DIFFERENTIAL GEOMETRY
BANACH CENTER PUBLICATIONS, VOLUME 12
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1984

NATURAL LAGRANGIAN STRUCTURES

DEMETER KRUPKA

Department of Mathematics, J. E. Purkyně University, Brno, Czechoslovakia

Contents

Introduction
Part I. Differential invariants
1. The r-frame lifting, associated liftings
2. Differential invariants
3. Prolongation of a lifting, associated with the r-frame lifting
Part II. Odd base forms and projectable vector fields on a fibered manifold
1. Odd base forms on a fibered manifold
2. Prolongations of projectable vector fields
Part III. Natural Lagrangian structures
1. Lagrangian structures
2. Natural Lagrangian structures
3. The Euler–Lagrange form of a natural Lagrangian structure
4. Induced variations
References

Introduction

The purpose of these lectures is to discuss the fundamental properties of a specific class of integral variational problems in fiber bundles, which are usually called *generally invariant*. Owing to their "categorical" properties and relations to the theory of natural bundles, we call these variational problems *natural Lagrangian structures*.

The lectures are concerned with the beginnings of the Lagrangian formulation of the calculus of variations in fiber bundles. Conceptually, we follow papers [17], [21], [24], [40] by Trautman and the author on differential geometry of the variational theory and the lifting theory, and, to a well-known extent, the classical ideas of Lepage (see e.g. [26]) and Noether [32] on differential forms and invariance of variational functionals. The definitions and theorems are usually stated in full generality. The proofs are only sketched or are omitted at all; the reader is then referred to the above papers or to some special articles.

We shall give a few references to the papers which complete the material of each of the three parts of these lectures. For abstract theory of natural bundles, the theory of lifting, associated with the r-frame lifting, and the theory of differential invariants, we refer to [15], [18], [20], [22], [27], [31], [39]. The geometric theory of Lagrangian structures, especially the first order ones, is developed in papers [4]–[8], [14], [40]. The general theory of higher order Lagrangian structures is treated in [1], [2], [16], [23], [25], [34], [38]. Various questions of this theory are discussed in the form of "concrete" variational problems, see e.g. [10]–[13], [19], [28]–[30], [33], [35]–[37]; many other references are listed and commented therein.

Part I

DIFFERENTIAL INVARIANTS

From now on, the category of real n-dimensional smooth manifolds and their embeddings is denoted by $\mathscr{D}_n$. The category whose objects are smooth (right) principal fiber bundles over the objects of $\mathscr{D}_n$, and whose morphisms are homomorphisms of principal fiber bundles over the morphisms of $\mathscr{D}_n$, is denoted by $\mathscr{PB}_n$.

The class of objects (resp. morphisms) of a category $\mathscr{C}$ is denoted by Ob $\mathscr{C}$ (resp. Mor $\mathscr{C}$). The projection functor from the category of fibered manifolds to the category of manifolds is denoted by proj.

The r-jet of a mapping α at a point x is denoted by $j^r_x\alpha$, and the composition of two r-jets $j^r_x\alpha$ and $j^r_y\beta$ such that $y = \alpha(x)$ is denoted by $j^r_y\beta \circ j^r_x\alpha$ or by $j^r_x\beta\alpha$.

L^r_n denotes the Lie group of invertible r-jets with source and target at the origin of the real n-dimensional Euclidean space R^n. When there is no need of specifying r and n, the group L^r_n is called simply the *differential group*. Recall that the group multiplication $(j^r_0\alpha, j^r_0\beta) \to j^r_0\alpha \circ j^r_0\beta$ in L^r_n is defined by the composition of jets. The group L^1_n is canonically identified with the general linear group $GL_n(R)$.

1. The r-frame lifting, associated liftings

A manifold endowed with a left action of a Lie group G is called a *G-manifold.* Let P and Q be two G-manifolds, U a G-invariant open subset of P. A mapping $F\colon U \to Q$ is called *G-equivariant* if $F(g \cdot p) = g \cdot F(p)$ for all $g \in G$ and $p \in U$.

Let Y_1 (resp. Y_2) be a principal G-bundle, P (resp. Q) a G-manifold. Denote by $Y_1 \times_G P$ (resp. $Y_2 \times_G Q$) the bundle of fiber-type P (resp. Q) associated with Y_1 (resp. Y_2). A point of $Y_1 \times_G P$ is an equivalence class $z = [y, p]$ of a pair $(y, p) \in Y_1 \times P$ relative to the action $((y, p), g) \to (y \cdot g, g^{-1} \cdot p)$ of G on $Y_1 \times P$. We introduce the following definition. A *homomorphism* of the bundle $Y_1 \times_G P$ into the bundle $Y_2 \times_G Q$ is a mapping $\Phi\colon Y_1 \times_G P \to Y_2 \times_G Q$ such that there exist a G-homomorphism σ of the principal G-bundle Y_1 into the principal G-bundle Y_2 and a G-equivariant mapping $F\colon P \to Q$ such that for each $z \in Y_1 \times_G P$, $z = [y, p]$, $\Phi(z) = [\sigma(y), F(p)]$. We write σ_F instead of Φ. The representation of σ_F by the pair (σ, F) is not unique. Clearly, proj σ_F = proj σ. Bundles over n-dimensional manifolds, associated with principal G-bundles, and homomorphisms of these bundles form a category denoted by $\mathscr{F}\mathscr{B}_n(G)$.

Let us consider a manifold $X \in \operatorname{Ob} \mathscr{D}_n$. An *r-frame* at a point $x \in X$ is by definition an invertible r-jet with source at $0 \in R^n$ and target at x. The set $\mathscr{F}^r X$ of r-frames at the points of X together with the canonical projection ϱ_X^r: $\mathscr{F}^r X \to X$, carries a natural structure of a principal L_n^r-bundle. This bundle is called the *bundle of r-frames* over X. The right action $(z, a) \to z \circ a$ of L_n^r on $\mathscr{F}^r X$ is defined by composition of jets.

Let $\alpha \in \operatorname{Mor} \mathscr{D}_n$, α: $X \to Y$, and let ϱ_X^r (resp. ϱ_Y^r) be the projection of $\mathscr{F}^r X$ (resp. $\mathscr{F}^r Y$). α defines a morphism $\mathscr{F}^r\alpha$ of the principal L_n^r-bundle $\mathscr{F}^r X$ into the principal L_n^r bundle $\mathscr{F}^r Y$ by the formula $\mathscr{F}^r\alpha(z) = j_{\zeta(0)}^r \alpha \circ j_0^r \zeta$, where $z = j_0^r \zeta$. The projection of $\mathscr{F}^r\alpha$ is α. The correspondence $X \to \mathscr{F}^r X$, $\alpha \to \mathscr{F}^r\alpha$, denoted by $\mathscr{F}^r$, is a covariant functor from the category $\mathscr{D}_n$ to the category $\mathscr{P}\mathscr{B}_n(L_n^r)$, called the *r-frame lifting*. We write $\mathscr{F}^1 = \mathscr{F}$, and call the 1-frame lifting simply the *frame lifting*. For $\alpha \in \operatorname{Mor} \mathscr{D}_n$, the morphism $\mathscr{F}^r\alpha \in \operatorname{Mor}\mathscr{P}\mathscr{B}_n(L_n^r)$ is called the *r-frame lift* (or simply the *lift*) of α.

Let $\mathscr{F}^r$ be the r-frame lifting, $X \in \operatorname{Ob} \mathscr{D}_n$, let P be an L_n^r-manifold. We denote by $\mathscr{F}_P^r X$ the fiber bundle of fiber-type P, associated with the principal L_n^r-bundle $\mathscr{F}^r X$ (via the action of L_n^r on P).

Consider a morphism $\alpha \in \operatorname{Mor} \mathscr{D}_n$, α: $X \to Y$. Then $\mathscr{F}^r\alpha \in \operatorname{Mor}\mathscr{P}\mathscr{B}_n(L_n^r)$, $\mathscr{F}^r\alpha$: $\mathscr{F}^r X \to \mathscr{F}^r Y$, and we have a well-defined mapping $\mathscr{F}_P^r X \ni z \to \mathscr{F}_P^r\alpha(z) = [\mathscr{F}^r\alpha(y), p] \in \mathscr{F}_P^r Y$, where $z = [y, p]$. This mapping is a morphism of the category $\mathscr{F}\mathscr{B}_n(L_n^r)$. The correspondence $X \to \mathscr{F}_P^r X$, $\alpha \to \mathscr{F}_P^r\alpha$, denoted by $\mathscr{F}_P^r$, is a covariant functor from the category $\mathscr{D}_n$ to the category $\mathscr{F}\mathscr{B}_n(L_n^r)$. We call this functor the *P-lifting*, associated with the r-frame lifting $\mathscr{F}^r$. For $r = 1$ we write $\mathscr{F}_P^1 = \mathscr{F}_P$. The morphism $\mathscr{F}_P^r\alpha \in \operatorname{Mor}\mathscr{F}\mathscr{B}_n(L_n^r)$ is called the *P-lift* (or simply the *lift*) of α.

2. Differential invariants

We state the following definition. A *differential invariant* is an L_n^r-equivariant mapping F: $P \to Q$ of an L_n^r-manifold P into an L_n^r-manifold Q.

Let P and Q be two L_n^r-manifolds, F: $P \to Q$ a differential invariant. For each $X \in \operatorname{Ob} \mathscr{D}_n$, the formula $F_X(z) = [y, F(p)]$, where $z = [y, p]$, establishes a morphism $F_X \in \operatorname{Mor}\mathscr{F}\mathscr{B}_n(L_n^r)$, F_X: $\mathscr{F}_P^r X \to \mathscr{F}_Q^r X$, whose projection is id_X. This morphism is called the *realization* of the differential invariant F on the manifold X.

Let $\mathscr{C}$ and $\mathscr{D}$ be two categories, $\mathscr{F}_1$ and $\mathscr{F}_2$ two covariant functors from $\mathscr{C}$ to $\mathscr{D}$. A *natural transformation* T of $\mathscr{F}_1$ to $\mathscr{F}_2$ is by definition a system of morphisms $T_X \in \operatorname{Mor} \mathscr{D}$, T_X: $\mathscr{F}_1 X \to \mathscr{F}_2 X$, where X runs over $\operatorname{Ob} \mathscr{C}$, such that for each $\alpha \in \operatorname{Mor} \mathscr{C}$, α: $X \to Y$, $\mathscr{F}_2\alpha \circ T_X = T_Y \circ \mathscr{F}_1\alpha$.

A natural transformation T of a P-lifting $\mathscr{F}_P^r$ to a Q-lifting $\mathscr{F}_Q^r$ satisfies the condition $\operatorname{proj} T_X = \mathrm{id}_X$ for each $X \in \operatorname{Ob} \mathscr{D}_n$. Obviously, for each $X \in \operatorname{Ob} \mathscr{D}_n$ and each local isomorphism of X, $\operatorname{proj} T_X \circ \alpha = \alpha \circ \operatorname{proj} T_X$; assuming that for some $x \in X$, $\operatorname{proj} T_X(x) \neq x$ and choosing α in such a way that $\alpha(x) = x$ and $\alpha(\operatorname{proj} T_X(x))$

$\neq \operatorname{proj} T_X(x)$, we obtain $\operatorname{proj} T_X(x) = \alpha(\operatorname{proj} T_X(x)) \neq \operatorname{proj} T_X(x)$ which is a contradiction.

The following three theorems are proved in [22].

THEOREM 1. *Let* $X \in \operatorname{Ob} \mathscr{D}_n$, $\Phi \in \operatorname{Mor} \mathscr{F}\mathscr{B}_n(L_n^r)$, Φ: $\mathscr{F}_P^r X \to \mathscr{F}_Q^r X$. *Then the following two conditions are equivalent*:

(1) *For each local isomorphism* α *of* X, $\mathscr{F}_P^r \alpha \circ \Phi = \Phi \circ \mathscr{F}_Q^r \alpha$.

(2) *There exists a unique differential invariant* F: $P \to Q$ *whose realization on* X *is* Φ, *i.e., such that* $F_X = \Phi$.

THEOREM 2. *Let* F: $P \to Q$ *be a differential invariant. Then for each* $X, Y \in \operatorname{Ob} \mathscr{D}_n$ *and each* $\alpha \in \operatorname{Mor} \mathscr{D}_n$, α: $X \to Y$, *the realizations* F_X *and* F_Y *of* F *satisfy* $\mathscr{F}_Q^r \alpha \circ F_X = F_Y \circ \mathscr{F}_P^r \alpha$. *In other words, the correspondence* T_F, *assigning to each manifold* $X \in \operatorname{Ob} \mathscr{D}_n$ *the morphism* $F_X \in \operatorname{Mor} \mathscr{F}\mathscr{B}_n(L_n^r)$, *is a natural transformation of* $\mathscr{F}_P^r$ *to* $\mathscr{F}_Q^r$.

In the following theorem we use the notation of Theorem 2.

THEOREM 3. *The correspondence* $F \to T_F$ *is a bijection between the set of differential invariants from an* L_n^r*-manifold* P *to an* L_n^r*-manifold* Q, *and the set of natural transformations of* $\mathscr{F}_P^r$ *to* $\mathscr{F}_Q^r$.

Let G be a Lie group, $e \in G$ the identity, l_G the Lie algebra of G, exp: $l_G \to G$ the exponential mapping. Consider a G-manifold P (resp. Q) and denote by Φ (resp. Ψ) the group action $(g, p) \to g \cdot p$ (resp. $(g, q) \to g \cdot q$) of G on P (resp. Q). Recall that for each $g_1, g_2 \in G$, $p \in P$, $\Phi(g_1, \Phi(g_2, p)) = \Phi(g_1 g_2, p)$, $\Phi(e, p) = p$, and analogous relations hold for Ψ. We define Φ_g and Φ_p by putting $\Phi_g(p) = \Phi_p(g) = \Phi(g, p)$. Let Tf denote the tangent mapping to a mapping f. For each $\xi \in l_G$ the relation $\xi_P(p) = T_e \Phi_p \cdot \xi$ defines a vector field ξ_P on P, called the *fundamental vector field* on the G-manifold P, relative to ξ. Analogous notations and definitions are used for the G-manifold Q.

Let F: $P \to Q$ be a mapping. For each $\xi \in l_G$ and each $p \in P$, we have a curve $t \to F_t(p)$ in Q defined by $F_t(p) = \exp(t\xi) \cdot F(\exp(-t\xi) \cdot p)$. Let $\partial_\xi F(p)$ be the tangent vector to this curve at $t = 0$:

$$\partial_\xi F(p) = \left\{ \frac{d}{dt} F_t(p) \right\}_0 .$$

The vector field $p \to \partial_\xi F(p)$ along the mapping F is called the *Lie derivative* of the mapping F relative to ξ.

Recall that for each $p \in P$, $s \in R$, and each $\xi \in l_G$,

$$\partial_\xi F(p) = \xi_Q(F(p)) - T_p F \cdot \xi_P(p),$$

$$\frac{d}{ds} F_s(p) = T_{F(\exp(-s\xi) \cdot p)} \Psi_{\exp(s\xi)} \cdot \partial_\xi F(\exp(-s\xi) \cdot p).$$

These formulas immediately imply the following infinitesimal criterion for a mapping of L_n^r-manifolds to be a differential invariant.

Denote by l_n^r the Lie algebra of the differential group L_n^r, and set for each $x \in R^n$, $x = (x^1, x^2, \dots, x^n)$, $\nu(x^1, x^2, \dots, x^n) = (-x^1, x^2, \dots, x^n)$. ν is an isomorphism of $R^n \in \mathrm{Ob}\,\mathscr{D}_n$ satisfying $\nu(0) = 0$, which implies that $j_0^r\nu \in L_n^r$; notice that $\det D\nu(0) = -1$, where $D\nu$ is the derivative of ν.

THEOREM 4. *Let P and Q be two L_n^r-manifolds, F: $P \to Q$ a mapping. The following two conditions are equivalent*:

(1) *F is a differential invariant.*

(2) *For each $\xi \in l_n^r$ and all $p \in P$,*

$$\partial_\xi F = 0, \qquad F(j_0^r\nu \cdot p) = j_0^r\nu \cdot F(p). \tag{1}$$

Let F: $P \to Q$ be a mapping of L_n^r-manifolds and assume that for some vectors $\xi, \eta \in l_n^r$, $\partial_\xi F = 0$, $\partial_\eta F = 0$. It is easily verified that then $\partial_{[\xi,\eta]}F = 0$, where $[\xi, \eta]$ is the Lie bracket of the vectors ξ and η in l_n^r. This shows that for F to be a solution of the system of equations (1) it is enough that F be a solution of this system for each ξ belonging to a vector subspace of l_n^r generating the Lie algebra l_n^r.

Let $X \in \mathrm{Ob}\,\mathscr{D}_n$, let P be an L_n^r-manifold. Each local isomorphism α of X defines a local isomorphism of the fiber bundle $\mathscr{F}_P^r X$, associated with $\mathscr{F}^r X$—the $\mathscr{F}_P^r$-lift $\mathscr{F}_P^r\alpha$ of α. The construction of $\mathscr{F}_P^r\alpha$ is directly extended to vector fields. Let ξ be a vector field on X and α_t its local one-parameter group. We set for each $z \in \mathscr{F}_P^r X$

$$\mathscr{F}_P^r\xi(z) = \left\{\frac{d}{dt}\mathscr{F}_P^r\alpha_t(z)\right\}_0.$$

This formula defines a vector field $\mathscr{F}_P^r\xi$ on $\mathscr{F}_P^r X$ which is called the *$\mathscr{F}_P^r$-lift* of ξ.

Let P and Q be two L_n^r-manifolds, $X \in \mathrm{Ob}\,\mathscr{D}_n$, $\Phi \in \mathrm{Mor}\,\mathscr{F}\mathscr{B}_n(L_n^r)$, Φ: $\mathscr{F}_P^r X \to \mathscr{F}_Q^r X$. Let ξ be a vector field on X, α_t its local one-parameter group. For each $z \in \mathscr{F}_P^r X$, $t \to (\mathscr{F}_Q^r\alpha_t \circ \Phi \circ \mathscr{F}_P^r\alpha_{-t})(z)$ is a curve in $\mathscr{F}_Q^r X$ passing through the point $\Phi(z)$. We define

$$\partial_\xi\Phi(z) = \left\{\frac{d}{dt}(\mathscr{F}_Q^r\alpha_t \circ \Phi \circ \mathscr{F}_P^r\alpha_{-t})(z)\right\}_0.$$

The vector field $z \to \partial_\xi\Phi(z)$ along the morphism Φ is called the *Lie derivative* of the morphism Φ relative to ξ. The following is an infinitesimal version of Theorem 1.

THEOREM 5. *Let $X \in \mathrm{Ob}\,\mathscr{D}_n$ be a connected manifold. Then each of conditions* (1) *and* (2) *of Theorem* 1 *is equivalent to the following condition*:

For each vector field ξ defined on an open subset of X, $\partial_\xi\Phi = 0$, and there exist a point $x_0 \in X$ and a local isomorphism α_0 of X defined on a neighborhood of x_0 such that $\alpha_0(x_0) = x_0$, $\mathscr{F}_Q^r\alpha_0 \circ \Phi = \Phi \circ \mathscr{F}_P^r\alpha_0$, and the Jacobian of α_0 at x_0 is negative.

3. Prolongation of a lifting, associated with the r-frame lifting

Recall that a *fibered manifold* is a surjective submersion of differential manifolds. Let π: $Y \to X$ be a fibered manifold. The set of r-jets of local sections of π endowed with the natural differential structure and denoted by $j^r Y$, together with the natural projection π_r: $j^r Y \to X$, is a fibered manifold called the *r-jet prolongation* of the fibered manifold π. We set $j^0 Y = Y$. For any integer s, $0 \leqslant s \leqslant r$, the natural projection $\pi_{r,s}$: $j^r Y \to j^s Y$ is a fibered manifold.

Let π_1: $Y_1 \to X_1$, and π_2: $Y_2 \to X_2$ be two fibered manifolds, α: $Y_1 \to Y_2$ an isomorphism of fibered manifolds, $\alpha_0 = \operatorname{proj}\alpha$. By the *r-jet prolongation* of α we mean the following isomorphism of fibered manifolds:

$$j^r Y_1 \ni j^r_x \gamma \to j^r \alpha(j^r_x \gamma) = j^r_{\alpha_0(x)} \alpha\gamma\alpha_0^{-1} \in j^r Y_2.$$

The correspondence $Y \to j^r Y$, $\alpha \to j^r \alpha$ is a covariant functor from the category of fibered manifolds and their isomorphisms into the same category. We denote this functor by j^r.

Let P be an L^s_n-manifold, $\mathscr{F}^s_P$ the corresponding P-lifting associated with the s-frame lifting $\mathscr{F}^s$. The correspondence $X \to j^r \mathscr{F}^s_P X$, $\alpha \to j^r \mathscr{F}^s_P \alpha$ may be regarded as a covariant functor from the category $\mathscr{D}_n$ to the category of fibered manifolds. We shall show that this correspondence is a Q-lifting, associated with the $(r+s)$-frame lifting $\mathscr{F}^{r+s}$.

Consider the set $T^r_n P$ of r-jets with source at $0 \in R^n$ and target in P, with its natural differential structure. Let $a \in L^{r+s}_n$, $p \in T^r_n P$ be any points, and denote by t_x the translation of R^n sending a point $x \in R^n$ to the origin. Choosing a representative α of the $(r+s)$-jet a, we may construct the mapping $x \to \alpha^s(x) = j^s_0(t_{\alpha(x)} \alpha t_{-x})$ of a neighborhood of $0 \in R^n$ into L^s_n. Similarly, choosing a representative μ of p, we may construct the mapping $x \to \mu\alpha^{-1}(x)$ of a neighborhood of $0 \in R^n$ into P. Then the mapping $x \to (\alpha^s \cdot \mu\alpha^{-1})(x)$ is defined on a neighborhood of $0 \in R^n$, and takes values in P. We set $a \cdot p = j^r_0(\alpha^s \cdot \mu\alpha^{-1})$. It is readily verified that this formula defines a left action of the group L^{r+s}_n on $T^r_n P$.

Let $X \in \operatorname{Ob} \mathscr{D}_n$. Consider the fiber bundle $\mathscr{F}^s_P X$ and its r-jet prolongation $j^r \mathscr{F}^s_P X$ which is a fibered manifold; on the other hand, consider the fiber bundle $\mathscr{F}^{r+s}_Q X$, where $Q = T^r_n P$, associated with the bundle $\mathscr{F}^{r+s} X$ of $(r+s)$-frames over X. It can be shown that there exists a canonical isomorphism of fibered manifolds between $j^r \mathscr{F}^s_P X$ and $\mathscr{F}^{r+s}_Q X$. Because of this the canonical isomorphism $j^r \mathscr{F}^s_P X$ has a natural structure of a fiber bundle of fiber-type Q, associated with $\mathscr{F}^{r+s} X$. We have the following more complete result:

THEOREM 6. *The correspondence $X \to j^r \mathscr{F}^s_P X$, $\alpha \to j^r \mathscr{F}^s_P \alpha$ is a $T^r_n P$-lifting, associated with the $(r+s)$-frame lifting $\mathscr{F}^{r+s}$.*

The $T^r_n P$-lifting $j^r \mathscr{F}^s_P$ is called the *r-jet prolongation* of the P-lifting $\mathscr{F}^s_P$.

Part II

ODD BASE FORMS AND PROJECTABLE VECTOR FIELDS ON A FIBERED MANIFOLD

In the next two sections, π: $Y \to X$ is a fibered manifold over n-dimensional base X. We denote $m = \dim Y - n$. The standard summation convention is used unless otherwise explicitly stated.

1. Odd base forms on a fibered manifold

In this section we introduce the concept of an odd base form on the total space of a fibered manifold. Roughly speaking, an odd *base* form is a field of geometric objects which has the property that its pull-back by a section of the underlying fibered manifold is an (ordinary) odd form in the sense of Bourbaki [3].

Let $\tilde{R}$ denote the real line R endowed with the left action $(a, t) \to \operatorname{sign}\det a \cdot t$ of the group $\mathrm{GL}_n(R)$.

Consider an n-dimensional manifold X. The bundle of fiber-type $\tilde{R}$, associated with the bundle of r-frames $\mathscr{F}X$ over X, is called the *bundle of odd scalars* over X, and is denoted by $\mathscr{F}_{\tilde{R}}X$. The points of this bundle are called *odd scalars*. Obviously, $\mathscr{F}_{\tilde{R}}X$ is a vector bundle. Let p be any non-negative integer, and let $\bigwedge^p T^*X$ be the bundle of (ordinary) p-forms over X. The tensor product of vector bundles $\mathscr{F}_{\tilde{R}}X \otimes \bigwedge^p T^*X$ is called the *bundle of odd p-forms* over X. A section of this bundle over an open set $U \subset X$ is called an *odd p-form* on U.

Consider the fibered manifold $\pi\colon Y \to X$. The vector bundle $\pi^*\mathscr{F}_{\tilde{R}}X$ is called the *bundle of odd base scalars* over Y. Let $p \geqslant 0$ be an integer. The tensor product $\pi^*\mathscr{F}_{\tilde{R}}X \otimes \bigwedge^p T^*Y$ of vector bundles is called the *bundle of odd base p-forms* over Y. A section of the bundle $\pi^*\mathscr{F}_{\tilde{R}}X$ (resp. $\pi^*\mathscr{F}_{\tilde{R}}X \otimes \bigwedge^p T^*Y$) defined on an open set $V \subset Y$ is called a *field of odd base scalars* (resp. an *odd base p-form*) over V. Clearly, if $\pi = \mathrm{id}_X$ then the odd base p-forms coincide with (ordinary) odd p-forms on X.

We shall now discuss the chart expressions of odd forms and odd base forms.

Let X be an n-dimensional manifold, (U, φ), $\varphi = (x^i)$, a chart on X. Let $x \to j_0(\varphi^{-1}t_{-\varphi(x)})$ be the field of frames, relative to (U, φ), where $t_{-\varphi(x)}$ denotes the translation $y \to y+\varphi(x)$ of R^n. We denote by $\tilde{\varphi}(x) = [j_0(\varphi^{-1}t_{-\varphi(x)}), 1]$ the equivalence class in $\mathscr{F}_{\tilde{R}}X$ over $x \in U$, defined by the pair $\big(j_0(\varphi^{-1}t_{-\varphi(x)}), 1\big) \in \mathscr{F}X \times \tilde{R}$. The section $x \to \tilde{\varphi}(x)$ of $\mathscr{F}_{\tilde{R}}X$ is called the *field of odd scalars* relative to the chart (U, φ). Let $\omega \in \mathscr{F}_{\tilde{R}}X \otimes \bigwedge^p T^*X$ be an odd form over $x \in U$. ω is uniquely expressible in the form $\omega = \tilde{\varphi}(x) \otimes \omega_\varphi$, where $\omega_\varphi \in \bigwedge^p T^*X$. In more detail,

$$\omega = \sum_{i_1 < \ldots < i_p} \omega_{i_1 i_2 \ldots i_p} \cdot \tilde{\varphi}(x) \otimes dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p},$$

where the expressions dx^i on the right-hand side are considered at the point x, and $\omega_{i_1 i_2 \ldots i_p}$ are the *components* of ω relative to the chart (U, φ). If $(\bar{U}, \bar{\varphi})$, $\bar{\varphi} = (\bar{x}^i)$, is another chart on X such that $x \in \bar{U}$, we easily obtain the following *transformation formula* for the components of ω:

$$\bar{\omega}_{j_1 \ldots j_p} = \operatorname{sign}\det D(\bar{\varphi}\varphi^{-1})\big(\varphi(x)\big) \cdot \frac{\partial x^{i_1}}{\partial \bar{x}^{j_1}} \ldots \frac{\partial x^{i_p}}{\partial \bar{x}^{j_p}} \cdot \omega_{i_1 \ldots i_p}.$$

In this formula, D denotes the derivative operator, and the partial derivatives on the right-hand side are considered at the point $\bar{\varphi}(x)$. In particular, if $p = n$ and ω is expressed by the formulas

$$\omega = f \cdot \tilde{\varphi}(x) \otimes dx^1 \wedge \ldots \wedge dx^n = \bar{f} \cdot \tilde{\bar{\varphi}}(x) \otimes d\bar{x}^1 \wedge \ldots \wedge d\bar{x}^n,$$

this transformation formula reads $f = |\det D(\bar{\varphi}\varphi^{-1})(\varphi(x))| \cdot \bar{f}$.

Consider the fibered manifold $\pi\colon Y \to X$. Let $\omega \in \pi^*\mathscr{F}_{\tilde{R}}X \otimes \bigwedge^p T^*X$ be an element over a point $y \in Y$, let (V_1, ψ_1) and (V_2, ψ_2) be two fiber charts on Y such that $y \in V_1 \cap V_2$, and let (U_1, φ_1) and (U_2, φ_2) be the corresponding charts on X. Then ω is uniquely expressible in the forms $\omega = \tilde{\varphi}_1(x) \otimes \omega_1 = \tilde{\varphi}_2(x) \otimes \omega_2$, where ω_1 and ω_2 are two elements of the space $\bigwedge^p T^*X$ over y, and $x = \pi(y)$. We obtain the following transformation formula:

$$\omega_1 = \operatorname{sign}\det D(\varphi_2\varphi_1^{-1})(\varphi_1\pi(y)) \cdot \omega_2 .$$

Substituting the expression of ω_1 (resp. ω_2) relative to (V_1, ψ_1) (resp. (V_2, ψ_2)) into this relation we directly obtain the corresponding transformation formulas for the components of the odd base p-form ω at y. These formulas differ from the usual ones for transformation of the components of (ordinary) p-forms on Y by a constant factor $\operatorname{sign}\det D(\varphi_2\varphi_1^{-1})(\varphi_1\pi(y))$.

It can be shown that all main operations of the calculus of forms (the *pull-back*, the *exterior multiplication*, the *contraction* by a vector field, the *exterior differential*, and the *Lie derivative*) are directly extended to odd base forms in such a way that the standard rules for computation remain valid. Moreover, an odd base form becomes an (ordinary) odd form on the base manifold when pulled back by a section; for these pull-backs, all main theorems of the integral calculus of odd forms remain valid. We shall discuss these questions in detail in another paper.

2. Prolongations of projectable vector fields

Let Ξ be a π-projectable vector field on Y, ξ its π-projection, and let Ξ_t be the local one-parameter group of Ξ. Ξ_t is a family of local isomorphisms of π. Let $j^r\Xi_t$ be the r-jet prolongation of Ξ_t. We set for each $j_x^r\gamma \in j^rY$

$$j^r\Xi(j_x^r\gamma) = \left\{\frac{d}{dt}\, j^r\Xi_t(j_x^r\gamma)\right\}_0 .$$

$j^r\Xi$ is a vector field on j^rY called the *r-jet prolongation* of the π-projectable vector field Ξ. $j^r\Xi$ is π_r-projectable and $\pi_{r,s}$-projectable for each non-negative integer $s \leqslant r$, and $T\pi_r \cdot j^r\Xi = \xi \circ \pi_r$, $T\pi_{r,s} \cdot j^r\Xi = j^s\Xi \circ \pi_{r,s}$.

THEOREM 7. *Let Ξ and Θ be two vector fields on Y. If both Ξ and Θ are π-projectable, then so is the Lie bracket $[\Xi, \Theta]$, and for each r, $j^r[\Xi, \Theta] = [j^r\Xi, j^r\Theta]$.*

We shall now give a coordinate formula for the vector field $j^r\Xi$. Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fiber chart on Y, (V_r, ψ_r), $\psi_r = (x^i, y^\sigma, y^\sigma_{j_1}, \ldots, y^\sigma_{j_1 \ldots j_r})$ the natural prolongation of (V, ψ) to j^rY. If $f\colon V_r \to R$ is a function, we define

$$d_k f = \frac{\partial f}{\partial x^k} + \frac{\partial f}{\partial y^\sigma} \cdot y^\sigma_k + \ldots + \sum \frac{\partial f}{\partial y^\sigma_{i_1 \ldots i_r}} \cdot y^\sigma_{i_1 \ldots i_r k},$$

where on the right-hand side we sum over all sequences of subscripts such that $1 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_r \leqslant n$. $d_k f$ is a function on $V_{r+1} = \pi_{r+1,0}^{-1}(V)$, called the *formal* *derivative* of f with respect to x^k. If f is defined on j^rY, then $d_kf \cdot dx^k$ is a 1-form on $j^{r+1}Y$.

THEOREM 8. *Let Ξ be a π-projectable vector field on Y, let*

$$\Xi = \xi^i \cdot \frac{\partial}{\partial x^i} + \Xi^\sigma \cdot \frac{\partial}{\partial y^\sigma}$$

be its expression for a fiber chart (V, ψ), $\psi = (x^i, y^\sigma)$, on Y. Then the expression of the vector field $j^r\Xi$ for the natural prolongation (V_r, ψ_r) of (V, ψ) is given by

$$j^r\Xi = \xi^i \cdot \frac{\partial}{\partial x^i} + \Xi^\sigma \cdot \frac{\partial}{\partial y^\sigma} + \ldots + \sum \Xi^\sigma_{i_1 \ldots i_r} \frac{\partial}{\partial y^\sigma_{i_1 \ldots i_r}},$$

where, for each non-negative integer $s \leqslant r-1$,

$$\Xi^\sigma_{i_1 \ldots i_s k} = d_k \Xi^\sigma_{i_1 \ldots i_s} - y^\sigma_{i_1 \ldots i_s j} \cdot d_k \xi^j,$$

and the summation over all sequences of subscripts satisfying $1 \leqslant i_1 \leqslant \ldots \leqslant i_r \leqslant n$ is assumed.

For the proof of this theorem we refer to [23].

Part III

NATURAL LAGRANGIAN STRUCTURES

Let us briefly recall the notation introduced in Part I and Part II of these lectures. The category of n-dimensional manifolds and their embeddings is denoted by $\mathscr{D}_n$. The category whose objects are fiber bundles over n-dimensional manifolds, with structure group G, and whose morphisms are homomorphisms of these fiber bundles over embeddings of n-dimensional manifolds, is denoted by $\mathscr{FB}_n(G)$. The r-frame lifting is denoted by $\mathscr{F}^r$, and we write for convenience $\mathscr{F}^1 = \mathscr{F}$. If L^r_n is a differential group and P an L^r_n-manifold, the P-lifting associated with $\mathscr{F}^r$ is denoted by $\mathscr{F}^r_P$. The s-jet prolongation of $\mathscr{F}^r_P$ is denoted by $j^s\mathscr{F}^r_P$. If $X \in \mathrm{Ob}\,\mathscr{D}_n$, then ϱ^r_X (resp. $\varrho^r_{X,P}$) denotes the projection of $\mathscr{F}^rX$ (resp. $\mathscr{F}^r_PX$). Unless otherwise stated, the standard summation convention is applied.

Let $\pi\colon Y \to X$ be a fibered manifold, $n = \dim X$, $m = \dim Y - n$. Recall that a *fiber chart* on Y is a chart (V, ψ) with the property that there exists a chart (U, φ), $\varphi = (x^i)$, on X such that $U = \pi(V)$ and $\psi = (x^i \circ \pi, y^\sigma)$, $1 \leqslant i \leqslant n$, $1 \leqslant \sigma \leqslant m$; we write simply $\psi = (x^i, y^\sigma)$. The *natural prolongation* of such a fiber chart (V, ψ), $\psi = (x^i, y^\sigma)$, to the r-jet prolongation j^rY of the fibered manifold π, is denoted by (V_r, ψ_r), $\psi_r = (x^i, y^\sigma, y^\sigma_{j_1}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$, $1 \leqslant i \leqslant n$, $1 \leqslant \sigma \leqslant m$, $1 \leqslant j_1 \leqslant j_2 \leqslant \ldots \leqslant j_r \leqslant n$. We denote by $\tilde{\Omega}^p(j^rY)$ (resp. $\tilde{\Omega}^p_X(j^rY)$, resp. $\tilde{\Omega}^p_Y(j^rY)$) the space of odd base p-forms (resp. odd base π_r-horizontal p-forms, resp. odd base $\pi_{r,0}$-horizontal p-forms) on j^rY. If $\varrho \in \tilde{\Omega}^p(j^sY)$ and $s \leqslant r$, the odd base p-form $\pi^*_{r,s}\varrho \in \tilde{\Omega}^p(j^rY)$ is also denoted by ϱ.

1. Lagrangian structures

Let $\pi\colon Y \to X$ be a fibered manifold, $n = \dim X$. A *Lagrangian of order* r for π is an odd base π_r-horizontal n-form defined on an open subset of the r-jet prolongation $j^r Y$ of π. A *Lagrangian structure of order* r is a pair (π, λ), where π is a fibered manifold and λ is a Lagrangian of order r for π.

Let (π, λ) be a Lagrangian structure of order r, where $\pi\colon Y \to X$ is a fibered manifold with n-dimensional base X. We write $m = \dim Y - n$. Let Ω be a piece of X, i.e., a compact n-dimensional submanifold of X with boundary $\partial\Omega$, let $C^\infty_\Omega(Y)$ be the set of smooth sections of π defined on Ω, and let $j^r\gamma \in C^\infty_\Omega(\pi_r)$ denote the *r-jet prolongation* of a section $\gamma \in C^\infty_\Omega(Y)$, defined by $j^r\gamma(x) = j^r_x\gamma$. The main problem of the theory of Lagrangian structures is to investigate the family of functions

$$C^\infty_\Omega(\pi) \ni \gamma \to \lambda_\Omega(\gamma) = \int_\Omega j^r\gamma^* \lambda \in R$$

labelled by Ω. The function λ_Ω is called the *action* of the Lagrangian structure (π, λ) over Ω.

For a general fibered manifold $\pi\colon Y \to X$ there is no natural algebraic and differential structure on the sets $C^\infty_\Omega(Y)$. To study the action λ_Ω one therefore applies specific methods which could be called *variational*. These methods consist in the use of one-parameter *deformations*, or *variations*, of each element $\gamma \in C^\infty_\Omega(\pi)$ separately, and in investigating the *induced deformations*, or *induced variations*, of the value $\lambda_\Omega(\gamma) \in R$ of λ_Ω at γ. We shall be concerned with the problem of determining those sections $\gamma \in C^\infty_\Omega(\pi)$ for which the value $\lambda_\Omega(\gamma)$ is, roughly speaking, non-sensitive to a wide class of such one-parameter deformations of γ.

Let $\pi\colon Y \to X$ be a fibered manifold, Ξ a π-projectable vector field on Y, ξ its π-projection, Ξ_t (resp. ξ_t) the local one-parameter group of Ξ (resp. ξ). To each section γ of π one can assign a one-parameter family γ_t of (local) sections of π by putting $\gamma_t = \Xi_t\gamma\xi_{-t}$. We call γ_t the *variation* of γ *induced* by Ξ. Given Ξ we obtain, for each piece $\Omega \subset X$, the following real-valued function of one real variable, defined for some $\varepsilon_\Omega > 0$:

$$(-\varepsilon_\Omega, \varepsilon_\Omega) \ni t \to \lambda_{\xi_t(\Omega)}(\Xi_t\gamma\xi_{-t}) = \int_{\xi_t(\Omega)} j^r(\Xi_t\gamma\xi_{-t})^* \lambda \in R.$$

Differentiating this function at $t = 0$ and using the standard rules of differential and integral calculus of odd base forms, we obtain

$$\left\{\frac{d}{dt}\lambda_{\xi_t(\Omega)}(\Xi_t\gamma\xi_{-t})\right\}_0 = \int_\Omega j^r\gamma^* \partial_{j^r\Xi}\lambda,$$

where $\partial_{j^r\Xi}\lambda$ is the *Lie derivative* of the odd base form λ relative to the vector field $j^r\Xi$. The arising function

$$C^\infty_\Omega(\pi) \ni \gamma \to (\partial_{j^r\Xi}\lambda)_\Omega(\gamma) = \int_\Omega j^r\gamma^* \partial_{j^r\Xi}\lambda \in R \tag{2}$$

is called the *(first) variation of the action* λ_Ω, *induced* by the π-projectable vector field Ξ.

In what follows, supp Ξ denotes the *support* of a vector field Ξ, i.e., the closure of the set where Ξ is different from zero.

Using our standard notation, consider a section $\gamma \in C_\Omega^\infty(\pi)$. γ is called *Ξ-stationary*, or *Ξ-stable* if the first variation $(\partial_{j^r\Xi}\lambda)_\Omega$ vanishes at γ, i.e.,

$$(\partial_{j^r\Xi}\lambda)_\Omega(\gamma) = 0.$$

γ is called *critical* (*over* Ω) if it is Ξ-stationary for each π-vertical vector field Ξ on Y such that supp $\Xi \subset \pi^{-1}(\Omega)$. A section γ of π is said to be a *critical section* of the Lagrangian structure (π, λ) if the restriction of γ to each piece $\Omega \subset X$ from the domain of definition of γ is critical over Ω.

Usually, the Lagrangian of order r for a fibered manifold $\pi\colon Y \to X$ with n-dimensional base X is defined as an *(ordinary)* π_r-horizontal n-form on an open subset of j^rY. Replacing the n-forms by odd base n-forms, we extend the notion of a Lagrangian structure to fibered manifolds with arbitrary, *not necessarily orientable*, base manifolds, that is, to the category of fibered manifolds over the objects of the category $\mathscr{D}_n$. As for our assumption of π_r-horizontality of the Lagrangian, it is easily seen that omitting it, we do not obtain more actions. To show it consider an element $\varrho \in \tilde{\Omega}^n(j^rY)$. There exists a unique element $h(\varrho) \in \tilde{\Omega}^n_X(j^{r+1}Y)$ such that for each section γ of π,

$$j^r\gamma^*\varrho = j^{r+1}\gamma^*h(\varrho). \tag{3}$$

This relation shows that extending the notion of Lagrangian to arbitrary (not necessarily π_r-horizontal) odd base forms, we do not obtain new actions of Lagrangian structures.

Nevertheless it is reasonable to take into account some odd base forms which are, in a sense, equivalent to the Lagrangians, and are not π_r-horizontal. Introducing these equivalents, we obtain a possibility of the free use of such invariant differential-geometrical operations as, for instance, the Lie derivative, and exterior differential, which does not preserve the π_r-horizontality of odd base forms.

Let $i_\xi\varrho$ denote the contraction of an odd base form ϱ by a vector field ξ, $d\varrho$ the exterior differential of ϱ.

THEOREM 9. *Let $\pi\colon Y \to X$ be a fibered manifold, $n = \dim X$, let λ be an odd base π_r-horizontal n-form defined on an open subset V_r of j^rY. Then there exist an integer $s \geqslant r$ and an odd base n-form θ_λ defined on $\pi_{s,r}^{-1}(V_r) \subset j^sY$ such that:*

(1) *For each π_s-vertical vector field Ξ on j^sY, the odd base $(n-1)$-form $i_\Xi\theta_\lambda$ is π_s-horizontal.*

(2) $h(\theta_\lambda) = \lambda$.

(3) *For each $\pi_{s,0}$-projectable, π_s-vertical vector field Ξ on j^sY, the odd base n-form $h(i_\Xi d\theta_\lambda)$ depends on the $\pi_{s,0}$-projection of Ξ only.*

The proof of Theorem 9 is based on the following three remarks. Firstly, define a mapping $h\colon \tilde{\Omega}^p(j^rY) \to \tilde{\Omega}^p_X(j^{r+1}Y)$ as follows. h is a unique R-linear, exterior-product-preserving mapping such that for each fiber chart (V, ψ), $\psi = (x^i, y^\sigma)$, on Y, and each function f on V_r,

$$h(f) = f\cdot \pi_{r+1,r}, \quad h(dx^i) = dx^i,$$

$$h(dy^\sigma) = y^\sigma_j dx^j, \quad \ldots, \quad h(dy^\sigma_{k_1\ldots k_r}) = y^\sigma_{k_1\ldots k_r j}dx^j.$$

This definition of h coincides with definition (3). Secondly, consider an odd base form $\varrho \in \tilde{\Omega}^n(j^sY)$, where $s = 2r-1$, and assume that ϱ has properties (1)–(3) of Theorem 9. By (1), the expression of ϱ for the fiber chart (V, ψ) is of the form

$$\varrho = \tilde{\varphi} \otimes \Big[f_0\omega_0 + \Big(\sum_{i=1}^{n}\sum_{k=0}^{s-1}\sum_{j_1\leqslant\ldots\leqslant j_k} f^{ij_1\ldots j_k}_\sigma \cdot \omega^\sigma_{j_1\ldots j_k}\Big)\wedge \omega_i\Big],$$

where $f_0, f^{ij_1\ldots j_k}_\sigma$ are functions, and

$$\text{(4)}\qquad \begin{aligned} &\omega_0 = dx^1\wedge dx^2\wedge \ldots \wedge dx^n, \quad \omega_i = (-1)^{i-1}dx^1\wedge \ldots \wedge \widehat{dx^i}\wedge \ldots \wedge dx^n,\\ &\omega^\sigma = dy^\sigma - y^\sigma_j dx^j, \quad \ldots, \quad \omega^\sigma_{k_1\ldots k_s} = dy^\sigma_{k_1\ldots k_s} - y^\sigma_{k_1\ldots k_s j}dx^j. \end{aligned}$$

In the formula for ω_i, the symbol $\widehat{dx^i}$ denotes that the factor dx^i is missing. Thirdly, apply requirements (2) and (3). Since λ is π_r-horizontal, it has an expression $\lambda = \tilde{\varphi}\otimes\mathscr{L}\cdot\omega_0$, where $\mathscr{L}$ is a function on V_r. Since $h(\omega^\sigma) = 0, \ldots, h(\omega^\sigma_{j_1\ldots j_s}) = 0$, condition (2) gives $f_0 = \mathscr{L}$. To apply condition (3), consider any $\pi_{s,0}$-projectable, π_s-vertical vector field Ξ on j^sY. In terms of the fiber chart (V_r, ψ_r), Ξ has an expression

$$\Xi = \sum_{k=1}^{n}\sum_{j_1\leqslant\ldots\leqslant j_k} \Xi^\sigma_{j_1\ldots j_k}\cdot\frac{\partial}{\partial y^\sigma_{j_1\ldots j_k}}.$$

After some calculation one obtains

$$\begin{aligned} h(i_\Xi d\varrho) = \tilde{\varphi}\otimes\Bigg[\Big(\frac{\partial\mathscr{L}}{\partial y^\sigma} - d_j f^j_\sigma\Big)\cdot\Xi^\sigma + &\\ + \sum_{k=1}^{s-1}\sum_{j_1\leqslant\ldots\leqslant j_k}\Big(\frac{\partial\mathscr{L}}{\partial y^\sigma_{j_1\ldots j_k}} - d_j f^{jj_1\ldots j_k}_\sigma - f^{j_1\ldots j_k}_\sigma\Big)\cdot\Xi^\sigma_{j_1\ldots j_k} + &\\ + \Big(\frac{\partial\mathscr{L}}{\partial y^\sigma_{j_1\ldots j_s}} - f^{j_1\ldots j_s}_\sigma\Big)\cdot\Xi^\sigma_{j_1\ldots j_s}\Bigg]\cdot\omega_0. & \end{aligned}$$

Hence condition (3) shows that we can take

$$f^{j_1\ldots j_s}_\sigma = 0, \quad f^{j_1\ldots j_k}_\sigma = \frac{\partial\mathscr{L}}{\partial y^\sigma_{j_1\ldots j_k}} - d_i f^{ij_1\ldots j_k}_\sigma$$

for all k, $1\leqslant k\leqslant s-1$. This proves the existence of a form ϱ satisfying requirements (1)–(3). Putting $\theta_\lambda = \varrho$ we obtain Theorem 9.

If λ is a Lagrangian for π, the odd base n-form θ_λ is called the *generalized Poincaré–Cartan form,* associated with λ, or the *generalized Poincaré–Cartan equivalent* of λ. If $\lambda \in \tilde{\Omega}^n_X(j^rY)$, then, in general, $\theta_\lambda \in \tilde{\Omega}^n(j^sY)$, where $s = 2r-1$.

Consider a Lagrangian structure (π, λ), where $\pi\colon Y \to X$ is a fibered manifold with n-dimensional base X, and $\lambda \in \tilde{\Omega}^n_X(j^rY)$. The first variation (2) of the action of (π, λ) over a piece $\Omega \subset X$ is easily expressed by means of the generalized Poincaré–Cartan equivalent of λ. Using the commutativity of the mapping h and the Lie derivative operator, property (2) of Theorem 9, and writing in accordance with our general convention λ instead of $\pi^*_{s,r}\lambda$, we obtain for each π-projectable vector field Ξ on Y

$$\partial_{j^r\Xi}\lambda = h(i_{j^r\Xi}d\theta_\lambda)+h(di_{j^r\Xi}\theta_\lambda). \tag{5}$$

This is the *(infinitesimal) first variation formula.* Let $\gamma \in C^\infty_\Omega(Y)$. By the Stokes theorem, for sufficiently small Ω,

$$(\partial_{j^r\Xi}\lambda)_\Omega(\gamma) = \int_\Omega j^r\gamma^* i_{j^r\Xi}d\theta_\lambda + \int_{\partial\Omega} j^r\gamma^* i_{j^r\Xi}\theta_\lambda. \tag{6}$$

This is the well-known *(integral) first variation formula.*

Let (π, λ) be as above. There exists a unique (global) odd base $(n+1)$-form $\mathscr{E}_\lambda$ on $j^{2r}Y$ such that, for each π-vertical vector field Ξ on Y,

$$i_{j^{2r}\Xi}\mathscr{E}_\lambda = h(i_{j^{2r-1}\Xi}d\theta_\lambda).$$

We call $\mathscr{E}_\lambda$ the *Euler–Lagrange form* of the Lagrangian structure (π, λ). By Theorem 9, (3), $\mathscr{E}_\lambda \in \tilde{\Omega}^{n+1}_Y(j^{2r}Y)$. If (V, ψ), $\psi = (x^i, y^\sigma)$, is a fiber chart on Y, (U, φ), $\varphi = (x^i)$, the corresponding chart on X, then $\mathscr{E}_\lambda$ has an expression

$$\mathscr{E}_\lambda = \tilde{\varphi}\otimes\mathscr{E}_\sigma(\mathscr{L})\cdot dy^\sigma \wedge \omega_0,$$

where ω_0 is defined by (4), $\mathscr{L}$ by the representation $\lambda = \tilde{\varphi}\otimes\mathscr{L}\cdot\omega_0$, and $\mathscr{E}_\sigma(\mathscr{L})$ are the *Euler–Lagrange expressions,* relative to (V, ψ),

$$\mathscr{E}_\sigma(\mathscr{L}) = \frac{\partial\mathscr{L}}{\partial y^\sigma} + \sum_{k=1}^{r}(-1)^k \sum_{i_1 \leqslant \ldots \leqslant i_k} d_{i_1}d_{i_2}\ldots d_{i_k}\frac{\partial\mathscr{L}}{\partial y^\sigma_{i_1 i_2\ldots i_k}}.$$

THEOREM 10. *Let (π, λ) be a Lagrangian structure of order r, γ a section of π. The following conditions are equivalent:*

(1) *γ is a critical section of (π, λ).*

(2) *The equality $j^{2r}\gamma^* i_{j^{2r}\Xi}d\theta_\lambda = 0$ holds on the domain of definition of γ.*

(3) *The Euler–Lagrange form $\mathscr{E}_\lambda$ vanishes along $j^{2r}\gamma$, i.e., $\mathscr{E}_\lambda \circ j^{2r}\gamma = 0$.*

(4) *For each fiber chart (V, ψ), $\psi = (x^i, y^\sigma)$, on Y, γ satisfies the system $\mathscr{E}_\sigma(\mathscr{L}) \circ j^{2r}\gamma = 0$, where $\mathscr{E}_\sigma(\mathscr{L})$ are the Euler–Lagrange expressions relative to (V, ψ).*

Theorem 10 is an immediate consequence of the integral first variation formula (6).

We state the following definition. Let $\pi_1\colon Y_1 \to X$ and $\pi_2\colon Y_2 \to X$ be two fibered manifolds over the same base, let $C^\infty(Y_i)$ be the set of (local) sections of π_i, $i = 1, 2$. A mapping $D\colon C^\infty(Y_1) \to C^\infty(Y_2)$ is said to be a *differential operator of order* r if there exists a morphism $D^r\colon j^rY_1 \to Y_2$ of fibered manifolds over id_X such that for each section $\gamma \in C^\infty(\pi_1)$, $D(\gamma) = D^r \circ j^r\gamma$.

Let (π, λ) be a Lagrangian structure of order r, where $\pi\colon Y \to X$ is a fibered manifold with n-dimensional base. Let $\mathscr{E}_\lambda$ be the Euler–Lagrange form of (π, λ). $\mathscr{E}_\lambda$ is an odd base $(n+1)$-form on $j^{2r}Y$, that is, a section of the vector bundle $\pi_{2r}^*\mathscr{F}_{\tilde{R}}X \otimes \bigwedge^{n+1} T^*j^{2r}Y$. In fact, $\mathscr{E}_\lambda$ is a $\pi_{2r,0}$-horizontal odd base form such that for each π_{2r}-vertical vector field Ξ on $j^{2r}Y$, $i_\Xi\mathscr{E}_\lambda$ is a π_{2r}-horizontal odd base n-form. These properties imply that $\mathscr{E}_\lambda$ is a section of a subbundle of $\pi_{2r}^*\mathscr{F}_{\tilde{R}}X \otimes \bigwedge^{n+1} T^*j^{2r}Y$ which may be characterized as follows. Since $\pi_{2r,0}$ is a submersion, we have an exact sequence $0 \to \pi_{2r,0}^* T^*Y \to T^*j^{2r}Y$, where the second arrow denotes the mapping $(j_x^{2r}\gamma, \omega) \to \omega \circ T\pi_{2r,0}$, where $T\pi_{2r,0}$ is considered on the tangent space to $j^{2r}Y$ at $j_x^{2r}\gamma$. This gives rise to the exact sequence

$$0 \to \pi_{2r,0}^*\big(\pi^*\mathscr{F}_{\tilde{R}}X \otimes (T^*Y \wedge \pi^* \textstyle\bigwedge^n T^*X)\big) \to \pi_{2r}^*\mathscr{F}_{\tilde{R}}X \otimes \bigwedge^{n+1} T^*j^{2r}Y,$$

in which the second arrow is the canonical inclusion. The image of the vector bundle

$$\pi_{2r,0}^*\big(\pi^*\mathscr{F}_{\tilde{R}}X \otimes (T^*Y \wedge \pi^* \textstyle\bigwedge^n T^*X)\big)$$

in

$$\pi_{2r}^*\mathscr{F}_{\tilde{R}}X \otimes \textstyle\bigwedge^{n+1} T^*j^{2r}Y$$

is the desired subbundle. An element

$$\Omega \in \pi_{2r}^*\mathscr{F}_{\tilde{R}}X \otimes \textstyle\bigwedge^{n+1} T^*j^{2r}Y$$

at a point $j_x^{2r}\gamma \in j^{2r}Y$ belongs to this subbundle if and only if there exists a fiber chart (V, ψ), $\psi = (x^i, y^\sigma)$, on Y such that $\gamma(x) \in V$ and Ω is expressible in the form

$$\Omega = \tilde{\varphi} \otimes \Omega_\sigma \cdot dy^\sigma \wedge dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n,$$

where $\Omega_\sigma \in R$, and the form on the right-hand side is considered at the point $j_x^{2r}\gamma$; here as usual, $(\pi(V), \varphi)$, $\varphi = (x^i)$, is the chart on X defined by (V, ψ). This discussion can be summarized in the following diagram:

$$\begin{tikzcd}
\pi_{2r}^*(\pi^*\mathscr{F}_{\tilde{R}}X \otimes (T^*Y \wedge \pi^* \bigwedge^n T^*X)) \arrow[rr] \arrow[d, shift right] & & \pi^*\mathscr{F}_{\tilde{R}}X \otimes (T^*Y \wedge \pi^* \bigwedge^n T^*X) \arrow[d] & \\
j^{2r}Y \arrow[u, dashed, shift right, "\mathscr{E}_\lambda"'] \arrow[urr, dashed] \arrow[rr, "\pi_{2r,0}"] & & Y \arrow[r, "\pi"] & X
\end{tikzcd}$$

This diagram shows that the Euler–Lagrange form $\mathscr{E}_\lambda$ may be canonically identified with a morphism of fibered manifolds

$$j^{2r}Y \to \pi^*\mathscr{F}_{\tilde{R}}X \otimes (T^*Y \wedge \pi^* \textstyle\bigwedge^n T^*X)$$

over $\pi_{2r,0}$. Considering the last bundle as a fiber bundle with base X, we immediately see that the mapping

$$C^\infty(Y) \ni \gamma \to \mathscr{E}_\lambda \circ j^{2r}\gamma \in C^\infty(\pi^* \mathscr{F}_{\tilde{R}} X \otimes (T^*Y \wedge \pi^* \textstyle\bigwedge^n T^*X))$$

is a differential operator of order $2r$.

Recall that a form ϱ on an open subset of j^rY is said to be *contact* if $j^r\gamma^*\varrho = 0$ for all sections γ of π. This definition applies directly to odd base forms.

The differential operator $\gamma \to \mathscr{E}_\lambda \circ j^{2r}\gamma$ may be simplified when we choose an everywhere non-zero odd n-form ω on X; such an odd n-form always exists, and may be constructed by means of a partition of unity. There exists a unique contact 1-form E_λ on $j^{2r}Y$ such that

$$\mathscr{E}_\lambda = E_\lambda \wedge \omega. \tag{7}$$

If in the fibered chart (V, ψ), $\omega = \tilde{\varphi} \otimes F \cdot \omega_0$, then

$$E_\lambda = \frac{1}{F}\, \mathscr{E}_\sigma(\mathscr{L}) \cdot \omega^\sigma,$$

where ω_0 and ω^σ are defined by (4), and $\mathscr{E}_\sigma(\mathscr{L})$ are the Euler–Lagrange expressions relative to (V, ψ). We have the following diagram:

$$\begin{tikzcd}
\pi^*_{2r,0}T^*Y \arrow[r] \arrow[d, shift right] & T^*Y \arrow[d] & \\
j^{2r}Y \arrow[u, dashed, shift right, "E_\lambda"'] \arrow[ur, dashed] \arrow[r, "\pi_{2r,0}"] & Y \arrow[r, "\pi"] & X
\end{tikzcd}$$

E_λ is canonically identified with a morphism $j^{2r}Y \to T^*Y$ of fibered manifolds over $\pi_{2r,0}$, and the corresponding differential operator is the mapping

$$C^\infty(Y) \ni \gamma \to E_\lambda \circ j^{2r}\gamma \in C^\infty(T^*Y),$$

where T^*Y is considered as a fibered manifold over X.

From now on we assume without loss of generality that each of the odd base forms λ (the Lagrangian), θ_λ (the generalized Poincaré–Cartan equivalent of λ), and $\mathscr{E}_\lambda$ (the Euler–Lagrange form) of a Lagrangian structure (π, λ) of order r are defined on the same space of jets, j^rY. If a Lagrangian structure of order r does not satisfy this assumption, we simply replace the Lagrangian by its canonical lift to the $2r$-jet prolongation of the underlying fibered manifold which does not alter the generalized Poincaré–Cartan equivalent and the Euler–Lagrange form.

Consider the Lagrangian structure (π, λ) and denote by $C^\infty_{\mathrm{crit}}(\pi)$ the set of its critical sections (with various domains of definition). A local isomorphism α of π with projection α_0 is called a *symmetry* of (π, λ) if, for each $\gamma \in C^\infty_{\mathrm{crit}}(Y)$, the section $\alpha\gamma\alpha_0^{-1}$ (over its domain of definition) belongs to $C^\infty_{\mathrm{crit}}(Y)$.

There are important special cases of symmetries of (π, λ) which we shall now describe. A local isomorphism α of π is called an *invariant transformation* of (π, λ)

if its r-jet prolongation preserves the Lagrangian λ, i.e., if $j^r\alpha^*\lambda = \lambda$. α is called a *generalized invariant transformation* of (π, λ) if its r-jet prolongation preserves the Euler–Lagrange form $\mathscr{E}_\lambda$, i.e., if $j^r\alpha^*\mathscr{E}_\lambda = \mathscr{E}_\lambda$.

THEOREM 11. *For each π-projectable vector field Ξ on Y,*

$$\partial_{j^r\Xi}\mathscr{E}_\lambda = \mathscr{E}_{\partial_{j^r\Xi}\lambda}. \tag{8}$$

This assertion follows from the equality $\mathscr{E}_{j^r\alpha^*\lambda} = j^r\alpha^*\mathscr{E}_\lambda$ holding for each local isomorphism α of π.

We say that a π-projectable vector field Ξ on Y *generates invariant transformations* (resp. *generalized invariant transformations*) of (π, λ) if its local one-parameter group consists of invariant transformations (resp. generalized invariant transformations) of (π, λ).

THEOREM 12. *Let Ξ be a π-projectable vector field on Y.*

(1) *Ξ generates invariant transformations of (π, λ) if and only if*

$$\partial_{j^r\Xi}\lambda = 0. \tag{9}$$

(2) *Ξ generates generalized invariant transformations of (π, λ) if and only if*

$$\mathscr{E}_{\partial_{j^r\Xi}\lambda} = 0. \tag{10}$$

Equation (9) (resp. (10)) is called the *generalized Noether equation* (resp. the *generalized Noether–Bessel–Hagen equation*).

The vector fields Ξ which are solutions of the Noether–Bessel–Hagen equation form a Lie algebra with respect to the bracket operation; the solutions of the Noether equation form a Lie subalgebra of this Lie algebra.

THEOREM 13. *A necessary and sufficient condition for the Euler–Lagrange form $\mathscr{E}_\lambda$ to vanish is that to each point $j^r_x\gamma \in j^rY$ there exist a neighborhood $V \subset j^rY$ of $j^r_x\gamma$ and a contact odd base n-form Ψ on V such that $d(\theta_\lambda + \Psi) = 0$.*

Obviously, only necessity of this condition needs proof. Writing $d\theta_\lambda = \mathscr{E}_\lambda + \mathscr{F}_\lambda$ we obtain a uniquely determined contact odd base $(n+1)$-form $\mathscr{F}_\lambda$. By assumption $\mathscr{E}_\lambda = 0$; hence $d\mathscr{F}_\lambda = 0$. To prove Theorem 13 we now apply an appropriate version of the Poincaré lemma.

Let η be an odd base $(n-1)$-form defined on an open subset V of j^rY. We say that η is a *first integral* of (π, λ) if there exists a π-projectable vector field Ξ on $\pi_{r,0}(V) \subset Y$ and a contact odd base n-form ν on V such that

$$i_{j^r\Xi}d\theta_\lambda = -d\eta + \nu.$$

Clearly, η is the first integral if and only if there exists a π-projectable vector field Ξ on $\pi_{r,0}(V) \subset Y$ such that $h(i_{j^r\Xi}d\theta_\lambda + d\eta) = 0$. If such a vector field Ξ exists, it is said to be *related* to the first integral η. In general, there may exist more vector fields related to a first integral, and vice versa.

The following is a modification of the well-known *first theorem of E. Noether*.

THEOREM 14. *Each π-projectable vector field on an open subset of Y, generating generalized invariant transformations of (π, λ), is related to a first integral of (π, λ). Conversely, each π-projectable vector field related to a first integral of (π, λ), generates generalized invariant transformations of (π, λ).*

The proof of Theorem 14 is based on (8), Theorem 13, and the standard formula for computation the Lie derivative of an odd base form relative to a vector field.

2. Natural Lagrangian structures

We shall now consider a class of Lagrangian structures whose underlying fibered manifolds are fiber bundles with structure group a differential group.

Let n, r, s be positive integers, P an L_n^s-manifold, $\mathscr{F}_P^s$ the P-lifting associated with the s-frame lifting $\mathscr{F}^s$, $X \in \mathrm{Ob}\,\mathscr{D}_n$, $\varrho_{X,P}^s$: $\mathscr{F}_P^s X \to X$ the projection. Consider a Lagrangian structure $(\varrho_{X,P}^s, \lambda)$ of order r, and denote by W the domain of definition of λ. We say that $(\varrho_{X,P}^s, \lambda)$ is a *natural Lagrangian structure of order r* if for each local isomorphism α of X, the P-lift $\mathscr{F}_P^s\alpha$ of α leaves W invariant, and is an invariant transformation of $(\varrho_{X,P}^s, \lambda)$, i.e., $(j^r\mathscr{F}_P^s\alpha)^*\lambda = \lambda$.

We note that a more general notion of a natural Lagrangian structure is obtained when we require that for each α, $\mathscr{F}_P^s\alpha$ is a *generalized invariant transformation* of $(\varrho_{X,P}^s, \lambda)$.

From now on we use the following convention. If $X \in \mathrm{Ob}\,\mathscr{D}_n$ and ξ is a vector field on X, then the Lie derivative of an odd base form ϱ on $j^r\mathscr{F}_P^s X$ relative to $j^r\mathscr{F}_P^s\xi$ (resp. the contraction of ϱ by $j^r\mathscr{F}_P^s\xi$) is denoted by $\partial_\xi\varrho$ (resp. $i_\xi\varrho$).

Let $\bar{R}$ be the L_n-manifold defined by the action $(a, t) \to |\det a|^{-1}\cdot t$ of $L_n^1 \approx \mathrm{GL}_n(R)$ on the real line R. Notice that, for each $X \in \mathrm{Ob}\,\mathscr{D}_n$, $\mathscr{F}_{\bar{R}}X \approx \mathscr{F}_R X \otimes \otimes \bigwedge^n T^*X$.

THEOREM 15. *Let $(\varrho_{X,P}^s, \lambda)$ be a Lagrangian structure of order r. The following three conditions are equivalent*:

(1) *$(\varrho_{X,P}^s, \lambda)$ is a natural Lagrangian structure.*

(2) *For each vector field ξ on X,*

$$\partial_\xi \lambda = 0,$$

and there exist a point $x_0 \in X$ and a local isomorphism α_0 of X such that α_0 is defined at x_0, $\alpha_0(x_0) = x_0$, $T_{x_0}\alpha$ changes orientation of the tangent space $T_{x_0}X$, and $\mathscr{F}_P^s\alpha_0$ is an invariant transformation of $(\varrho_{X,P}^s, \lambda)$.

(3) *There exists a unique differential invariant L from an open L_n^{r+s}-invariant subset of $T_n^r P$ to $\bar{R}$ whose realization on X is equal to λ,*

$$L_X = \lambda.$$

Theorem 15 is an immediate consequence of Theorem 1 and Theorem 5.

Let $X \in \mathrm{Ob}\,\mathscr{D}_n$. Theorem 15 and Theorem 3 imply that there exists a one-to-one correspondence between the set of Lagrangians λ of natural Lagrangian structures

$(\varrho^s_{P,X}, \lambda)$, the set of differential invariants L from open, L^{r+s}_n-invariant subsets of $T^r_n P$ into $\bar{R}$, and the set of natural transformations of the lifting $j^r\mathscr{F}^s_P$ to the lifting $\mathscr{F}_{\bar{R}}$. Accordingly, a natural Lagrangian structure of order r may be looked upon in three equivalent ways: (1) as a Lagrangian structure of certain invariant properties as defined above, (2) as a differential invariant between appropriate L^{r+s}_n-manifolds, and (3) as a natural transformation of a certain $T^r_n P$-lifting to the $\bar{R}$-lifting $\mathscr{F}_{\bar{R}}$.

In particular, given a natural Lagrangian structure $(\varrho^s_{X,P}, \lambda)$ and the corresponding differential invariant L from $T^r_n P$ to $\bar{R}$, this correspondence allows us to assign to each $X \in \operatorname{Ob} \mathscr{D}_n$ a natural Lagrangian structure $(\varrho^s_{X,P}, L_X)$. In this sense we say that a natural Lagrangian structure is *canonically extended* to the whole category $\mathscr{D}_n$.

Let $(\varrho^s_{X,P}, \lambda)$ be a Lagrangian structure whose underlying manifold is a fiber bundle $\varrho^s_{X,P}\colon \mathscr{F}^s_P X \to X$. Then the r-jet prolongation $j^r\mathscr{F}^s_P X$ of $\mathscr{F}^s_P X$ is a fiber bundle of fiber-type $T^r_n P$ associated with $\mathscr{F}^{r+s}X$. Denote by $\Xi^i_j, \Xi^i_{j_1 j_2}, \ldots, \Xi^i_{j_1 j_2 \ldots j_{r+s}}$ the fundamental vector fields on $T^r_n P$ relative to the elements of the canonical basis of the Lie algebra l^{r+s}_n of L^{r+s}_n.

THEOREM 16. *Let U be an L^{r+s}_n-invariant open subset in $T^r_n P$, $L\colon U \to \bar{R}$ a mapping. L is a differential invariant if and only if the following two conditions hold*:

(1) *For each* $i, j, j_1, \ldots, j_{r+s}$, $1 \leqslant i,\ j \leqslant n$, $1 \leqslant j_1 \leqslant j_2 \leqslant \ldots \leqslant j_{r+s} \leqslant n$,

$$\begin{gathered} \Xi^i_j(L) + \delta^i_j \cdot L = 0, \\ \Xi^i_{j_1 j_2}(L) = 0, \quad \ldots, \quad \Xi^i_{j_1 j_2 \ldots j_{r+s}}(L) = 0. \end{gathered} \tag{11}$$

(2) *There exists an element* $a_0 \in L^{r+s}_n$, $a_0 = j^{r+s}_0 \alpha_0$, *such that* $\det D\alpha_0(0) < 0$, *and for each* $p \in U$,

$$L(a_0 \cdot p) = |\det D\alpha_0(0)|^{-1} \cdot L(p). \tag{12}$$

Theorem 16 is a modification of Theorem 4. Let a^i_j be the canonical coordinates on the Lie group L^1_n, and write, for each $a \in L^1_n$, $b^i_j(a) = a^i_j(a^{-1})$. Then $a^i_k b^k_j = \delta^i_j$. Since $\partial(\det a)/\partial a^i_j = b^j_i \cdot \det a$, we have $\partial|\det a|^{-1}/\partial a^i_j = -b^j_i \cdot |\det a|^{-1}$. The fundamental vector field $(\xi^i_j)_{\bar{R}}$ on $\bar{R}$, defined by the vector $\xi^i_j = (\delta^i_j) \in l^1_n$, is of the form

$$(\xi^i_j)_{\bar{R}} = \left\{\frac{\partial}{\partial a^i_j} |\det a|^{-1}\right\}_e \cdot t\frac{d}{dt} = -\delta^i_j t \cdot \frac{d}{dt}.$$

To derive (11) from Theorem 4, we apply this vector field.

Theorem 16 implies that the problem of existence of a nontrivial natural Lagrangian structure $(\varrho^s_{X,P}, \lambda)$ is equivalent to the problem of existence of a nontrivial solution of the system (11) of partial differential equations, satisfying the additional condition (12).

3. The Euler–Lagrange form of a natural Lagrangian structure

Let P and Q be two L^s_n-manifolds, $X \in \operatorname{Ob} \mathscr{D}_n$, and let $\mathscr{F}^s_P X$ (resp. $\mathscr{F}^s_Q X$) be the corresponding fiber bundle of fiber-type P (resp. Q) associated with the bundle of s-frames $\mathscr{F}^s X$. Let D: $C^\infty(\mathscr{F}^s_P X) \to C^\infty(\mathscr{F}^s_Q X)$ be a differential operator of order r. We say that D is a *natural differential operator of order r*, if for each local isomorphism α of X, $D(\mathscr{F}^s_P \alpha \circ \gamma \circ \alpha^{-1}) = \mathscr{F}^s_Q \alpha \circ D(\gamma) \circ \alpha^{-1}$. A necessary and sufficient condition for D to be a natural differential operator of order r is that there exist a morphism D^r: $j^r \mathscr{F}^s_P X \to \mathscr{F}^s_Q X$ over id_X, and a differential invariant Δ: $T^r_n P \to Q$ whose realization on X is D^r, i.e., $\Delta_X = D^r$.

Let $\mathscr{F}_Q$: $\mathscr{D}_n \to \mathscr{F}\mathscr{B}_n(L^1_n)$ be a lifting such that $Q = \bigwedge^m R^n$ is a vector space of exterior forms on R^n. Recall that if $\alpha \in \operatorname{Mor} \mathscr{D}_n$, α: $X_1 \to X_2$, then for each $\omega \in \mathscr{F}_Q X_1$ over a point $x \in X_1$, $\mathscr{F}_Q \alpha(\omega) = \omega \cdot (T\alpha^{-1})^m$, where the tangent mapping $T\alpha^{-1}$ is considered at the point $\alpha(x) \in X_2$.

THEOREM 17. *The Euler–Lagrange form of a natural Lagrangian structure is a natural differential operator.*

Theorem 17 is a reformulation of the well-known transformation properties of the Euler–Lagrange expressions. Let $(\varrho^s_{X,P}, \lambda)$ be a natural Lagrangian structure of order r, $X \in \operatorname{Ob} \mathscr{D}_n$, let $\mathscr{E}_\lambda$ denote the Euler–Lagrange form of $(\varrho^s_{X,P}, \lambda)$. Since for each local isomorphism α of X,

$$\mathscr{E}_{(j^r \mathscr{F}^s_P \alpha)^* \lambda} = (j^r \mathscr{F}^s_P \alpha)^* \mathscr{E}_\lambda,$$

and the Lagrangian structure $(\varrho^s_{X,P}, \lambda)$ is natural, we have $(j^r \mathscr{F}^s_P \alpha)^* \mathscr{E}_\lambda = \mathscr{E}_\lambda$ or, which is the same,

$$\mathscr{E}_\lambda \circ j^r \mathscr{F}^s_P \alpha = \mathscr{E}_\lambda \cdot (Tj^r \mathscr{F}^s_P \alpha^{-1})^{n+1}.$$

The right-hand side expression is equal to $\mathscr{F}_Q \alpha \circ \mathscr{E}_\lambda$, where Q denotes the fiber of the bundle

$$\varrho^*_{X,P} \mathscr{F}_{\bar{R}} X \otimes (T^* \mathscr{F}^s_P X \wedge \varrho^*_{X,P} \textstyle\bigwedge^n T^* X)$$

in accordance with the first diagram of Section 1. This shows that $\mathscr{E}_\lambda$ defines a natural differential operator. We note that in general, this differential operator is of order $2r$ (compare with the convention introduced in Section 1).

Our last remark in this section is concerned with an important class of natural Lagrangian structures appearing frequently in practice.

THEOREM 18. *Let* $(\varrho^s_{X,P}, \lambda)$ *be a natural Lagrangian structure of order* r, $X \in \operatorname{Ob} \mathscr{D}_n$. *Assume that the following two conditions hold*:

(1) *P is a vector space endowed with a linear representation of the group* L^s_n.

(2) *The base manifold X is endowed with an everywhere non-zero odd n-form* ω.

Define a contact 1*-form E_λ on $j^r \mathscr{F}^s_P X$ by the formula $\mathscr{E}_\lambda = E_\lambda \wedge \omega$. Then E_λ defines a natural differential operator from $C^\infty(\mathscr{F}^s_P X)$ to $C^\infty(\mathscr{F}^s_{P^*} X)$, where $\mathscr{F}^s_{P^*} X$ is the dual bundle of the vector bundle $\mathscr{F}^s_P X$.*

To show it we may proceed as follows. Assume that we have a Lagrangian structure (π, λ) of order r such that $\pi\colon E \to X$ is a vector bundle, and there exists a π_r-horizontal odd base n-form ω on j^rE such that for each section $\gamma \in C^\infty(Y)$, $j^r\gamma^*\omega$ is an everywhere non-zero odd form on X. Roughly speaking, $j^r\gamma^*\omega$ is a "volume element" on X. Since E is a vector bundle, we have the following exact sequence of vector bundles over E: $0 \to \pi^*E \to TE \to \pi^*TX \to 0$. The kernel of the morphism $TE \to \pi^*TX$ is the bundle VTE of π-vertical vectors over E; that is, π^*E is isomorphic with VTE. Define E_λ as above (see also (7)). Then the forms $\mathscr{E}_\lambda$ and E_λ may be interpreted as the following morphisms of vector bundles:

$$\begin{array}{ccc} \pi^*_{r,0}VTE & \xrightarrow{\mathscr{E}_\lambda} & \mathscr{F}_{\tilde{R}}X \otimes \bigwedge^n T^*X \\ \downarrow & & \downarrow \\ j^rE & \xrightarrow{\pi_r} & X \end{array} \qquad \begin{array}{ccc} \pi^*_{r,0}VTE & \xrightarrow{E_\lambda} & X \times R \\ \downarrow & & \downarrow \\ j^rE & \xrightarrow{\pi_r} & X \end{array}$$

Since $\pi^*_{r,0}VTE \approx \pi^*_{r,0}\pi^*E \approx \pi^*_rE$, E_λ is a linear form on π^*_rE, i.e., a section of π^*_rE, and we have the following diagram:

$$\begin{array}{ccc} \pi^*_rE & \to & E^* \\ \downarrow\uparrow E_\lambda & & \downarrow \\ j^rE & \longrightarrow & X \end{array}$$

This shows that E_λ defines a differential operator from $C^\infty(E)$ to $C^\infty(E^*)$. Now we take $E = \mathscr{F}^s_PX$. Then it is easily seen that under the assumptions of Theorem 18 E_λ is a natural differential operator.

4. Induced variations

Let P be an L^s_n-manifold, $X \in \mathrm{Ob}\,\mathscr{D}_n$, and consider a (not necessarily natural) Lagrangian structure $(\varrho^s_{X,P}, \lambda)$ of order r. Each vector field ξ on X defines its $\mathscr{F}^s_P$-lift $\mathscr{F}^s_P\xi$ which is a $\varrho^s_{X,P}$-projectable vector field on $\mathscr{F}^s_PX$ whose projection is ξ. Let $\gamma \in C^\infty(\mathscr{F}^s_PX)$. The variation of γ generated by $\mathscr{F}^s_P\xi$ is called the variation *induced* by ξ (via the lifting $\mathscr{F}^s_P$). The variation of the action of $(\varrho^s_{X,P}, \lambda)$ *induced* by ξ is the variation of the action generated by $\mathscr{F}^s_P\xi$. In this section we discuss the infinitesimal first variation formula (5) for the induced variations of $(\varrho^s_{X,P}, \lambda)$.

Let (U, φ), $\varphi = (x^i)$, be a chart on X, (V, ψ), a chart on $\mathscr{F}^s_PX$ such that $V = \pi^{-1}(U)$ and $\psi = (x^i, p^\sigma)$. We shall express the first variation formula in terms of these charts and their natural prolongations to $j^r\mathscr{F}^s_PX$.

Let ξ be a vector field on X, let

$$\xi = \xi^i \cdot \frac{\partial}{\partial x^i} \tag{13}$$

be its expression for the chart (U, φ). We shall determine the expression of $\mathscr{F}^s_P\xi$ for the fiber chart (V, ψ). Denote by $(a, p) \to \Psi(a, p)$ the action of L^s_n on P. Let $z \in V$, $x = \pi(z)$, let t_y be the translation of R^n sending $y \in R^n$ to the origin. Write z in the form $z = [j^s_0(\varphi^{-1}t_{-\varphi(x)}), p]$. Then for all sufficiently small t,

$$\mathscr{F}_P^s \alpha_t(z) = \left[j_0^s(\varphi^{-1}t_{-\varphi\alpha_t(x)}),\ \Psi\big(j_0^s(t_{\varphi\alpha_t(x)}\varphi\alpha_t\varphi^{-1}t_{-\varphi(x)}), p\big)\right].$$

Hence

$$x^i \circ \mathscr{F}_P^s \alpha_t(z) = x^i \circ \alpha_t(x),$$
$$p^\sigma \circ \mathscr{F}_P^s \alpha_t(z) = p^\sigma \circ \Psi\big(j_0^s(t_{\varphi\alpha_t(x)}\varphi\alpha_t\varphi^{-1}t_{-\varphi(x)}), p\big).$$

This implies

$$\left\{\frac{d}{dt}(x^i \circ \mathscr{F}_P^s \alpha_t)\right\}_0 = \left\{\frac{d}{dt}(x^i \circ \alpha_t)\right\}_0 = \xi^i,$$

and it remains to compute the coefficients at $\partial/\partial p^\sigma$ in the expression of $\mathscr{F}_P^s \xi$ for the fiber chart (V, ψ). Denote by $a_{j_1}^i, \ldots, a_{j_1 j_2 \ldots j_s}^i$ the canonical coordinates on the group L_n^s and by e the identity of L_n^s. Denoting by $t_{\varphi\alpha_t(x)}^i$ the ith component of the mapping $t_{\varphi\alpha_t(x)}$ we obtain immediately

$$\left\{\frac{d}{dt} p^\sigma \circ \Psi\big(j_0^s(t_{\varphi\alpha_t(x)}\varphi\alpha_t\varphi^{-1}t_{-\varphi(x)}), p\big)\right\}_0$$
$$= \sum_{k=1}^{s} \frac{1}{k!}\left\{\frac{\partial p^\sigma \circ \psi}{\partial a_{j_1 \ldots j_k}^i}\right\}_e \frac{\partial^k}{\partial x^{j_1} \ldots \partial x^{j_k}} \left\{\frac{d}{dt}(t_{\varphi\alpha_t(x)}^i \varphi\alpha_t\varphi^{-1}t_{-\varphi(x)})\right\}_0 .$$

Differentiating the mapping $(t, y) \to (t_{\varphi\alpha_t(x)}^i \varphi\alpha_t\varphi^{-1}t_{-\varphi(x)})(y)$ we obtain the expression $(\partial^k\xi^i)/(\partial x^{j_1} \ldots \partial x^{j_k})$. Consequently,

$$\mathscr{F}_P^s \xi = \xi^i \frac{\partial}{\partial x^i} + \sum_{k=1}^{s} \frac{1}{k!} F_i^{j_1 \ldots j_k \sigma} \cdot \frac{\partial^k \xi^i}{\partial x^{j_1} \ldots \partial x^{j_k}} \cdot \frac{\partial}{\partial p^\sigma}, \tag{14}$$

where $F_i^{j_1 \ldots j_k \sigma}$ are functions on V, uniquely determined by the lifting $\mathscr{F}_Q^s$ and the charts considered.

Consider, for example, a vector space P endowed with a linear representation of the group L_n^1. Let p^σ be the global coordinates on P defined by a basis of the vector space P. Let the linear representation of L_n^1 on P be expressed by the formulas $p^\nu(a \cdot p) = \Psi_\sigma^\nu(a) \cdot p^\sigma(p)$. Then $F_i^{j\sigma} = C_{i\nu}^{j\sigma} \cdot p^\nu$, where $C_{i\nu}^{j\sigma} = \{\partial\Psi_\nu^\sigma/\partial a_j^i\}_e$ are some constants satisfying $C_{k\lambda}^{i\sigma} \cdot C_{p\sigma}^{j\nu} = \delta_p^i \cdot C_{k\lambda}^{j\nu}$.

We are now in a position to express the infinitesimal first variation formula of the Lagrangian structure $(\varrho_{X,P}^s, \lambda)$ in terms of the fiber chart (V, ψ) and its natural prolongation (V_r, ψ_r) to $j^r\mathscr{F}_P^s X$. As usual, $\lambda = \tilde{\varphi} \otimes \mathscr{L}\omega_0$, where $\omega_0 = dx^1 \wedge \ldots \wedge dx^n$ and $\mathscr{L}$: $V_r \to R$ is a function. Similarly, $\mathscr{E}_\lambda = \tilde{\varphi} \otimes \mathscr{E}_\sigma(\mathscr{L}) \cdot dp^\sigma \wedge \omega_0$, where $\mathscr{E}_\sigma(\mathscr{L})$ are the Euler–Lagrange expressions relative to (V, ψ). Let θ_λ be the generalized Poincaré–Cartan equivalent of λ. Recall that in our coordinates,

$$\theta_\lambda = \tilde{\varphi} \otimes \Big(\mathscr{L}\omega_0 + \sum_{i=1}^{n} \sum_{k=0}^{r-1} \sum_{j_1 \leqslant \ldots \leqslant j_k} f_\sigma^{ij_1 \ldots j_k} \omega_{j_1 \ldots j_k}^\sigma \wedge \omega_i\Big),$$

where for all k, $1 \leqslant k \leqslant r-1$,

$$\omega_i = (-1)^{i-1} dx^1 \wedge \ldots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \ldots \wedge dx^n,$$

$$\omega^\sigma_{j_1 \dots j_k} = dp^\sigma_{j_1 \dots j_k} - p^\sigma_{j_1 \dots j_k i} dx^i,$$

$$f^{j_1 \dots j_r}_\sigma = 0, \quad f^{j_1 \dots j_k}_\sigma = \frac{\partial \mathscr{L}}{\partial p^\sigma_{j_1 \dots j_k}} - d_i f^{ij_1 \dots j_k}_\sigma.$$

Let ξ be a vector field on X. Using (13) and (14) we obtain after some calculation

$$h(i_\xi d\theta_\lambda) = \tilde{\varphi} \otimes \mathscr{E}_\sigma(\mathscr{L}) \cdot \left(\sum_{k=1}^{s} \frac{1}{k!} F^{j_1 \dots j_k \sigma}_i \frac{\partial^k \xi^i}{\partial x^{j_1} \dots \partial x^{j_k}} - p^\sigma_i \xi^i \right) \cdot \omega_0.$$

But

$$\mathscr{E}_\sigma(\mathscr{L}) \cdot F^{j_1\sigma}_i \frac{\partial \xi^i}{\partial x^j} = d_j(\mathscr{E}_\sigma(\mathscr{L}) \cdot F^{j_1\sigma}_i \cdot \xi^i) - d_j(\mathscr{E}_\sigma(\mathscr{L}) \cdot F^{j_1\sigma}_i) \cdot \xi^i,$$

$$\mathscr{E}_\sigma(\mathscr{L}) \cdot F^{j_1 j_2 \sigma}_i \frac{\partial^2 \xi^i}{\partial x^{j_1} \partial x^{j_2}} = d_j \left(\mathscr{E}_\sigma(\mathscr{L}) \cdot F^{j_1 j_2 \sigma}_i \frac{\partial \xi^i}{\partial x^{j_2}} \right) -$$
$$- d_{j_2}(d_{j_1}(\mathscr{E}_\sigma(\mathscr{L}) \cdot F^{j_1 j_2 \sigma}_i)\xi^i) + d_{j_2} d_{j_1}(\mathscr{E}_\sigma(\mathscr{L}) \cdot F^{j_1 j_2 \sigma}_i)\xi^i, \quad \dots,$$

so we have

$$h(i_\xi d\theta_\lambda) = \tilde{\varphi} \otimes \Big[\big(-\mathscr{E}_\sigma(\mathscr{L}) \cdot p^\sigma_i - d_{j_1}(\mathscr{E}_\sigma(\mathscr{L}) \cdot F^{j_1\sigma}_i) +$$
$$+ d_{j_2} d_{j_1}(\mathscr{E}_\sigma(\mathscr{L}) \cdot F^{j_1 j_2 \sigma}_i) - \dots \big) \cdot \xi^i + d_{j_1}(\mathscr{E}_\sigma(\mathscr{L}) \cdot F^{j_1\sigma}_i \xi^i) +$$
$$+ d_{j_1}\left(\mathscr{E}_\sigma(\mathscr{L}) \cdot F^{j_1 j_2 \sigma}_i \frac{\partial \xi^i}{\partial x^{j_2}} \right) - d_{j_2}(d_{j_1}(\mathscr{E}_\sigma(\mathscr{L}) \cdot F^{j_1 j_2 \sigma}_i \cdot \xi^i) + \dots) \Big] \cdot \omega_0.$$

Similarly

$$h(di_\xi \theta_\lambda) = \tilde{\varphi} \otimes d_i \left(\mathscr{L}\xi^i + \sum_{i=1}^{n} \sum_{k=0}^{r-1} \sum_{j_1 \leqslant \dots \leqslant j_k} f^{ij_1 \dots j_k}_\sigma \cdot d_{j_1} \dots d_{j_k} \left(F^{m\sigma}_j \frac{\partial \xi^j}{\partial x^m} - p^\sigma_m \xi^m \right) \right) \cdot \omega_0.$$

Summarizing these calculations, we obtain the following chart expression for the Lie derivative of the Lagrangian λ relative to the vector field $j^r \mathscr{F}^s_P \xi$:

$$\text{(15)} \quad \partial_\xi \lambda = \tilde{\varphi} \otimes \Big[\Big(-\mathscr{E}_\sigma(\mathscr{L}) \cdot p^\sigma_i - d_{j_1}(\mathscr{E}_\sigma(\mathscr{L}) \cdot F^{j_1\sigma}_i) + d_{j_2} d_{j_1}(\mathscr{E}_\sigma(\mathscr{L}) \cdot F^{j_1 j_2 \sigma}_i - \dots \Big) \cdot \xi^i +$$
$$+ d_i \left(\mathscr{E}_\sigma(\mathscr{L}) \cdot F^{i\sigma}_m \xi^m + \mathscr{E}_\sigma(\mathscr{L}) \cdot F^{ij_1\sigma}_m \frac{\partial \xi^m}{\partial x^{j_1}} \right) - d_{j_1}(\mathscr{E}_\sigma(\mathscr{L}) \cdot F^{ij_1\sigma}_m \xi^m) + \dots + \mathscr{L}\xi^i +$$
$$+ \sum_{i=1}^{n} \sum_{k=0}^{r-1} \sum_{j_1 \leqslant \dots \leqslant j_k} f^{ij_1 \dots j_k}_\sigma d_{j_1} \dots d_{j_k} \left(F^{m\sigma}_j \frac{\partial \xi^j}{\partial x^m} - p^\sigma_m \xi^m \right) \Big) \Big] \cdot \omega_0.$$

Notice that this representation of the Lie derivative of λ contains two terms, the first one depending linearly on ξ, and the second one of the form of exterior derivative of an odd base $(n-1)$-form.

In the following theorem the chart expressions of ξ, $\mathscr{F}^s_P \xi$, λ, and $\mathscr{E}_\lambda$ are assigned to a fiber chart (V, ψ), $\psi = (x^i, p^\sigma)$, on $\mathscr{F}^s_P X$ in the same manner as above.

THEOREM 19. *The following four conditions are equivalent*:

(1) *For each vector field* ξ *on* X, $\partial_\xi \lambda = 0$.

(2) *For each vector field* ξ *on* X,

$$h(i_\xi d\theta_\lambda) + h(di_\xi \theta_\lambda) = 0.$$

(3) *For each fiber chart* (V, ψ), $\psi = (x^i, p^\sigma)$ *on* $\mathscr{F}^s_P X$,

$$\text{(16)} \qquad -\mathscr{E}_\sigma(\mathscr{L}) \cdot p^\sigma_i - d_{j_1}(\mathscr{E}_\sigma(\mathscr{L}) \cdot F^{j_1\sigma}_i) + d_{j_2} d_{j_1}(\mathscr{E}_\sigma(\mathscr{L}) \cdot F^{j_1 j_2 \sigma}_i) - \ldots = 0.$$

(4) *For each vector field* ξ *on* X *and each fiber chart* (V, ψ), $\psi = (x^i, p^\sigma)$ *on* $\mathscr{F}^s_P X$,

$$\text{(17)} \qquad d_i\Big(\mathscr{L}\xi^i + \sum_{l=1}^{n} \sum_{k=0}^{r-1} \sum_{j_1 \leqslant \ldots \leqslant j_k} f^{lj_1 \ldots j_k}_\sigma d_{j_1} \ldots d_{j_k}\Big(F^{m\sigma}_j \frac{\partial \xi^j}{\partial x^m} - p^\sigma_m \xi^m\Big) +$$

$$+ \mathscr{E}_\sigma(\mathscr{L}) \cdot F^{i\sigma}_m \xi^m + \mathscr{E}_\sigma(\mathscr{L}) \cdot F^{ij_1\sigma}_m \frac{\partial \xi^m}{\partial x^j} - d_{j_1}(\mathscr{E}_\sigma(\mathscr{L}) \cdot F^{ij_1\sigma}_m \xi^m) + \ldots\Big) = 0.$$

The equivalence of (1) and (2) follows from the first variation formula. To prove the equivalence of (1) and (3), notice that for each piece $\Omega \subset X$ such that $\Omega \subset U$, each section γ of $\mathscr{F}^s_P X$ over Ω, and each vector field ξ on U such that all the partial derivatives of its components ξ^i relative to (U, φ), of order $\leqslant r$, vanish along the boundary $\partial\Omega$ of Ω,

$$\int_\Omega j^r\gamma^* \partial_\xi \lambda = \int_\Omega j^r\gamma^* A_i \cdot \xi^i \cdot \tilde{\varphi} \otimes \omega_0,$$

where A_i is the expression on the left-hand side of (16). This relation is a direct consequence of the chart expression of $\partial_\xi \lambda$ and the Stokes theorem. Now if (1) holds, then obviously (3) must hold in the interior of each Ω, and hence everywhere. Conversely, assume that (3) holds and there exist a vector field ξ and a point of $j^r \mathscr{F}^s_P X$ at which $\partial_\xi \lambda \neq 0$. Then there exists a piece $\Omega \subset X$ containing the projection of this point, and a section γ of $\mathscr{F}^s_P X$ over Ω such that

$$\int_\Omega j^r\gamma^* \partial_\xi \lambda \neq 0$$

which is a contradiction. Hence (3) implies (1). The equivalence of (4) and (1) follows from the chart expression of $\partial_\xi \lambda$ (15).

A *natural* Lagrangian structure satisfies each of the four equivalent conditions of Theorem 19.

Let us assume that we have a natural Lagrangian structure $(\varrho^s_{X,P}, \lambda)$. With the notation of Theorem 19, put for each vector field ξ on X, locally,

$$\eta_\xi = i_\xi \theta_\lambda.$$

Then, by condition (2),

$$i_\xi d\theta_\lambda = -d\eta_\xi + \nu_\xi,$$

where ν_ξ is a contact form. That is, η_ξ is a first integral of $(\varrho^s_{X,P}, \lambda)$ and the vector field $\mathscr{F}^s_P \xi$ is related to this first integral. This assertion is in fact the first theorem of E. Noether applied to the transformations induced by the lifting.

We may easily obtain a chart expression for η_ξ over a point $x \in X$, where $\xi(x) \neq 0$. About such a point there exists a chart (U, φ), $\varphi = (x^l)$, such that $\xi = \partial/\partial x^1$, and we have

$$i_\xi \theta_\lambda = \tilde{\varphi} \otimes \Big(\mathscr{L}\omega_0 - \sum_{l=1}^{n} \sum_{k=0}^{r-1} \sum_{j_1 \leqslant \ldots \leqslant j_k} f_\sigma^{lj_1 \ldots j_k} (p^\sigma_{j_1 \ldots j_k 1} \cdot \omega_l + \omega^\sigma_{j_1 \ldots j_k} \wedge i_{(\partial/\partial x^1)} \omega_l)\Big).$$

Let ξ be a vector field on X. Then for each $\gamma \in C^\infty_{\mathrm{crit}}(\mathscr{F}^s_P X)$, $dj^r\gamma^* \eta_\xi = 0$. This relation is called the *weak conservation law* associated with ξ.

Consider conditions (3) and (4) of Theorem 19. Relations (16) are called *generalized Bianchi identities* of the natural Lagrangian structure $(\varrho^s_{X,P}, \lambda)$. Since the expressions on the left-hand side of (17) are of "divergence type", that is, are of the form $h(d\chi)$ for some odd base $(n-1)$-form χ, relations (17) may also be regarded as some "conservation laws". In accordance with standard terminology, we call them *strong conservation laws* of $(\varrho^s_{X,P}, \lambda)$. Each strong conservation law may be rewritten in the form $h(d\Phi_\xi) = 0$, where Φ_ξ is an odd base $(n-1)$-form on $j^r\mathscr{F}^s_P X$. By an appropriate Poincaré lemma, this is equivalent to saying that there exists an odd base $(n-2)$-form ψ_ξ such that $\Phi_\xi = d\psi_\xi + \chi_\xi$, where χ_ξ is a contact form. Each such a form ψ_ξ is called a *superpotential* of the natural Lagrangian structure $(\varrho^s_{X,P}, \lambda)$.

Note that both systems of identities (16) and (17) hold universally in the considered category $\mathscr{D}_n$ of differential manifolds.

The assertion that if $(\varrho^s_{X,P}, \lambda)$ is natural then the generalized Bianchi identities, or equivalently, the strong conservation laws hold, is the *second theorem of E. Noether* for natural Lagrangian structures.

References

[1] V. Aldaya, J. A. de Azcárraga, *Variational principles on r-th order jets of fiber bundles in field theory*, J. Math. Phys. 19 (1978), 1869–1875.

[2] —, —, *Vector bundles, r-th order Noether invariants and canonical symmetries in Lagrangian field theory*, ibid. 19 (1978), 1876–1880.

[3] N. Bourbaki, *Variétés differéntielles et analytiques. Fascicule de résultats*, Paragraphes 1 à 7, Hermann, Paris 1967; Paragraphes 8 à 15, Hermann, Paris 1971.

[4] P. Dedecker, *On the generalization of symplectic geometry to multiple integrals in the calculus of variations*, Lecture Notes in Math. Vol. 570, Springer, Berlin 1977, 395–456.

[5] P. L. Garcia, *The Poincaré-Cartan invariant in the calculus of variations*, Symposia Math. 14 (1974), 219–246.

[6] P. L. Garcia, A. Peréz-Rendón, *Symplectic approach to the theory of quantized fields*, I, Comm. Math. Phys. 13 (1969), 24–44; Arch. Rational Mech. Anal. 43 (1971), 101–124.

[7] H. Goldschmidt, S. Sternberg, *The Hamilton–Cartan formalism in the calculus of variations*, Ann. Inst. Fourier 23 (1973), 203–267.

[8] R. Hermann, *Geometry, Physics and Systems*, Dekker, New York 1973.

[9] D. Hilbert, *Die Grundlagen der Physik*, Nachrichten von der Kön. Ges. der Wissenschaften zu Göttingen, Math.-Phys. Kl. (1915), 395–407.

[10] M. Horák, D. Krupka, *First order invariant Einstein–Cartan structures*, Internat. J. Theoret. Phys. 17 (1978), 573–583.

[11] G. W. Horndeski, D. Lovelock, *Scalar-tensor field theories*, Tensor, N.S. 24 (1972), 79–92.

[12] J. Horský, J. Novotný, *Conservation laws in general relativity*, Czechoslovak J. Phys. B19 (1969), 419–442.

[13] J. Kijowski, *On a new variational principle in general relativity and the energy of the gravitational field*, General Relativity and Gravitation 9 (1978), 857–877.

[14] I. Kolář, *On the Euler–Lagrange differential in fibered manifolds*, Rep. Math. Phys. 12 (1977), 301–305.

[15] —, *Structure morphisms of prolongation functors*, Math. Slovaca 30 (1980), 83–93.

[16] J. Komorowski, *A geometric formulation of the general free boundary problems in the calculus of variations and the theorems of E. Noether connected with them*, Rep. Math. Phys. 1 (1970), 105–132.

[17] D. Krupka, *A geometric theory of ordinary first order variational problems in fibered manifolds*, I. *Critical sections*; II. *Invariance*, J. Math. Anal. Appl. 49 (1975), 180–206; 469–476.

[18] —, *A setting for generally invariant Lagrangian structures in tensor bundles*, Bull. Acad. Polon. Sci. Sér. Sci Math. Astronom. Phys. 22 (1974), 967–972.

[19] —, *A theory of generally invariant Lagrangians for the metric fields*, II, Internat. J. Theoret. Phys. 15 (1976), 949–959; I, Internat. J. Theoret. Phys. 17 (1978), 359–368.

[20] —, *Elementary theory of differential invariants*, Arch. Math. (Brno) 14 (1978), 207–214.

[21] —, *Natural Lagrangian structures* (Czech), DrSc-dissertation, Mathematical Institute of the ČSAV, Prague 1978.

[22] —, *Reducibility theorems for differentiable liftings in fibre bundles*, Arch. Math. (Brno) 15 (1979), 93–106.

[23] —, *Some geometric aspects of variational problems in fibered manifolds*, Folia Fac. Sci. Nat. UJEP Brunensis (Physica) 14 (1973), 1–65.

[24] D. Krupka, A. Trautman, *General invariance of Lagrangian structures*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 22 (1974), 207–211.

[25] B. A. Kupershmidt, *Lagrangian formalism in the calculus of variations* (Russian), Functional Anal. Appl. 10 (1976), 77–78.

[26] Th. H. J. Lepage, *Sur les champs geodésiques du calcul des variations*, I, II, Bull. Acad. R. Belg. Cl. des Sc. 22 (1936), 716–793; 1036–1046.

[27] P. Libermann, *Prolongements des fibrés principiaux et des grupoides différentiables*, Séminaire Analyse Globale, Montreal 1969.

[28] D. Lovelock, *Degenerate Lagrange densities involving geometric objects*, Arch. Rational Mech. Anal. 36 (1970), 293–304.

[29] —, *The Euler–Lagrange expressions and degenerate Lagrange densities*, J. Australian Math. Soc. 14 (1972), 482–495.

[30] —, *Vector-tensor field theories and the Einstein–Maxwell field equations*, Proc. Roy. Soc. London A341 (1974), 285–297.

[31] A. Nijenhuis, *Natural bundles and their general properties, Geometric objects revisited*, in: *Differential Geometry*, in honour of K. Yano, Kinokuniya, Tokyo 1972, 317–334.

[32] E. Noether, *Invariante Variationsprobleme*, Nachrichten von der Kön. Ges. der Wissenschaften zu Göttingen, Math.-Phys. Kl. (1918), 235–258.

[33] J. N o v o t n ý, *On the generally invariant Lagrangians for the metric field and other tensor fields*, Internat. J. Theoret. Phys. 17 (1978), 677–684.

[34] P. R. R o d r i g u e s, *On generating forms of k-generalized Lagrangian and Hamiltonian systems*, J. Math. Phys. 18 (1977), 1720–1723.

[35] H. R u n d, *Invariant theory of variational problems for geometric objects*, Tensor, N.S. 18 (1967), 239–258.

[36] H. R u n d, D. L o v e l o c k, *Variational principles in the general theory of relativity*, Über Deutsch. Math.-Verein. 74 (1972), 1–65.

[37] W. S z c z y r b a, *On the geometric structure of the set of solutions of Einstein equations*, Diss. Math. 150, Warszawa 1977.

[38] F. T a k e n s, *Symmetries, conservation laws and variational principles*, Lecture Notes in Math. Vol. 597, Springer, Berlin 1978.

[39] C h u u - L i a n T e r n g, *Natural vector bundles and natural differential operators*, Amer. J. Math. 100 (1978), 775–828.

[40] A. T r a u t m a n, *Invariance of Lagrangian systems*, in: *General Relativity*, in honour of J.L. Synge, Clarendon Press, Oxford 1972, 86–99.

[41] —, *Foundations and current problems of general relativity*, in: A. Trautman, F. A. E. Pirani, H. Bondi, *Lectures on General Relativity*, Brandeis Summer Institute in Theoretical Physics, Prentice Hall, N.J. 1965.

Presented to the Semester
Differential Geometry
(September 17–December 15, 1979)