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CONNECTION ON DIFFERENTIAL MODULES

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It is not difficult to observe that in many definitions and theorems concerning a differentiable manifold M only algebraic properties of the ring of all smooth real functions $C^{\infty}(M)$ on that manifold play an essential role. This point of view suggests a possibility of generalization of the concept of a differentiable manifold. The notion of a differential space was introduced, for example, by R. Sikorski [6], J. W. Smith [8], M. M. Postnikov [5], M. A. Mostow [4]. Their definitions differ and are not equivalent.

We consider differential spaces in the sense of R. Sikorski. Let us recall some definitions.

Let (M, τ) be a topological space and let P be a set of functions defined on M. A function f defined on a subset A of M is said to be a *local P-function* if for every $p \in A$ there exist a neighbourhood B of p in $(A, \tau|A)$ and a function $g \in P$ such that $f|B = g|B$. The set of all local P-functions defined on the set $A \subset M$ will be denoted by P_A. Let C be a non-empty set of real functions defined on the set M. We shall consider M as a topological space with the weakest topology τ_C such that all functions of C are continuous. Any pair (M, C) is called a *differential space* [6] if

(a) $C_M = C$ (with respect to τ_C);

(b) if $\omega\colon R^n \to R$ is a smooth function (i.e., an infinitely differentiable function) and $\alpha_1, \ldots, \alpha_n$ belong to C, then the composition $\omega(\alpha_1(\cdot), \ldots, \alpha_n(\cdot))$ belongs to C.

It is obvious that for any $A \subset M$ the pair (A, C_A) is a differential space, namely (A, C_A) is a differential subspace of (M, C).

Let (M, C) be a differential space. By a *vector tangent to* (M, C) *at a point* $p \in M$ we shall mean any linear mapping $v\colon C \to R$ such that

$$v(\alpha \cdot \beta) = v(\alpha) \cdot \beta(p) + \alpha(p) \cdot v(\beta) \quad \text{for} \quad \alpha, \beta \in C.$$

The set $(M, C)_p$ (or simply M_p) of all tangent vectors at a point $p \in M$ is a linear space in a natural way. By $\mathscr{X}(M)$ we shall denote the set of all smooth vector fields on (M, C), i.e., $X \in \mathscr{X}(M)$ if and only if $X\colon C \to C$ is linear and

$$X(\alpha \cdot \beta) = \alpha \cdot X(\beta) + \beta \cdot X(\alpha) \quad \text{for} \quad \alpha, \beta \in C.$$

Let ϕ be a mapping which assigns a linear space $\phi(p)$ to any point $p \in M$. By a *ϕ-field on M* we shall mean any function W which assigns an element $W(p) \in \phi(p)$ to any $p \in M$. There is no general definition of smooth ϕ-fields in the case of an arbitrary ϕ. Such a definition is possible for many special ϕ's.

Let $\mathscr{W}$ be a C-module of ϕ-fields on M. A sequence $V_1, \ldots, V_n \in \mathscr{W}$ is said to be a *vector basis of the C-module $\mathscr{W}$* if

(i) $V_1, \ldots, V_n$ is a C-basis of the C-module $\mathscr{W}$;

(ii) for every point $p \in M$ the sequence $V_1(p), \ldots, V_n(p)$ is a basis of the linear space $\phi(p)$.

Let A be a subset of a differential space (M, C). Then we say that $V_1, \ldots, V_n \in \mathscr{W}_A$ is a *vector basis of the C-module $\mathscr{W}$ on A* if the sequence $V_1, \ldots, V_n$ is a vector basis of the C_A-module $\mathscr{W}_A$, i.e.,

(i′) $V_1, \ldots, V_n$ is a C_A-basis of the C_A-module $\mathscr{W}_A$;

(ii′) for every $p \in A$ the sequence $V_1(p), \ldots, V_n(p)$ is a basis of the linear space $\phi(p)$.

A C-module $\mathscr{W}$ of ϕ-fields on a differential space (M, C) is said to be a *differential module* if $\mathscr{W}$ is closed with respect to localization (i.e., $\mathscr{W}_M = \mathscr{W}$) and $\mathscr{W}$ has locally a vector basis composed of n ϕ-fields [7]. The number n is called the *dimension* of the differential module $\mathscr{W}$.

EXAMPLES. 1. We set

$$\phi(p) = (M, C)_p \quad \text{for} \quad p \in M$$

and

$$\tilde{\mathfrak{X}}(M) = \{\tilde{X}\colon \tilde{X}(p) \in (M, C)_p \text{ for } p \in M \text{ and } X(\alpha) \in C \text{ for } \alpha \in C\},$$

where

$$(X(\alpha))(p) = \tilde{X}(p)(\alpha) \quad \text{for} \quad p \in M.$$

It is obvious that $\tilde{\mathfrak{X}}(M)_M = \tilde{\mathfrak{X}}(M)$ and, if M is a differentiable manifold, then $\tilde{\mathfrak{X}}(M) = \Gamma(\xi)$, where $\xi = (TM, \pi, M)$ is the tangent bundle of M. Thus $\tilde{\mathfrak{X}}(M)$ is a differential module in this case.

2. If C is a differential structure on R generated by $\{\sin(x/n), \cos(x/n)\}_{n=1,2,\ldots}$, then $\tilde{\mathfrak{X}}(R)$ is a 1-dimensional differential module [3].

3. If M is a dense subset of a differential manifold, then $\tilde{\mathfrak{X}}(M) = \Gamma(\xi)_M$ is a differential module.

4. Let us consider a differential space (R, C), where C is a differential structure on R generated by the functions $\{f_a\}_{a \in R}$, $f_a(x) = |x-a|$ for $x \in R$. Then $\dim(R, C)_p = 2$ for every $p \in R$ and $\tilde{\mathfrak{X}}(R) = 0$. Thus the C-module $\tilde{\mathfrak{X}}(R)$ is not a differential module.

The following theorems are true:

THEOREM [7]. *Let (M, C) be a differential space. Then $\tilde{\mathfrak{X}}(M)$ is a differential module if and only if*

(a) *for every $p \in M$ and $v \in (M, C)_p$ there exists a smooth vector field V on M such that $V(p) = v$,*

or, equivalently,

(a′) *for every $p \in M$ and $v \in (M, C)_p$ there exists a neighbourhood A of p and smooth tangent vector field V on A such that $V(p) = v$.*

THEOREM [7]. *If $\mathscr{W}$ is a differential module of ϕ-fields on a differential space (M, C), then for every $p \in M$ and for every $w \in \phi(p)$ there exists a $W \in \mathscr{W}$ such that $w = W(p)$.*

Let $\mathscr{W}$ be an n-dimensional differential module of ϕ-fields on a differential space (M, C). We set

$$Q = \{(p, w)\colon\ p \in M \text{ and } w \in \phi(p)\}.$$

By the projection of Q onto M we shall mean the mapping $\pi\colon Q \to M$ defined by

$$\pi(p, w) = p \quad \text{for} \quad (p, w) \in Q. \tag{1}$$

If A is an open subset of M and $W_1, \ldots, W_n$ is a vector basis of $\mathscr{W}$ on the set A, then the mapping $f\colon A \times R^n \to Q$ defined by

$$f(p, (x^1, \ldots, x^n)) = \big(p, x^i W_i(p)\big) \quad \text{for} \quad p \in A \text{ and } (x^1, \ldots, x^n) \in R^n$$

is said to be *fundamental with respect to the vector basis $W_1, \ldots, W_n$*. Let F be the set of all functions $\alpha\colon Q \to R$ such that

$$\alpha \circ f\colon A \times R^n \to R$$

is smooth for every fundamental mapping $f\colon A \times R^n \to Q$. We have the following proposition:

PROPOSITION [7]. *(Q, F) is a differential space.*

The differential space (Q, F) is said to be the *differential space of the differential module $\mathscr{W}$*. If for every $W \in \mathscr{W}$ we set

$$\overline{W}(p) = \big(p, W(p)\big) \quad \text{for} \quad p \in M,$$

then we get the following theorem:

THEOREM [7]. *Let $\mathscr{W}$ be a differential module. Then $W \in \mathscr{W}$ if and only if $\overline{W}\colon M \to Q$ is smooth.*

If $\tilde{\mathfrak{X}}(M)$ is a differential module, then the differential space of $\tilde{\mathfrak{X}}(M)$ will be denoted by (TM, TC). The following theorem is true:

THEOREM [2]. *If $\mathscr{W}$ is a differential module of the dimension n on a differential space (M, C) of dimension m, then the dimension of (Q, F) equals $m+n$.*

COROLLARY. *If* (M, C) *is of dimension* m, *then* (TM, TC) *is of dimension* $2m$.

Assume that (M, C) is a differential space of dimension m and $\mathscr{W}$ is a differential module of ϕ-fields on M, $\dim\mathscr{W} = n$. By a *covariant derivative* in the C-module $\mathscr{W}$ we shall mean a bilinear function $\nabla\colon \mathscr{X}(M)\times\mathscr{W} \to \mathscr{W}$ such that

$$\nabla_{\alpha X} W = \alpha\nabla_X W \quad \text{and} \quad \nabla_X \alpha W = X(\alpha) W + \alpha\cdot\nabla_X W$$
$$\text{for} \quad \alpha \in C,\ X\in\mathscr{X}(M),\ W\in\mathscr{W}.$$

We can define $\nabla_v W$ for $v \in (M, C)_p$ and $W\in\mathscr{W}$ by the formula

$$\nabla_v W = (\nabla_V W)(p), \tag{2}$$

where $V\in\mathscr{X}(M)$ is a smooth vector field such that $v = V(p)$. It is easy to see that the right side of (2) does not depend on the choice of V.

We denote by $h_a\colon Q \to Q$ the mapping given by the formula

$$h_a(p, w) = (p, aw) \quad \text{for} \quad (p, w)\in Q \quad (a\in R).$$

Let

$$V_{(p,w)} = \ker\pi_{*(p,w)} \quad \text{for} \quad (p, w)\in Q.$$

The following theorem is true:

THEOREM [1]. *Let* ∇ *be a covariant derivative in* $\mathscr{W}$. *Then there is exactly one smooth mapping* $\bar{K}\colon (TQ, TF) \to (Q, F)$, $\bar{K}\big((p, w), z\big) = \big(p, K(z)\big)$ *for* $(p, w)\in Q$ *and* $z\in(Q, F)_{(p,w)}$, *such that*

(i) $K(W_{*p}v) = \nabla_v W$ *for* $W\in\mathscr{W}$, $v\in(M, C)_p$, $p\in M$;

(ii) $K|(Q, F)_{(p,w)}\colon (Q, F)_{(p,w)} \to \phi(p)$ *is a linear mapping for* $(p, w)\in Q$;

(iii) $(Q, F)_{(p,w)} = V_{(p,w)}\oplus H_{(p,w)}$ *for* $(p, w)\in Q$ *and the distribution* $H\colon Q\to \bigcup_{(p,w)\in Q}(Q, F)_{(p,w)}$ *given by* $H(p, w) = H_{(p,w)}$ *for* $(p, w)\in Q$ *is smooth, where* $H_{(p,w)} = \ker\big(K|(Q, F)_{(p,w)}\big)$;

(iv) $(h_a)_{*(p,w)}H_{(p,w)} = H_{(p,aw)}$ *for* $a\in R-\{0\}$ *and* $(p, w)\in Q$.

The unique K satisfying the condition of the above theorem will be called the *mapping of the covariant derivative* ∇ and the vectors belonging to the distribution H will be called *horizontal with respect to the covariant derivative* ∇.

Also the following theorem holds:

THEOREM [1]. *Let* $\{HQ_{(p,w)}\}_{(p,w)\in Q}$ *be a smooth distribution on the differential space* (Q, F) *such that*

(a) $(Q, F)_{(p,w)} = V_{(p,w)}\oplus HQ_{(p,w)}$ *for* $(p, w)\in Q$;

(b) $(h_a)_{*(p,w)}HQ_{(p,w)} = HQ_{(p,aw)}$ *for* $(p, w)\in Q$ *and* $a\in R-\{0\}$.

Then there is exactly one covariant derivative ∇ *in the differential module* $\mathscr{W}$ *such that the horizontal vectors with respect to* ∇ *belong to* $\{HQ_{(p,w)}\}_{(p,w)\in Q}$.

Let (M, C) be a differential space of dimension m. Let ∇ be a covariant derivative on the the differential space (M, C). It follows from the above theorem that if $X \in \mathscr{X}(M)$ then there exist unique vector fields X^h, $X^v \in \mathscr{X}(TM)$ such that

$$\pi_{*(p,w)} X^h(p, w) = X(p) \quad \text{and} \quad K(p, w)\big(X^h(p, w)\big) = 0 \quad \text{for} \quad (p, w) \in TM;$$

$$\pi_* X^v = 0 \quad \text{and} \quad K(p, w)\big(X^v(p, w)\big) = X(p) \quad \text{for} \quad (p, w) \in TM.$$

The vector fields X^h and X^v will be called the *horizontal* and *vertical lifts* of the vector field X, respectively. Next, it follows from the above theorem that there is exactly one C-linear mapping J: $\mathscr{X}(TM) \to \mathscr{X}(TM)$ such that

$$\pi_* \circ J = -K \quad \text{and} \quad K \circ J = \pi_*.$$

The tensor field J is an almost complex structure of (TM, TC). The following theorem is true:

THEOREM [1]. *Let T and R be the torsion and the curvature tensors of the covariant derivative ∇ on (M, C), respectively. Let J be the almost complex structure of (TM, TC) defined above and let N be the torsion of J. Then*

$$\pi_{*(p,z)}\big(N(X^v, Y^v)\big)(p, z) = T(X, Y)(p) \quad \textit{for} \quad X, Y \in \mathscr{X}(M),\ (p, z) \in TM,$$

$$K(p, z)\big(N(X^v, Y^v)\big)(p, z) = R_{X(p), Y(p)} z \quad \textit{for} \quad X, Y \in \mathscr{X}(M),\ (p, z) \in TM.$$

As a corollary to this theorem we obtain:

COROLLARY. $N = 0$ *if and only if* $T = 0$ *and* $R = 0$.

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