]):

$$(2.8)\qquad \left|\frac{f_2(z_1)-f_2(z_2)}{z_1-z_2}\right| \geqslant 3/8.$$

Therefore (see [5]) the partial sum $\sigma_n^{(2)}(z)$ is a univalent function in $|z| < 1/\sqrt{3}$, if

$$(2.9)\qquad \left|\sum_{\nu=n+1}^{\infty} c_{2\nu+1}\frac{z_1^{2\nu+1}-z_2^{2\nu+1}}{z_1-z_2}\right| < 3/8.$$

The above inequality is fulfilled if

$$(2.10)\qquad \sum_{\nu=n+1}^{\infty} |c_{2\nu+1}|(2\nu+1)r^{2\nu} < 3/8,$$

where $r = 1/\sqrt{3}$. According to V. Levin [15], $|c_{2\nu+1}| < 3.4$ for every ν and, especially $|c_9| < 1.4$, $|c_{11}| < 1.7$, so that (2.10) is fulfilled for $n \geqslant 3$, if

$$(2.11)\qquad 9\cdot 1.4\cdot r^8+11\cdot 1.7\cdot r^{10}+3.4\sum_{\nu=6}^{\infty}(2\nu+1)r^{2\nu} < 3/8,$$

i.e., if

$$(2.12)\qquad 9\cdot 1.4\cdot r^8+11\cdot 1.7\cdot r^{10}+3.4\frac{13(1-r^2)+2r^2}{(1-r^2)^2}r^{12} < 3/8,$$

or, if

$$9\cdot 1.4/3^4+11\cdot 1.7/3^5+3.4\cdot 7/3^5 < 3/8.$$

Since the above inequality is true, in view of the remarks after the formulation of the theorem, its first part is already proved. The fact that the constant $1/\sqrt{3}$ is the best one is known from the papers mentioned.

Proof of Theorem 2. Let $f_3(z) \in S_3$. Using the method of V. Levin [15], we obtain the inequality

$$|c_n^{(3)}|^2 \leqslant \sum_{\nu=1}^{n} \frac{|c_{n-\nu}^{(3)}|^2}{3\nu-1}, \tag{2.13}$$

where $c_0^{(3)} = 1$.

As is well known [16]:

$$|c_2^{(k)}| \leqslant \frac{2}{k} e^{-2\frac{k-1}{k+1}} + \frac{1}{k}, \tag{2.14}$$

so that

$$|c_2^{(3)}| \leqslant \frac{2}{3e} + \frac{1}{3} < 0.579. \tag{2.15}$$

Since $|c_1^{(3)}| \leqslant 2/3$, (2.13) and (2.15) yield

$$\begin{aligned} &|c_2^{(3)}| < 0.579, \quad |c_3^{(3)}| < 0.618, \quad |c_4^{(3)}| < 0.636,\\ &|c_5^{(3)}| < 0.658, \quad |c_6^{(3)}| < 0.683, \quad |c_7^{(3)}| < 0.711,\\ &|c_8^{(3)}| < 0.741, \quad |c_9^{(3)}| < 0.774. \end{aligned} \tag{2.16}$$

In view of the inequality of Buniakovski for $0 \leqslant r < 1$, we obtain

$$\begin{aligned} \sum_{\nu=n+1}^{\infty} \frac{r^{3\nu}}{(3\nu+1)^{1/3}} &= \sum_{\nu=n+1}^{\infty} (3\nu+1)^{1/2} r^{3\nu} \frac{1}{(3\nu+1)^{1/2+1/3}} \\ &\leqslant \left\{ \sum_{\nu=n+1}^{\infty} (3\nu+1) r^{6\nu} \sum_{\nu=n+1}^{\infty} \frac{1}{(3\nu+1)^{1+2/3}} \right\}^{1/2}. \end{aligned} \tag{2.17}$$

Since

$$\begin{aligned} &\sum_{\nu=n+1}^{\infty} \frac{1}{(3\nu+1)^{1+2/3}} \\ &< \sum_{k=0}^{\infty} \int_{n+k}^{n+k+1} \frac{d\nu}{(3\nu+1)^{1+2/3}} = \int_{n}^{\infty} \frac{d\nu}{(3\nu+1)^{1+2/3}} = \frac{1}{2} \frac{1}{(3n+1)^{2/3}} \end{aligned} \tag{2.18}$$

and

$$\sum_{\nu=n+1}^{\infty} (3\nu+1) r^{6\nu} = r^{6n+6} \frac{(3n+4)(1-r^6)+3r^6}{(1-r^6)^2}, \tag{2.19}$$

then

$$\sum_{\nu=n+1}^{\infty} \frac{r^{3\nu}}{(3\nu+1)^{1/3}} < \frac{1}{2^{1/2}} \frac{r^{3n+3}}{(3n+1)^{1/3}} \frac{\{(3n+4)(1-r^6)+3r^6\}^{1/2}}{1-r^6}. \tag{2.20}$$

The Hölder inequality yields analogously:

$$\begin{aligned} \sum_{\nu=n+1}^{\infty} (3\nu+1)^{2/3} r^{3\nu} &= \sum_{\nu=n+1}^{\infty} (3\nu+1)^{2/3} r^{2\nu} r^{\nu} \\ &\leqslant \left\{ \sum_{\nu=n+1}^{\infty} (3\nu+1) r^{3\nu} \right\}^{2/3} \left\{ \sum_{\nu=n+1}^{\infty} r^{3\nu} \right\}^{1/3} \\ &= \left\{ r^{3n+3} \frac{(3n+4)(1-r^3)+3r^3}{(1-r^3)^2} \right\}^{2/3} \left\{ \frac{r^{3n+3}}{1-r^3} \right\}^{1/3} \\ &= r^{3n+3} \frac{\{(3n+4)(1-r^3)+3r^3\}^{2/3}}{(1-r^3)^{5/3}}. \end{aligned} \tag{2.21}$$

From (1.1), by the Szegö method, we obtain that the partial sum $\sigma_n^{(3)}(z)$ for $n > 2$ is univalent in the disc $|z| < \sqrt[3]{3}/2$ if

$$\sum_{\nu=4}^{\infty} |c_\nu^{(3)}|(3\nu+1) r^{3\nu} < \frac{4(4-\sqrt[3]{9})}{11\sqrt[3]{11}}, \qquad \text{where } r^3 = 3/8. \tag{2.22}$$

As was shown by K. Joh [10], for an arbitrary ν the inequality exists: (1)

$$(3\nu+1)^{1/3} |c_{3\nu+1}'| < 7.96. \tag{2.23}$$

From (2.16), (2.21) and (2.23) it follows that (2.22) will hold true if

$$\begin{aligned} &0.636 \cdot 13 \cdot (3/8)^4 + 0.658 \cdot 16 \cdot (3/8)^5 + 0.683 \cdot 19 \cdot (3/8)^6 + \\ &\quad + 0.711 \cdot 22 \cdot (3/8)^7 + 0.741 \cdot 25 \cdot (3/8)^8 + 0.774 \cdot 28 \cdot (3/8)^9 + \\ &\quad + 7.96 \cdot \frac{8}{5} \cdot \left(\frac{164}{5}\right)^{2/3} \left(\frac{3}{8}\right)^{10} < 0.312. \end{aligned} \tag{2.24}$$

Since the inequality (2.24) holds and the partial sum $\sigma_1^{(3)}(z)$ is univalent in the disc $|z| < \sqrt{3}/2$, then Theorem 2 has been proved for $n \neq 2$.

The proof of the theorem for $n = 2$ with the help of the method of K. Löwner is given by L. Iliev in [14].

According to the method of Löwner [17], in order to establish that the partial sum

$$\sigma_2^{(3)}(z) = z + c_1^{(3)} z^4 + c_2^{(3)} z^7 \tag{2.\alpha}$$

of the function $f_3(z) \in S_3$ is univalent in the disc $|z| < \sqrt[3]{3}/2$, it is sufficient to find that concerning the functions for which

$$\begin{aligned} c_1^{(3)} &= -\tfrac{2}{3} \int_0^\infty \varkappa(\tau) e^{-\tau} d\tau, \\ c_2^{(3)} &= \tfrac{8}{9} \left(\int_0^\infty \varkappa(\tau) e^{-\tau} d\tau \right)^2 - \tfrac{2}{3} \int_0^\infty \varkappa^2(\tau) e^{-2\tau} d\tau, \end{aligned} \tag{2.\beta}$$

(1) This inequality has been later improved.

where $\varkappa(\tau)$ denotes an arbitrary continuous function whose absolute value equals 1 for any $\tau \geqslant 0$.

Following well known methods (see, for instance [10], p. 10, 11) it can be easily shown that the proposition will be established in case we prove that

$$\mathrm{Re}(1+4c_1^{(3)}z^3+7c_2^{(3)}z^6) > 0 \tag{2.γ}$$

for $|z| < \sqrt[3]{3}/2$, where $c_1^{(3)}$ and $c_2^{(3)}$ are determined by (2.β).

For that purpose, it is sufficient to prove that the minimum on the left-hand side of (2.γ) on the circle $|z| = \sqrt[3]{3}/2$ is positive. Let this minimum be obtained for $z = \dfrac{\sqrt[3]{3}}{2}e^{i\varphi}$, i.e. let

$$\min_{|z|=\sqrt[3]{3}/2} \mathrm{Re}(1+4c_1^{(3)}z^3+7c_2^{(3)}z^6) = \mathrm{Re}(1+\tfrac{3}{2}c_1^{(3)}e^{3i\varphi}+\tfrac{63}{64}c_2^{(3)}e^{6i\varphi}),$$

where

$$\tfrac{3}{2}c_1^{(3)}e^{3i\varphi} = -\int_0^\infty \varkappa(\tau)e^{3i\varphi}e^{-\tau}d\tau = \int_0^\infty \varkappa_1(\tau)e^{-\tau}d\tau = x_1+iy_1,$$

$$\begin{aligned}\tfrac{63}{64}c_2^{(3)}e^{6i\varphi} &= \tfrac{7}{8}\Big(\int_0^\infty \varkappa(\tau)e^{3i\varphi}e^{-\tau}d\tau\Big)^2-\tfrac{21}{32}\int_0^\infty \varkappa^2(\tau)e^{6i\varphi}e^{-2\tau}d\tau \\ &= \tfrac{7}{8}\Big(\int_0^\infty \varkappa_1(\tau)e^{-\tau}d\tau\Big)^2-\tfrac{21}{32}\int_0^\infty \varkappa_1^2(\tau)e^{-2\tau}d\tau = x_2+iy_2\end{aligned} \tag{2.δ}$$

and

$$\varkappa_1(\tau) = x(\tau)+iy(\tau), \quad x^2(\tau)+y^2(\tau) = 1. \tag{2.ε}$$

The proposition will be proved provided we prove that

$$1+x_1+x_2 > 0.$$

The equalities (2.δ) and (2.ε) yield:

$$\begin{aligned}1+x_1+x_2 &= 1+x_1+\tfrac{7}{8}\mathrm{Re}\Big(\int_0^\infty \varkappa_1(\tau)e^{-\tau}d\tau\Big)^2-\tfrac{21}{32}\mathrm{Re}\int_0^\infty \varkappa_1(\tau)e^{-2\tau}d\tau \\ &= 1+x_1+\tfrac{7}{8}(x_1^2-y_1^2)-\tfrac{21}{32}\int_0^\infty [x^2(\tau)-y^2(\tau)]e^{-2\tau}d\tau \\ &= 1+x_1+\tfrac{7}{8}x_1^2-\tfrac{7}{8}\Big(\mathrm{Im}\int_0^\infty \varkappa_1(\tau)e^{-\tau}d\tau\Big)^2-\tfrac{21}{32}\int_0^\infty [1-2y^2(\tau)]e^{-2\tau}d\tau \\ &= \tfrac{43}{64}+x_1+\tfrac{7}{8}x_1^2-\tfrac{7}{8}\Big(\int_0^\infty y(\tau)e^{-\tau}d\tau\Big)^2+\tfrac{21}{16}\int_0^\infty y^2(\tau)e^{-2\tau}d\tau\end{aligned}$$

$$\begin{aligned}&> 0.38-\tfrac{7}{8}\int_0^\infty y^2(\tau)e^{-\tau}d\tau+\tfrac{21}{16}\int_0^\infty y^2(\tau)e^{-2\tau}d\tau \\ &= 0.38-\int_0^\infty [\tfrac{7}{8}e^{-\tau}-\tfrac{21}{16}e^{-2\tau}]y^2(\tau)\,d\tau \\ &> 0.38-\int_{\ln 3/2}^\infty (\tfrac{7}{8}e^{-\tau}-\tfrac{21}{16}e^{-2\tau})d\tau > 0.38-\tfrac{7}{24} > 0,\end{aligned}$$

which had namely to be proved.

Remark. After this proof [14] was sent to Dokl. Acad. Nauk of the USSR, 1 (1954) Acta Mathematica Sinica was received at the Institute of Mathematics at the Bulgarian Academy of Sciences, where there is an analogous proof of the proposition for (2.α) given by Kung Sun [18].

On the proof of Theorem S I. In paper [5], G. Szegö established Theorem S I using the method exposed as well as the inequalities (2.3). As he said, "The proof of this theorem is easy for $n = 2$ and $n \geqslant 5$, but is not so simple for $n = 3$ and $n = 4$."

Actually, with the help of his method and using the inequality $|c_n| < en$ of J. E. Littlewood [19] for the functions from the class S, the fact that the theorem holds follows easily for $n \geqslant 6$, and using the inequality $|c_n| < n^n/(n-1)^{n-1}$ (cf. [5]) the same is true for $n = 5$. The proof of Szegö for $n = 2, 3$ and 4 is direct. Besides, for the last two cases it is rather complicated. With the help of the inequalities (1.1) and one result of Goluzin, the proof of the theorem can be considerably simplified.

For $|z_1| < 1/4$, $|z_2| < 1/4$, $z_1 \neq z_2$ for the functions of the class S the left inequality (1.1) yields, namely:

$$\left|\frac{f(z_1)-f(z_2)}{z_1-z_2}\right| \geqslant \frac{48}{125}. \tag{2.25}$$

Therefore, the statement of the theorem for the nth partial sum of the function (2.1) from the class S holds true if the following inequality is fulfilled:

$$\sum_{\nu=n+1}^{\infty} \frac{\nu|c_\nu|}{4^{\nu-1}} < \frac{48}{125}. \tag{2.26}$$

As was shown by Goluzin [20], for the functions of the class S the inequality $|c_n| < \frac{3}{4}en$ is fulfilled so that (2.26) is fulfilled, if

$$\frac{3}{4}\sum_{\nu=n+1}^{\infty} \frac{\nu^2}{4^{\nu-1}} < \frac{48}{125}, \tag{2.27}$$

i.e., if

$$\frac{3}{4}e\,\frac{9n^2+24n+20}{27\cdot 4^{n-1}} < \frac{48}{125}. \tag{2.28}$$

For $n = 4$ the inequality (2.28) holds true and thus the theorem is established for $n \geqslant 4$. In this way, the rather complicated direct proof of Szegö for $n = 4$ is needless. We will also note that with the method of Szegö cannot be achieved further simplification of the proof. Indeed, starting from the inequality (2.25), with the method of Szegö, the theorem cannot be established for $n = 3$ even if the Bieberbach conjecture is proved that for the functions of the class S the inequality $|c_n| \leqslant n$ holds. On the other hand, the left inequality in (1.1) for $k = 1$ is exact, so that (2.25) cannot be improved.

3. Using the Szegö method, V. Levin [21] proved the following

THEOREM L. *Let the function*

$$f(z) = z + c_2 z^2 + \ldots \tag{3.1}$$

belong to the class S. *The partial sum*

$$\sigma_n(z) = z + c_2 z^2 + \ldots + c_n z^n \tag{3.2}$$

for $n \geqslant 17$, *is univalent at least in the disc* $|z| < 1 - 6\dfrac{\ln n}{n}$.

With the help of the inequalities (1.1), following the considerations of V. Levin in [22], [14] the following theorems have been established:

THEOREM 3. *The partial sum* (3.2) *of the function* (3.1) *is univalent for* $n \geqslant 15$ *at least in the disc* $|z| < 1 - 4\dfrac{\ln n}{n}$.

THEOREM 4. *Let the function*

$$f_2(z) = z + c_3 z^3 + \ldots \tag{3.3}$$

belong to the class S_2. *The partial sum*

$$\sigma_n^{(2)}(z) = z + c_3 z^3 + \ldots + c_{2n+1} z^{2n+1} \tag{3.4}$$

for $n \geqslant 12$ *is univalent at least in the disc* $|z| < \left(1 - 3\dfrac{\ln n}{n}\right)^{1/2}$.

THEOREM 5. *If the function*

$$f_3(z) = z + c_1^{(3)} z^4 + \ldots \tag{3.5}$$

belongs to the class S_3, *the partial sum*

$$\sigma_n^{(3)}(z) = z + c_1^{(3)} z^4 + \ldots + c_n^{(3)} z^{3n+1}, \quad n = 1, 2, \ldots \tag{3.6}$$

is univalent at least in the disc $|z| < \left\{1 - \dfrac{8}{3}\,\dfrac{\ln \theta(n+1)}{n+1}\right\}^{1/3}$, $\theta = 7.96^{3/8} \cdot 3^{1/4} \cdot 2^{7/8}$.

Proof of Theorem 3. For $k = 1$, if $|z_1| \leqslant r$, $|z_2| \leqslant r$, $r < 1$, $z_1 \neq z_2$, for the functions of the class S the left inequality (1.1) yields:

$$\left|\frac{f(z_1) - f(z_2)}{z_1 - z_2}\right| \geqslant \frac{1 - r}{(1 + r)^3}. \tag{3.7}$$

Therefore, if for $|z_1| \leqslant r_n$, $|z_2| \leqslant r_n$, $z_1 \neq z_2$, $r_n < 1$, we have

$$\left|\sum_{\nu=n+1}^{\infty} c_\nu \frac{z_1^\nu - z_2^\nu}{z_1 - z_2}\right| < \frac{1 - r_n}{(1 + r_n)^3} \tag{3.8}$$

or

$$\sum_{\nu=n+1}^{\infty} |c_\nu| \nu r_n^{\nu-1} < \frac{1 - r_n}{(1 + r_n)^3}, \tag{3.9}$$

then (3.2) is univalent.

Taking into account the inequality of J. E. Littlewood: $|c_\nu| < e\nu$, we see that (3.9) is fulfilled, if

$$e \sum_{\nu=n+1}^{\infty} \nu^2 r_n^{\nu-1} < \frac{1 - r_n}{(1 + r_n)^3}. \tag{3.10}$$

But

$$\sum_{\nu=n+1}^{\infty} \nu^2 r_n^{\nu-1} = \frac{r_n^n}{(1 - r_n)^2}\left[n^2(1 - r_n)^2 - (2n - 1) r_n + 2n + 1\right], \tag{3.11}$$

since, if we set $\sum\limits_{\nu=n+1}^{\infty} \nu^2 r^{\nu-1} = S(r)$, then

$$\int_0^{r_n} \left\{\frac{1}{x}\int_0^x S(t)\,dt\right\} dx = \sum_{\nu=n+1}^{\infty} r_n^\nu = \frac{r_n^{n+1}}{1 - r_n}. \tag{3.12}$$

Thus, the inequality (3.10) assumes the form

$$\frac{r_n^n}{(1 - r_n)^4}\left[n^2(1 - r_n)^2 - (2n - 1) r_n + 2n + 1\right] < \frac{1}{e(1 + r_n)^2}. \tag{3.13}$$

The above inequality holds if

$$\frac{r_n^n}{(1 - r_n)^4}\left[n^2(1 - r_n)^2 - (2n - 1) r_n + 2n + 1\right] < 1/4e. \tag{3.14}$$

Let $r_n = 1 - \alpha/n$, $0 < \alpha < n$, so that

$$r_n^n = \left(1 - (\alpha/n)\right)^n < e^{-\alpha} \tag{3.15}$$

and

$$n^2(1 - r_n)^2 - (2n - 1) r_n + 2n + 1 < \alpha^2 + 2\alpha + 2. \tag{3.16}$$

From (3.15) and (3.16) it follows that the inequality (3.14) is true if

$$n^4 e^{-\alpha} \frac{\alpha^2 + 2\alpha + 2}{\alpha^4} < \frac{1}{4e}. \tag{3.17}$$

If $\alpha = 4\ln n$, the inequality (3.17) is fulfilled for $n \geqslant 3$. For the theorem to be valuable, however, $r_n > 1/4$ must be fulfilled. The last condition is satisfied for $n \geqslant 15$, and thus the theorem is established.

Proof of Theorem 4. For $k = 2$, $|z_1| < r$, $|z_2| < r$, $z_1 \neq z_2$, $r < 1$ for the function $f_2(z)$ the inequalities (1.1) yield:

$$\left|\frac{f_2(z_1)-f_2(z_2)}{z_1-z_2}\right| \geqslant \frac{1-r^2}{(1+r^2)^2}. \tag{3.18}$$

Taking into account that for the functions of the class S_2 according to V. Levin, as we already noted, $|c_{2\nu+1}| < 3.4$, then, as in the case of the proof of Theorem 3, we find that the partial sum (3.4) is a univalent function in the disc $|z| < r_n$, if the following inequality is fulfilled:

$$3.4 \sum_{\nu=n+1}^{\infty} (2\nu+1)r_n^{2\nu} < \frac{1-r_n^2}{(1+r_n^2)^2}. \tag{3.19}$$

Since

$$\sum_{\nu=n+1}^{\infty} (2\nu+1)r_n^{2\nu} = \frac{r_n^{2n+2}}{(1-r_n^2)^2}[(2n+3)(1-r_n^2)+2r_n^2], \tag{3.20}$$

the inequality (3.19) assumes the form

$$\frac{r_n^{2n+2}}{(1-r_n^2)^3}[(2n+3)(1-r_n^2)+2r_n^2] < \frac{1}{3.4(1+r_n^2)^2}. \tag{3.21}$$

The above inequality holds if

$$\frac{r_n^{2n+2}}{(1-r_n^2)^2}[(2n+3)(1-r_n^2)+2r_n^2] < \frac{1}{4\cdot 3.4}. \tag{3.22}$$

Let $r_n^2 = 1-\alpha/n$, $0 < \alpha < n$, so that

$$r_n^{2n+2} = \left(1-\frac{\alpha}{n}\right)^{n+1} < \left(1-\frac{\alpha}{n+1}\right)^{n+1} < e^{-\alpha} \tag{3.23}$$

and

$$(2n+3)(1-r_n^2)+2r_n^2 = 2\alpha+2+\alpha/n < 2\alpha+3. \tag{3.24}$$

The inequality (3.22) assumes the form

$$n^3e^{-\alpha}(2\alpha+3)/\alpha^3 < 1/4\cdot 3.4. \tag{3.25}$$

If $\alpha = 3\ln n$, this inequality holds for $n \geqslant 8$, so that the function (3.4) for $n \geqslant 8$ is univalent in the disc $|z| < r_n = \left(1-\frac{3\ln n}{n}\right)^{1/2}$. Having in mind Theorem 1, the last result is valuable only in the case when $r_n = \left(1-\frac{3\ln n}{n}\right)^{1/2} > \frac{1}{\sqrt{3}}$. The last inequality holds for $n \geqslant 12$, and thus the theorem is proved.

Proof of Theorem 5. By the method of Levin the proposition of the theorem is established in an analogous way with the help of the inequalities (1.1) for $k = 3$ and (2.21).

4. The application of Theorem 1 allows to find, keeping the notations introduced, the following results [23]:

THEOREM 6. *If the function $f_1(z) \in S_1$, the polynomial*

$$\sigma_n^{(1)}(z)/z = 1+c_2z+\ldots+c_nz^{n-1} \tag{4.1}$$

for $n \geqslant 1$ does not vanish in the dics $|z| < 1-2\frac{\ln 3n}{n}$, and for $n \geqslant 55$ it does not vanish in the disc $|z| < 1-2\frac{\ln n}{n}$. In general, for every $\varepsilon > 0$, a number N exists such that (4.1) does not vanish in the disc $|z| < 1-2\frac{\ln \varepsilon n}{n}$ for $n \geqslant N$.

THEOREM 7. *If the function $f_2(z) \in S_2$, the polynomial*

$$\sigma_n^{(2)}(z)/z = 1+c_2z+\ldots+c_{2n+1}z^{2n} \tag{4.2}$$

for $n \geqslant 1$ does not vanish in the disc $|z| < \left(1-\frac{\ln 4.3n}{n}\right)^{1/2}$. For every $\varepsilon > 0$ a number N exists such that (4.2) for $n \geqslant N$ does not vanish in the disc $|z| < \left(1-\frac{\ln \varepsilon n}{n}\right)^{1/2}$.

THEOREM 8. *If the function $f_3(z) \in S_3$, the polynomial*

$$\sigma_n^{(3)}(z)/z = 1+c_1^{(3)}z^3+\ldots+c_n^{(3)}z^{3n}, \quad n = 1, 2, \ldots \tag{4.3}$$

does not vanish in the disc $|z| < \left\{1-\frac{4}{3}\frac{\ln a(n+1)}{n+1}\right\}^{1/6}$, where $a = 7.96^{3/2}\cdot 3^{3/4}\cdot 2^{1/2}$.

Proof. Let us set

$$f_1(z) = \sigma_n^{(1)}(z)+p_n^{(1)}(z). \tag{4.4}$$

In view of the Bieberbach theorem, if $|z| \leqslant r < 1$, then

$$\left|\frac{f(z)}{z}\right| \geqslant \frac{1}{(1+r)^2}. \tag{4.5}$$

If, therefore, for $|z| \leqslant r_n < 1$ the following inequality is fulfilled:

$$\left|\frac{p_n^{(1)}(z)}{z}\right| < \frac{1}{(1+r_n)^2}, \tag{4.6}$$

i.e.

$$\left|\sum_{\nu=n+1}^{\infty} c_\nu z^{\nu-1}\right| < \frac{1}{(1+r_n)^2}, \tag{4.7}$$

then $\sigma_n^{(1)}(z)/z$ does not vanish in the disc $|z| < r_n$.

The inequality (4.7) is fulfilled if

$$\sum_{\nu=n+1}^{\infty} |c_\nu| r_n^{\nu-1} < \frac{1}{(1+r_n)^2}. \tag{4.8}$$

According to the result of Goluzin $|c_\nu| < 3e/4$, $\nu = 2, 3, \ldots$, so that (4.8) is satisfied if

$$\frac{3}{4} e \sum_{\nu=n+1}^{\infty} \nu r_n^{\nu-1} < \frac{1}{(1+r_n)^2}, \tag{4.9}$$

or, if

$$3e \sum_{\nu=n+1}^{\infty} \nu r_n^{\nu-1} < 1. \tag{4.10}$$

From the identity $\sum_{\nu=n+1}^{\infty} r^\nu = r^{n+1}/(1-r)$ where $0 < r < 1$ we get:

$$\sum_{\nu=n+1}^{\infty} \nu r^{\nu-1} = r^n \frac{(n+1)(1-r)+r}{(1-r)^2}. \tag{4.11}$$

According to (4.11) the inequality (4.10) assumes the form:

$$3er_n^n \frac{(n+1)(1-r_n)+r_n}{(1-r_n)^2} < 1. \tag{4.12}$$

If $r_n = 1-\alpha/n$, $0 < \alpha < n$, then $r_n^n = (1-\alpha/n)^n < e^{-\alpha}$ and $(n+1)(1-r_n)+ + r_n = \alpha+1$. Then the inequality (4.12) is fulfilled if

$$3en^2 e^{-\alpha} \frac{\alpha+1}{\alpha^2} < 1. \tag{4.13}$$

If $\alpha = 2\ln 3n$, i.e. $r_n = 1-2\dfrac{\ln 3n}{n}$, then (4.13) is fulfilled for $n \geqslant 1$; if $\alpha = 2\ln n$ it is fulfilled for $n \geqslant 55$. In general, for every $\varepsilon > 0$ there is a number N such that (4.13) is satisfied if $\alpha = 2\ln \varepsilon n$ and $n \geqslant N$.

Theorem 7 is proved with the help of the same method, having in mind that if $f_2(z) \in S_2$, then $|f_2(z)/z| \geqslant 1/(1+r^2)$ and that in view of the result of V. Levin: $|c_{2n+1}| < 3.4$; $n = 1, 2, \ldots$

Analogously, when $f_3(z) \in S_3$, $f_3(z) = \sigma_n^{(3)}(z) + p_n^{(3)}(z)$, if for $|z| \leqslant r_n < 1$ the inequality $|p_n^{(3)}(z)/z| \leqslant 1/(1+r_n^3)^{2/3}$ holds, then the polynomial $\sigma_n^{(3)}(z)$ does not vanish in the disc $|z| \leqslant r_n$.

This inequality is fulfilled if

$$\sum_{\nu=n+1}^{\infty} |c_\nu^{(3)}| r_n^{3\nu} < 1/(1+r_n^3)^{2/3}. \tag{4.14}$$

Since, according to the result of K. Joh for any integer $\nu > 1$ the inequality $(3\nu+1)^{1/3}|c_{3\nu+1}| < 7.96$ holds, then (4.14) is fulfilled if

$$7.96 \sum_{\nu=n+1}^{\infty} \frac{r_n^{3\nu}}{(3\nu+1)^{1/3}} < 1/2^{2/3}. \tag{4.15}$$

Taking into account (2.20), the above inequality will hold when $n \geqslant 1$, if

$$\frac{r_n^{3n+3}}{(3n+1)^{1/3}} \frac{[(3n+4)(1-r_n^6)+3r_n^6]^{1/2}}{1-r_n^6} < \frac{1}{7.96 \cdot 2^{1/6}}. \tag{4.16}$$

Let us set $r_n^6 = 1-\alpha/(n+1)$, $0 < \alpha < n+1$, so that

$$r_n^{3n+3} = \left(1-\frac{\alpha}{n+1}\right)^{(n+1)/2} < e^{-\alpha/2} \tag{4.17}$$

and

$$(3n+4)(1-r_n^6)+3r_n^6 < 3\alpha+3. \tag{4.18}$$

The inequality (4.16) is fulfilled if for $n \geqslant 1$:

$$e^{-\alpha/2} \frac{\sqrt{\alpha+1}}{\alpha} (n+1)^{2/3} \leqslant \frac{2^{1/6}}{7.96 \cdot 3^{1/2}}. \tag{4.19}$$

The above inequality is fulfilled if $\alpha = \frac{4}{3}\ln a(n+1)$, where $a = 7.96^{3/2} \cdot 3^{3/4} \cdot 2^{1/3}$.

5. Let the function

$$f_k(z) = z + c_1^{(k)} z^{k+1} + \ldots \tag{5.1}$$

belong to the class S_k, $k = 1, 2, \ldots$ In this case, the function

$$\begin{aligned} f_1(\zeta) = f_1(z^k) = [f_k(z)]^k &= (z + c_1^{(k)} z^{k+1} + \ldots)^k \\ &= z^k + kc_1^{(k)} z^{2k} + \ldots = \zeta + kc_1^{(k)} \zeta^2 + \ldots \end{aligned} \tag{5.2}$$

belongs to the class S_1. But then the function

$$F(\zeta) = [f_1(\zeta^2)]^{1/2} = \zeta + \frac{k}{2} c_1^{(k)} \zeta^3 + \ldots \tag{5.3}$$

is univalent and holomorphic in $|\zeta| < 1$, while the function

$$\Phi(\zeta) = 1/F(1/\zeta) \tag{5.4}$$

is univalent and holomorphic in the domain $|\zeta| > 1$ except for the simple pole at the improper point. The latter has a Laurent expansion beginning as follows:

$$\zeta - \frac{k}{2} \frac{c_1^{(k)}}{\zeta} + \ldots \tag{5.5}$$

Taking into consideration the area theorem (Flächensatz) (see [2], pp. 72, 73) we find that

$$|c_1^{(k)}| \leqslant 2/k. \tag{5.6}$$

The inequality (5.6) is exact for any k. Indeed, for the class S_k the sign of equality is reached by the function

$$z/(1-z^k)^{2/k}. \tag{5.7}$$

Now we will establish that the partial sum $\sigma_1^{(k)}(z)$, $k = 1, 2, \ldots$ is univalent in the domain $|z| < \varrho_k$, $k = 1, 2, \ldots$, where

$$\varrho_k = \left[\frac{k}{2(k+1)}\right]^{1/k}. \tag{5.8}$$

The constant ϱ_k is exact and is reached by the partial sum of the function (5.7) again.

Indeed, if for $|z_1| < \varrho_k$ and $|z_2| < \varrho_k$, $z_1 \neq z_2$, we have the equality

$$z_1 + c_1^{(k)} z_1^{k+1} = z_2 + c_1^{(k)} z_2^{k+1} \tag{5.9}$$

then, taking into consideration (5.6) we obtain the absurd inequality

$$1 = |c_1^{(k)}| \left|\frac{z_1^{k+1} - z_2^{k+1}}{z_1 - z_2}\right| < (k+1)|c_1^{(k)}|\varrho_k^k \leqslant \frac{2(k+1)}{k}\varrho_k^k = 1, \tag{5.10}$$

and thus the proposition is established.

Remark. In view of Theorem SI, Theorem 1 and Theorem 2, the constants ϱ_1, ϱ_2 and ϱ_3 are the greatest values of the radii of the circular domains around the origin, where all partial sums of the functions of the classes S_1, S_2 and S_3, respectively, are univalent. It is quite natural to ask the question whether the constant ϱ_k, for an arbitrary value of k, possesses the same property? Taking into consideration that an assumption of Szegö for a certain restriction of the coefficients of the functions of the class S_k (see the following item) turned to be untrue in the general case when $k > 3$, it is improbable for the assumption that the partial sums of the functions of the class S_k are univalent in the disc $|z| < \varrho_k$ to turn out to be true when $k > 3$.

6. G. Szegö assumed that for the functions

$$f_k(z) = z + c_1^{(k)} z^{k+1} + \ldots \tag{6.1}$$

of the class S_k, $k = 1, 2, \ldots$, the following relation holds:

$$|c_n^{(k)}| = O(n^{-1+2/k}). \tag{6.2}$$

The fact that, for $k = 1$, we have the inequality of Littlewood

$$|c_{n-1}^{(1)}| = |c_n| \leqslant en, \tag{6.3}$$

gave grounds to make this assumption, while for $k = 2$, J. E. Littlewood and R. E. A. C. Paley were the first to establish ([24], see also E. Landau [25]) that

$$|c_n^{(2)}| = O(1). \tag{6.4}$$

V. Levin [26] verified the assumption for $k = 3$ as well. On the contrary, as was shown by J. E. Littlewood [27], the assumption is not true for $k > 3$, even if $f_k(z)$ is bounded in the unit disc.

Therefore, we can assume that there exist three positive constants A_1, A_2, A_3 not depending on n, for which, for any n, the following inequality holds:

$$|c_n^{(k)}| \leqslant A_k n^{\frac{2}{k}-1}, \quad k = 1, 2, 3. \tag{6.5}$$

By A_1, A_2, A_3 denote the smallest of these constants.

The inequality of Littlewood yields that $A_1 < e$; in view of the already used inequality of G. M. Goluzin: $A_1 < \frac{3}{4}e$. The assumption of L. Bieberbach is that $A_1 = 1$.

V. Levin [15] established the inequality that we have already used:

$$A_2 < 2^{1/4} \cdot 3^{1/2} \cdot e^{3/2} < 3.39 < 3.4. \tag{6.6}$$

K. K. Chen [28] first showed that

$$A_3 < 12.1 e^{1/3} = 16.89\ldots, \tag{6.7}$$

and K. Joh found that

$$A_3 < 7.96. \tag{6.8}$$

The indicated upper bounds for A_1, A_2 and A_3 are determined but the values A_1, A_2, and A_3 are not found.

Thus, the problem (conjecture) of Bieberbach concerning the coefficients of univalent functions is extended as follows: Which are the values of A_1, A_2, A_3?

At the same time, the problem of Szegö still remains open: To determine the order of $|c_n^{(k)}|$ for $k > 3$.

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FOLIATIONS AND THE GENERALIZED COMPLEX MONGE–AMPÈRE EQUATIONS

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Introduction

The generalized complex Monge–Ampère equations arise when looking for a complex analogue of the principles of Dirichlet and Thomson, including the inhomogeneity (weight) functions on the space (complex manifold) and the hermitian structure. These equations were introduced by the second named author in 1975 [10] and then studied by him, partially together with A. Andreotti [2], [3]. A considerable part of the results included in this paper is due to the first named author. His Theorem 3 ensures the existence of a foliation corresponding to a generalized complex Monge–Ampère equation, and even to a more general equation. This enables him to obtain a weak maximum principle and some corollaries.

We begin with the formulation of well known Dirichlet's and Thomson's principles. Then we introduce on hermitian manifolds the capacities due to the second named author [10], [11], [12], give their basic properties and explain their connection with the generalized complex Monge–Ampère equations. Before formulating Theorems 3 and 4 we give some preliminaries on foliations.

1. The principles of Dirichlet and Thomson (the case of R^2)

We begin with the formulation of Dirichlet's and Thomson's principles (cf. [16] and [14]).

DIRICHLET'S PRINCIPLE. The energy of a constant electric field in a smooth condenser (D, γ_0, γ_1) has the minimal value among the energies of all irrotational fields $\tilde{E} \in \mathscr{E}$, $\mathscr{E}$ being the class of all functions of the form $\tilde{E} = -\operatorname{grad}\tilde{V}$, $\tilde{V} \in \mathscr{V}$, and $\mathscr{V}$ consisting of all $\tilde{V} \in C^2(\operatorname{cl} D)$, such that $\tilde{V}|\gamma_0 = V_0$, $\tilde{V}|\gamma_1 = V_1$, and the normal derivative of $\tilde{V}$ along $D\setminus\gamma_0\setminus\gamma_1$ vanishes. In other words, we have

$$W = \tfrac{1}{2}\inf_{\tilde{E}\in\mathscr{E}} \iint_D \varepsilon_0\, \varepsilon \tilde{E}^2 dx\, dy,$$

where $\tilde{E}^2 = \tilde{E}\cdot\tilde{E}$ and $\varepsilon' = \varepsilon_0\varepsilon$ denotes the electric permeability.