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# CHARACTERISTIC CLASSES OF FOLIATIONS PRESERVED BY A TRANSVERSE $k$-FIELD

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A vector field $X$ is said to *preserve a smooth foliation* $F$ if, for any vector field $Y$ tangent to $F$, the Lie bracket $[X, Y]$ is again tangent to $F$. In [4] and [5] C. Lazarov and H. Shulman studied consequences of existence of sufficiently many vector fields preserving a given foliation. They proved the following:

([5]) If there exists a transverse $k$-field $X_1, \ldots, X_k$ preserving a foliation $F$, and there are constant coefficients $c_{ij}^h$, $i, j, h = 1, \ldots, k$, such that $[X_i, X_j] = c_{ij}^h X_h$, then the exotic classes $\alpha_F[h_I C_J]$ corresponding to $h_I C_J \in \mathrm{WO}_q$ vanish whenever $\deg C_J > 2(q-k)$. Here $q = \operatorname{codim} F$ and $\alpha_F$ is the Bott characteristic homomorphism for $F$ ([2]).

([4]) If the $k$-field satisfies $\forall\, i, j,\ [X_i, X_j] = 0$, then $\alpha_F[h_I C_J] = 0$ for $[h_I C_J] \in \ker\big(H(\mathrm{WO}_q) \to H(\mathrm{WO}_{q-k})\big)$.

The aim of the present paper is to remove unnecessary conditions from the above statements and to prove the following, much stronger, result:

THEOREM. *Let a smooth foliation $F$ of a manifold $M$ have a codimension $q$. Assume that $F$ admits a transverse $k$-field $X_1, \ldots, X_k$ composed of vector fields preserving $F$. Assume moreover that there is another foliation $F'$ of $M$ whose tangent bundle is the Whitney sum of the tangent bundle of $F$ and the bundle spanned by the $k$-field. Then the characteristic homomorphism*

$$\alpha_F \colon H(\mathrm{WO}_q) \to H^*(M)$$

*of $F$ is a superposition of*

$$H(\mathrm{WO}_q) \to H(\mathrm{WO}_{q-k}) \quad \textit{and} \quad \alpha_{F'}.$$

*If, moreover, $F$ is framed by a homotopy class $t$ of trivializations of its dual normal bundle, then $F'$ carries a canonical frame structure $t'$ such that the characteristic homomorphism*

$$\alpha_{(F,t)} \colon H(\mathrm{W}_q) \to H^*(M)$$

*of $(F, t)$ is a superposition of*

$$H(\mathrm{W}_q) \to H(\mathrm{W}_{q-k}) \quad \text{and} \quad \alpha_{(F', t')}.$$

COROLLARY. *Under the assumptions of the Theorem, $\alpha_F$ annihilates* $\ker\big(H(\mathrm{WO}_q) \to H(\mathrm{WO}_{q-k})\big)$, *and respectively $\alpha_{(F,t)}$ annihilates* $\ker\big(H(\mathrm{W}_q) \to H(\mathrm{W}_{q-k})\big)$.

Before proving the Theorem we shall give a short exposition of some necessary notions and results of [1].

*Pontryagin classes of smooth foliations* ([1]). Let us fix an arbitrary smooth foliation $F$ of a codimension $q$ of a manifold $M$. Let $E_F \subset TM$ and $Q_F^* \subset T^*M$ be respectively its tangent bundle and its dual normal bundle ($Q_F^*$ consists of covectors annihilating the fibers of $E_F$). We shall denote by $I_F = I_F^1$ (resp. $I_F^h$, $h = 2, 3, \ldots$) the ideal in the algebra $A^*(M)$ of differential forms on $M$, generated by all global sections of $Q_F^*$ (resp. by forms that are locally products of at least $h$ sections of $Q_F^*$). We also put $I_F^0 := A^*(M)$. Since the exterior differentiation $d$ preserves all the above ideals, one can consider the appropriate homologies. Note that the direct sum $\bigoplus_{i=0}^{q} H^{2i}(I_F^i)$ carries a canonical structure of a commutative algebra over $\boldsymbol{R}$, the multiplication being induced from the skew product of differential forms:

$$H^{2i}(I_F^i) \otimes H^{2j}(I_F^j) \ni [\varphi]_i \otimes [\psi]_j \mapsto [\varphi \wedge \psi]_{i+j} \in H^{2(i+j)}(I_F^{i+j}).$$

Here and forever we accept the notation $[\mu]_h$ for the class of $\mu \in I_F^h$ in $H^*(I_F^h)$, $h = 0, 1, \ldots$

The exterior differentiation $d$ induces also a boundary operator in the quotient algebras $A^*(M)/I_F^h$ and so the homology groups $H^*(A^*(M)/I_F^h)$, $h = 1, 2, \ldots$, are well defined.

With each $\mathrm{Gl}(q)$-invariant polynomial $a \in I^h(\mathrm{Gl}(q))$ ($h = \deg a$) we associate the element $[\lambda(\nabla)(a)]_h$ of $H^{2h}(I_F^h)$, $\nabla$ being any Bott connection in $Q_F^*$, which means that $\nabla_v \varphi = i_v d\varphi$ for $v \in E_F$ and all sections $\varphi$ of $Q_F^*$ (see [2] for the definition of the operation $\lambda$). In particular, each Pontryagin polynomial $p_i^{(q)} \in I^i(\mathrm{Gl}(q))$, $i = 0, 1, \ldots$ ($\sum p_i^{(q)}(A) z^i = \det(I + zA)$; $p_0^{(q)} \equiv 1$ and $p_i^{(q)} = 0$ if $i > q$), corresponds to an element of $H^{2i}(I_F^i)$ called the *i-th Pontryagin class of $F$* and denoted by $p_i(F)$ ([1]). We shall use the notation $p(F)$ for the formal sum $1 + p_1(F) + \ldots + p_q(F) \in \bigoplus H^{2i}(I_F^i)$.

With those polynomials (which vanish when restricted to the Lie subalgebra $\mathfrak{o}(q) \subset \mathfrak{gl}(q)$ of skew-symmetric matrices) we associate secondary invariants of $F$. The rule is as follows:

$$\ker\big(I^h(\mathrm{Gl}(q)) \to I^h(\mathrm{O}(q))\big) \ni b \mapsto [\lambda(\nabla^1, \nabla)(b) + I_F^h] \in H^{2h-1}\big(A^*(M)/I_F^h\big)$$

for $h = 1, 2, \ldots$, where $\nabla$ and $\nabla^1$ stand respectively for any Bott connection and any metric connection in $Q_F^*$. Secondary invariants corresponding with odd Pontryagin polynomials will be called *secondary Pontryagin classes of $F$* and denoted by $s_{2i+1}(F)$:

$$s_{2i+1}(F) = [\lambda(\nabla^1, \nabla)(p_{2i+1}^{(q)}) + I_F^{2i+1}] \in H^{4i+1}\big(A^*(M)/I_F^{2i+1}\big), \quad i = 0, 1, \ldots$$

If the foliation $F$ is framed by a homotopy class $t$ of trivializations of $Q_F^*$, then there are well-defined secondary invariants

$$s_i(F, t) = [\lambda(\nabla^2, \nabla)(p_i^{(q)}) + I_F^i], \quad i = 1, 2, \ldots,$$

called *secondary Pontryagin classes of* $(F, t)$. Here $\nabla^2$ stands for any connection flat with respect to a trivialization from $t$. Moreover, there is $s_{2i+1}(F, t) = s_{2i+1}(F)$ for all $i$.

The exotic characteristic classes of $F$ (resp. of $(F, t)$) are determined completely by its Pontryagin classes and secondary Pontryagin classes.

PROPOSITION 1. *If* $\alpha_F\colon H(\mathrm{WO}_q) \to H^*(M)$ *(resp.* $\alpha_{(F,t)}\colon H(\mathrm{W}_q) \to H^*(M)$*) is the Bott characteristic homomorphism of* $F$ *(resp. of* $(F, t)$*) and the class of* $h_I C_J \in \mathrm{WO}_q$ *(resp.* $h_I C_J \in \mathrm{W}_q$*),* $I = (i_1, \ldots, i_m)$, $J = (j_1, \ldots, j_n)$, *is one of the generators of the homology group, then* $\alpha_F[h_I C_J]$ *(resp.* $\alpha_{(F,t)}[h_I C_J]$*) is the image of*

$$s_{i_1} \bullet (s_{i_2} \bullet \ldots (s_{i_m} \bullet p_{j_1} \ldots p_{j_n}) \ldots) \in H^*(I_F^{|J|}),$$

$|J| = j_1 + \ldots + j_n$, *under the mapping* $H^*(I_F^{|J|}) \to H^*(M)$ *induced by the inclusion* $I_F^{|J|} \hookrightarrow A^*(M)$. *Here* $s_i = s_i(F)$ *(resp.* $s_i = s_i(F, t)$*),* $p_j = p_j(F)$, *and the bold dot denotes the following product operation*:

$$H^a(A^*(M)/I_F^b) \otimes H^c(I_F^d) \to H^{a+c}(I_F^d), \qquad [\varphi + I_F^b] \otimes [\psi]_d \mapsto [\varphi \wedge \psi]_d,$$

*which is defined for all positive integers* $a$, $b$, $c$, $d$ *such that* $b+d > q$.

Generalizing the notion of a Bott connection for a single foliation to the case of a pair composed of a foliation and its subfoliation, we are able to construct characteristic invariants of such pairs. Indeed, let $F$ and $F'$ be such foliations of a manifold $M$ that $E_F \subset E_{F'}$ or, equivalently, $Q_{F'}^* \subset Q_F^*$. Let us consider the quotient vector bundle $Q_{F/F'}^* := Q_F^*/Q_{F'}^*$. Its fibre dimension $q''$ equals $\operatorname{codim} F - \operatorname{codim} F'$. We shall denote the cosets of elements and sections of $Q_F^*$ by adding the mark "$\sim$" (tilde).

A connection $\nabla$ in the vector bundle $Q_{F/F'}^*$ will be called a *Bott connection* ([1]) if it satisfies the condition

$$\nabla_v \tilde{\varphi} = [i_v(d\varphi)]^\sim \quad \text{for} \quad \varphi \in \Gamma(Q_F^*) \quad \text{and} \quad v \in E_F.$$

Modifying the definition of Pontryagin classes and secondary Pontryagin classes of a foliation, we shall call the invariants

$$p_i(F/F') := [\lambda(\nabla)(p_i^{(q'')})]_i \in H^{2i}(I_F^i), \quad i = 0, 1, \ldots, q'', \ldots,$$

and

$$s_{2h+1}(F/F') := [\lambda(\nabla^1, \nabla)(p_{2h+1}^{(q'')}) + I_F^{2h+1}] \in H^{4h+1}(A^*(M)/I_F^{2h+1}),$$

$h = 0, 1, \ldots,$ respectively relative Pontryagin classes and relative secondary Pontryagin classes of $F$ modulo $F'$. Here $\nabla$ stands for any Bott connection and $\nabla^1$ for any metric connection in $Q_{F/F'}^*$. We shall use the notation $p(F/F')$ for the formal sum

$$1 + p_1(F/F') + \ldots + p_{q''}(F/F') \in \bigoplus_{i=0}^{q} H^{2i}(I_F^i).$$

If $t''$ is a homotopy class of trivializations of $Q^*_{F/F'}$ (if exists any), then the formula

$$s_i(F/F', t'') := [\lambda(\nabla^2, \nabla)(p_i^{(q'')}) + I_F^i], \quad i = 1, 2, \ldots,$$

$\nabla^2$ being any connection in $Q^*_{F/F'}$, flat with respect to a section from $t''$, defines relative secondary Pontryagin classes of $F$ modulo $F'$ with respect to $t''$. As in the previous case, there is $s_{2h+1}(F/F', t'') = s_{2h+1}(F/F')$ for all $h$.

The condition $Q^*_{F'} \subset Q^*_F$ implies $I^i_{F'} \subset I^i_F$ for all $i$, that induces an algebra homomorphism

$$i_F^{F'}\colon \oplus H^{2i}(I^i_{F'}) \to \oplus H^{2i}(I^i_F).$$

We shall also use the symbol $i_F^{F'}$ to denote any (the appropriate one) of the homomorphisms

$$H^{2i-1}(A^*(M)/I^i_{F'}) \to H^{2i-1}(A^*(M)/I^i_F)$$

induced from the quotient mapping $A^*(M)/I^i_{F'} \to A^*(M)/I^i_F$, $i = 1, 2, \ldots$

To formulate the next proposition it is necessary to introduce one more multiplication. It is induced by the mappings

$$H^{2i-1}(A^*(M)/I^i_F) \otimes H^{2j}(I^j_F) \to H^{2(i+j)-1}(A^*(M)/I^{i+j}_F),$$
$$[\varphi + I^i_F] \otimes [\psi] \mapsto [\varphi \wedge \psi + I^{i+j}_F],$$

$i = 1, 2, \ldots; j = 0, 1, \ldots$, and will be denoted by a dot.

PROPOSITION 2. *If $F$ and $F'$ are smooth foliations of a manifold $M$, such that $E_F \subset E_{F'}$, then the following formulas hold*:

$$p(F) = p(F/F') \cdot i_F^{F'} p(F'),$$

*and*

$$s_{2h+1}(F) = \sum_{i+j=h} i_F^{F'} s_{2i+1}(F') \cdot p_{2j}(F/F') + s_{2j+1}(F/F') \cdot i_F^{F'} p_{2i}(F')$$

*for $h = 0, 1, \ldots$ If there are homotopy classes $t$, $t'$, $t''$ of trivializations of $Q^*_F$, $Q^*_{F'}$ and $Q^*_{F/F'}$, respectively, related in the sense that there is a trivialization $(\psi^1, \ldots, \psi^q) \in t$ such that $(\psi^1, \ldots, \psi^{q'}) \in t'$ and $(\tilde{\psi}^{q'+1}, \ldots, \tilde{\psi}^q) \in t''$, $q = \operatorname{codim} F$ and $q' = \operatorname{codim} F'$, then, moreover,*

$$s_i(F, t) = i_F^{F'} s_i(F', t') + \sum_{j=0}^{i-1} s_{i-j}(F/F', t'') \cdot i_F^{F'} p_j(F')$$

*for $i = 1, 2, \ldots$*

P r o o f of the Theorem. Let us observe that the $k$-field $X_1, \ldots, X_k$ gives rise to a trivialization of the quotient vector bundle $Q^*_{F/F'}$. Namely, splitting $TM$ into $E_{F'}$ and some vector bundle $N$, one can find 1-forms $\varphi^1, \ldots, \varphi^k \in A^*(M)$ that annihilate $E_F \oplus N$ and such that for all $i, j \leqslant k$, $\varphi^i(X_j) = \delta^i_j$. Consequently, $\varphi^1, \ldots, \varphi^k$ are sections of $Q^*_F$ and $(\tilde{\varphi}^1, \ldots, \tilde{\varphi}^k)$ is a trivialization of $Q^*_{F/F'}$ independent, as one can easily check, of the choice of $N$. Let $t''$ denote its homotopy class.

LEMMA. *The only connection $\nabla$ in $Q^*_{F/F'}$ such that $\nabla\tilde{\varphi}^i = 0$, $i = 1, \ldots, k$, is a Bott connection.*

Indeed, if $Y$ is any vector field tangent to $F$, then there is

$$(i_Y d\varphi^i)(X_j) = d\varphi^i(Y, X_j) = Y\varphi^i(X_j) - X_j\varphi^i(Y) - \varphi^i([Y, X_j]) = 0$$

for $i, j \leqslant k$, as the vector fields $X_j$ preserve the foliation. Consequently, the forms $i_Y d\varphi^i$, $i \leqslant k$, are sections of $Q^*_{F'}$, and so

$$\nabla_Y \tilde{\varphi}^i = 0 = (i_Y d\varphi^i)^{\sim}.$$

It is an immediate consequence of the Lemma that $p(F/F') = 1$ and all the relative secondary Pontryagin classes of $F$ modulo $F'$ with respect to $t''$ must vanish. By Proposition 2, this implies $p(F) = i_F^{F'} p(F')$ and $s_{2h+1}(F) = i_F^{F'} s_{2h+1}(F')$ for $h = 0, 1, \ldots$ (respectively if $F$ is framed by $t$, and $t'$ comes of $t$ and $\varphi^1, \ldots, \varphi^k$, then $s_i(F, t) = i_F^{F'} s_i(F', t')$ for $i = 1, 2, \ldots$; note that $t'$ depends on $t$ and the $k$-field only). Finally, applying Proposition 1 we conclude that $\alpha_F[h_I C_J] = \alpha_{F'}[h_I C_J]$ for all admissible $h_I C_J \in \mathrm{WO}_q$ (respectively $\alpha_{(F,t)}[h_I C_J] = \alpha_{(F',t')}[h_I C_J]$ for admissible $h_I C_J \in \mathrm{W}_q$). This ends the proof of the Theorem.

*Added in print.* Lately, the same generalization of the Lazarov–Shulman theorem has been independently proved by Cordero and Masa ([3]).

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