

PROJECTIVELY INVARIANT DISTANCES FOR AFFINE AND PROJECTIVE STRUCTURES

SHOSHICHI KOBAYASHI*

Department of Mathematics, University of California, Berkeley, Calif., U.S.A.

1. Introduction

The Poincaré metric

$$(1.1) \quad ds^2 = \frac{4dz d\bar{z}}{(1-|z|^2)^2}$$

in the unit disk gives a model for hyperbolic geometry in which the geodesics are the circular arcs meeting the boundary of the disk perpendicularly. In Klein's model of hyperbolic geometry, the distance between two points p, q in the disk is given by

$$(1.2) \quad d(p, q) = |\log(ab; pq)|,$$

where a, b are the points where the line \overline{pq} intersects the boundary of the disk and $(ab; pq)$ denotes the cross ratio of these four points. In this model, the geodesics are ordinary lines. While the Poincaré metric is conformally invariant and can be generalized to arbitrary complex analytic spaces [12], the distance (1.2) is manifestly a projective invariant. As observed by Hilbert in his letter to Klein [10], formula (1.2) defines a projectively invariant distance for any convex domain in affine space. The ordinary lines are geodesics with respect to this distance and, if the domain is strictly convex, they are the only geodesics. Roughly speaking, the fourth problem of Hilbert was to determine all metrics in domains in projective space whose geodesics are all lines, see [7], [24].

In [13] I generalized the Hilbert metric to all domains and also to affine and projective manifolds. This was further generalized in [14] to affine connections. The purpose of this article is to review [13], [14], [15], [17] and Wu's recent paper [30] by supplying more details where desirable and to relate our results with the work of others. To make the paper accessible to topologists as well as to differential geometers, I consider domains and affine or projective manifolds first. The case

* Partially supported by NSF Grant MCS 79-02552.

of affine connection is discussed in the last section. For the same reason, I use the concept of projective equivalence of affine connections instead of the less known concept of projective connection.

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2. Schwarz lemma

In the construction of the intrinsic distance in the complex case, the unit disk and the upper half-plane with Poincaré metric play the fundamental role. In the projective case, this role is played by the "unit" interval I and the half-line R^+ defined as follows:

$$I = \{u \in R; -1 < u < 1\},$$

$$R^+ = \{x \in R; x > 0\}.$$

Both I and R^+ should be regarded as intervals in the real projective line. As such, they are projectively equivalent just as the unit disk and the upper half-plane are conformally equivalent. In fact, the linear fractional transformation

$$(2.1) \quad x = (1+u)/(1-u) \quad (\text{or } u = (x-1)/(x+1))$$

establishes a one-to-one correspondence between I and R^+ , in which the origin $0 \in I$ corresponds to $1 \in R^+$. The corresponding Poincaré metrics are given by

$$(2.2) \quad ds_I = 2du/(1-u^2), \quad ds_{R^+} = dx/x.$$

By integrating these metrics, we obtain the corresponding distance functions:

$$\varrho_I(u_1, u_2) = |\log(-1 \ 1; u_1 u_2)|,$$

where

$$(2.3) \quad (-1 \ 1; u_1 u_2) = (1+u_1)(1-u_2)/(1-u_1)(1+u_2);$$

$$\varrho_{R^+}(x_1, x_2) = |\log x_1 - \log x_2|.$$

A linear fractional transformation

$$(2.4) \quad x \rightarrow (ax+b)/(cx+d), \quad a, b, c, d \in R, \quad ad-bc \neq 0$$

sends R^+ into itself if and only if a, b, c, d are all nonnegative or all nonpositive. It induces an automorphism of R^+ if and only if $b = c = 0$ or $a = d = 0$. Let $\text{End}(R^+)$ denote the semigroup of linear fractional transformations (2.4) sending R^+ into itself. We shall include in $\text{End}(R^+)$ those linear fractional transformations with $ad-bc = 0$, i.e., the constant maps sending R^+ into itself. The group of automorphisms of R^+ , i.e., linear fractional transformations mapping R^+ one-to-one onto itself, will be denoted by $\text{Aut}(R^+)$. This group is transitive on R^+ and leaves

the metric ds_{R^+} and the distance ϱ_{R^+} invariant. The following is a projective analogue of the Schwarz lemma.

(2.5) LEMMA. *We have*

$$|f^*(ds_{R^+})| \leq |ds_{R^+}| \quad \text{for } f \in \text{End}(R^+),$$

$$\varrho_{R^+}(f(x_1), f(x_2)) \leq \varrho_{R^+}(x_1, x_2) \quad \text{for } x_1, x_2 \in R^+, \quad f \in \text{End}(R^+),$$

and the inequalities are strict unless f is an automorphism of R^+ .

Proof. The second inequality is the integrated form of the first. So it suffices to prove the first. We express f by

$$y = (ax+b)/(cx+d).$$

Then

$$|f^*(ds_{R^+})| = |dy|/y = |ad-bc| |dx|/(ax+b)(cx+d).$$

Hence the proof is reduced to showing that

$$|ad-bc|x \leq (ax+b)(cx+d) \quad \text{for } x \in R^+,$$

and the equality holds for some x if and only if $a = d = 0$ or $b = c = 0$. But this can be easily verified by considering the two separate cases, $ad-bc > 0$ and $ad-bc < 0$. ■

Although we can describe $\text{End}(I)$ and $\text{Aut}(I)$ in a similar fashion, their descriptions are not as simple as in the case of R^+ . From (2.5) we obtain also the Schwarz lemma for ds_I and ϱ_I .

3. Domains in P^n

In this section we shall generalize the Hilbert distance for a convex domain given by (1.2) to an arbitrary domain in R^n . Actually it is more natural to consider a domain in the real projective space P^n .

Let x^0, x^1, \dots, x^n be a homogeneous coordinate system for P^n . Let I be the "unit" interval as in § 1, i.e.,

$$I = \{u \in R; -1 < u < 1\}.$$

It is more natural to consider I as an interval in the projective line P^1 . A projective map $f: I \rightarrow P^n$ is given by equations of the following form:

$$(3.1) \quad x^i = a^i u + b^i, \quad i = 0, 1, \dots, n.$$

This concept is invariant under projective transformations in I (i.e., linear fractional transformations in I) as well as under projective transformations in P^n . For instance, if

$$u = (at+b)/(ct+d),$$

then

$$x^i = a^i(at+b) + b^i(ct+d) = (a^i a + b^i c)t + (a^i b + b^i d)$$

since x^0, x^1, \dots, x^n are *homogeneous* coordinates. Since the positive half-line R^+ is projective-equivalent to I by (2.1), we can also speak of projective maps from R^+ into P^n as well as projective maps from I into P^n .

Let D be a domain in the projective space P^n . We shall define a pseudo-distance d_D on D . Given two points p, q of D , we consider a *chain of line segments* from p to q , i.e., a chain of points $p = p_0, p_1, \dots, p_k = q$, pairs of points $a_1, b_1, \dots, a_k, b_k$ of the interval I , and projective maps f_1, \dots, f_k from I into the domain D such that

$$(3.2) \quad f_i(a_i) = p_{i-1} \quad \text{and} \quad f_i(b_i) = p_i \quad \text{for} \quad i = 1, \dots, k.$$

We denote this chain by $\alpha = (p_i, a_i, b_i, f_i)$. The length of this chain is defined to be

$$(3.3) \quad L(\alpha) = \varrho_I(a_1, b_1) + \dots + \varrho_I(a_k, b_k),$$

where ϱ_I is the Poincaré distance in the interval I defined by (2.3). The pseudo-distance $d_D(p, q)$ between the points p and q is now defined by

$$(3.4) \quad d_D(p, q) = \inf L(\alpha),$$

where the infimum is taken over all chains α from p to q .

It is a simple matter to verify that d_D satisfies the usual axioms of a pseudo-distance, i.e., the symmetricity and the triangular inequality. However, we can only claim $d_D(p, q) \geq 0$ in general even for two distinct points p, q . From the construction of d_D which made use of only projective concepts of the Poincaré distance and projective map it is clear that the pseudo-distance d_D is projectively invariant, i.e., invariant under the projective automorphisms of the domain D . This pseudo-distance is a real projective analogue of the intrinsic pseudo-distance introduced for complex analytic spaces in [12].

In order to study properties of d_D , it is convenient to consider the real projective analogue of the Carathéodory pseudo-distance at the same time. To define the Carathéodory pseudo-distance c_D for D , we need projective maps from D into I . In general, a projective map from a domain D in P^n into another projective space P^m is given in terms of homogeneous coordinate systems x^0, x^1, \dots, x^n of P^n and y^0, y^1, \dots, y^m of P^m as follows:

$$(3.5) \quad y^i = \sum_{j=0}^n a_j^i x^j \quad \text{for} \quad i = 0, 1, \dots, m.$$

In particular, a projective map from a domain D in P^n into P^1 with the inhomogeneous coordinate $u = y^1/y^0$ is given by

$$(3.6) \quad u = \left(\sum_{j=0}^n a_j x^j \right) / \left(\sum_{j=0}^n b_j x^j \right).$$

Considering I as an interval in the projective line P^1 , we can speak of projective maps from D into I . Now, the Carathéodory pseudo-distance c_D is defined by

$$(3.7) \quad c_D(p, q) = \sup_f \varrho_I(f(p), f(q)) \quad \text{for} \quad p, q \in D,$$

where the supremum is taken over all projective maps f from D into I .

These pseudo-distances c_D and d_D possess properties similar to those in the complex case. The proofs of the following propositions are straightforward from the definitions of c_D and d_D and from the Schwarz lemma (2.5). (They are identical to the proofs of the corresponding propositions in the complex case, see [12].)

(3.8) PROPOSITION. *Let D and D' be domains in P^n and $P^{n'}$, respectively. Then*

$$c_{D'}(f(p), f(q)) \leq c_D(p, q) \quad \text{and} \quad d_{D'}(f(p), f(q)) \leq d_D(p, q)$$

for all projective maps $f: D \rightarrow D'$.

(3.9) PROPOSITION. *For the interval I , we have*

$$c_I = d_I = \varrho_I.$$

(3.10) PROPOSITION. (1) *If δ is a pseudo-distance on D such that*

$$\varrho_I(f(p), f(q)) \leq \delta(p, q), \quad p, q \in D$$

for all projective maps $f: D \rightarrow I$, then

$$c_D \leq \delta;$$

(2) *If δ is a pseudo-distance on D such that*

$$\delta(f(a), f(b)) \leq \varrho_I(a, b), \quad a, b \in I$$

for all projective maps $f: I \rightarrow D$, then

$$\delta \leq d_D.$$

In particular, we obtain

(3.11) PROPOSITION. *We have always*

$$c_D \leq d_D.$$

We shall consider several simple examples and show, in particular, that both c_D and d_D agree with the Hilbert distance when D is a convex domain.

(3.12) EXAMPLE. For the real line R , we have $c_R = d_R = 0$.

It is easy to verify that, given any two points p, q of R and any positive ε , we can find two points a, b in the interval I with $d_I(a, b) < \varepsilon$ and a projective map (in fact, even an affine map) $f: I \rightarrow R$ such that $f(a) = p$ and $f(b) = q$. Hence, $d_R(p, q) < \varepsilon$. Since ε is arbitrary, we have $d_R(p, q) = 0$. By (3.11), $c_R(p, q) = 0$.

From (3.8) and (3.12) we obtain the following trivial but useful

(3.13) LEMMA. *If a domain D contains affine lines L_1, \dots, L_k such that $L_i \cap L_{i+1} \neq \emptyset$, then $c_D(p, q) = d_D(p, q) = 0$ for any points $p \in L_1$ and $q \in L_k$.*

The following two assertions follow from (3.13).

(3.14) EXAMPLE. If D is an affine space R^n or a projective space P^n , then $c_D = d_D = 0$.

(3.15) EXAMPLE. If D is the complement of a finite number of points in R^n or P^n for $n \geq 2$, then $c_D = d_D = 0$. More generally, if D is the complement of a finite number of compact convex sets in R^n or P^n for $n \geq 2$, then $c_D = d_D = 0$.

For the complement of a closed unbounded convex set in R^n , (3.15) is no longer true. For example,

(3.16) EXAMPLE. If D is the upper half-plane $R \times R^+ = \{(x, y) \in R^2; y > 0\}$, then

$$c_D((x, y), (x', y')) = d_D((x, y), (x', y')) = \varrho_{R^+}(y, y').$$

To prove this assertion, consider an affine map $f: R^+ \rightarrow D$ such that $f(y) = (x, y)$ and $f(y') = (x', y')$. Then, by (3.8),

$$c_D((x, y), (x', y')) \leq \varrho_{R^+}(y, y').$$

Applying (3.8) to the projection $R \times R^+ \rightarrow R^+$, we obtain

$$\varrho_{R^+}(y, y') \leq c_D((x, y), (x', y')).$$

The proof for d_D is exactly the same.

(3.17) EXAMPLE. If D is a convex domain in P^n , then both c_D and d_D coincide with the Hilbert pseudo-distance.

We shall prove this assertion by induction on the dimension n . Although we can start the induction with the trivial case of $n = 1$, we consider first the case $n = 2$ in order to give the general idea of the proof. Let p and q be points of D in P^2 . Let l be the line through p and q . Let a and b be the points where the line l meets the boundary of D . We shall show that

$$(3.18) \quad c_D(p, q) = d_D(p, q) = |\log(ab; pq)|,$$

where $(ab; pq)$ is the cross ratio of the four points a, b, p, q .

Let l_1 and l_2 be the supporting lines of D at a and b , respectively. Let o be the intersection point of l_1 and l_2 . Let f be the perspectivity from o to the line l . It maps D onto the open interval $J = (a, b)$ of the line l . By (3.8), we have

$$c_J(p, q) = c_J(f(p), f(q)) \leq c_D(p, q).$$

The opposite inequality follows also from (3.8) applied to the natural injection $J \rightarrow D$. Hence,

$$c_J(p, q) = c_D(p, q).$$

Similarly,

$$d_J(p, q) = d_D(p, q).$$

On the other hand, by identifying the interval $J = (a, b)$ with the interval $I = (-1, 1)$, we see that

$$c_J(p, q) = d_J(p, q) = |\log(ab; pq)|.$$

This completes the proof for $n = 2$.

Let $n > 2$. Given two points p and q of D , we define the line l and the points a and b as above. Let H_1 and H_2 be the supporting hyperplanes of D at a and b , respectively. Let H be a hyperplane containing l . We take a point o in $H_1 \cap H_2$ but outside the hyperplane H . Let f be the perspectivity from o to the hyperplane H . Let D' be the image $f(D) \subset H$. Then, by (3.8) we have

$$c_{D'}(p, q) = c_{D'}(f(p), f(q)) \leq c_D(p, q).$$

By the induction hypothesis, we have

$$c_{D'}(p, q) = c_J(p, q).$$

Hence,

$$c_J(p, q) \leq c_D(p, q).$$

The opposite inequality follows also from (3.8) applied to the injection $J \rightarrow D$. Hence,

$$c_D(p, q) = c_J(p, q).$$

Similarly for d_D . This completes the proof of (3.17).

(3.19) DEFINITION. A domain D in P^n is said to be *projective-hyperbolic* or, simply, *hyperbolic* if the pseudo-distance d_D is a distance, i.e., if $d_D(p, q) > 0$ for any $p \neq q$. A hyperbolic domain D is said to be *complete* if it is Cauchy-complete with respect to the distance d_D . We say that D is *finitely compact* if every closed ball of finite radius with respect to d_D is compact. For a general metric space, the finite compactness is a stronger condition than the completeness. But it turns out that for the distance function d_D these two conditions are equivalent to each other. The proof is exactly the same as in the complex hyperbolic case (see [12]).

(3.20) PROPOSITION. A convex domain D in P^n is hyperbolic if and only if it contains no affine line R (i.e., there is no non-constant projective map of R into D). A hyperbolic convex domain is always complete.

Proof. This is immediate from (3.18). Given two distinct points p, q in D , we consider the line l through p, q . If the portion of l contained in D is neither l nor a complete affine line, then $|\log(ab; pq)|$ in (3.18) is positive. On the other hand, it is zero if the portion of the projective line l contained in D is either l or an affine line (i.e., l minus one point).

If D is hyperbolic and convex, then from formula (3.18) it is clear that the closed ball of finite radius about p does not touch the boundary of D and hence is compact. ■

(3.21) EXAMPLE. A bounded domain D in R^n is hyperbolic. In fact, even the Carathéodory pseudo-distance c_D is a distance. A bounded convex domain D in R^n is complete hyperbolic.

To see this, we consider a cubical domain $E = I_1 \times \dots \times I_n$ in R^n (where each I_k is a finite interval in R) containing D . Then c_E is a distance. By (3.8) c_D is also a distance.

(3.22) EXAMPLE. A sharp convex affine cone is complete hyperbolic. We recall that a domain D in R^n is called an *affine cone* if, whenever D contains a point p , it contains also the half-line $\{tp; t > 0\}$. It is said to be *sharp* if it contains no complete affine lines. So the assertion above is immediate from (3.20).

If a domain D in R^n is not convex, we consider the smallest convex domain E containing D , the open convex hull of D . (The domain E exists since the interior of a convex set is convex.)

(3.23) PROPOSITION. Let D be a domain in P^n and E the open convex hull of D . Then

$$c_E(p, q) = c_D(p, q) \quad \text{for } p, q \in D.$$

In particular, a non-convex domain D cannot be complete with respect to c_D .

Proof. By (3.8) we have

$$c_E(p, q) \leq c_D(p, q).$$

Let f be a projective map of D into the half-line R^+ . Since f is the restriction of a projective map \tilde{f} of P^n into P^1 (see Definition (3.6)), we can consider the half-space $\tilde{f}^{-1}(R^+)$ which contains obviously D . The open convex hull E is then nothing but the intersection of all these half-spaces:

$$(3.24) \quad E = \bigcap \tilde{f}^{-1}(R^+).$$

Hence, every $f: D \rightarrow R^+$ extends to $\tilde{f}: E \rightarrow R^+$. Now, from the definition of the Carathéodory pseudo-distance (3.7), it is clear that

$$c_E(p, q) \geq c_D(p, q).$$

(In Definition (3.7), we used the interval I instead of R^+ . But they are projectively equivalent, see (2.1).) ■

Although the relationship between d_D and d_E seems to be more complicated (except for the obvious inequality $d_E(p, q) \leq d_D(p, q)$), we can at least claim the following:

(3.25) PROPOSITION. Let D be a domain in P^n . If it is not convex, then it cannot be complete with respect to d_D .

Proof. The proof can be best described by the following figure (see Figure 1). The distance from p to r_k is bounded by a fixed constant M independent of k . Hence, the limit point r on the boundary is at a finite distance from p .

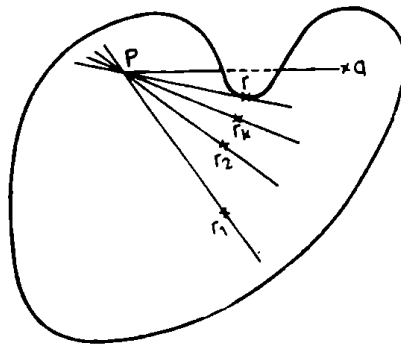


Fig. 1

(3.26) EXAMPLE. In P^n with homogeneous coordinates $\xi^0, \xi^1, \dots, \xi^n$, we take a domain D defined by

$$\xi^0 \xi^0 - \xi^1 \xi^1 - \dots - \xi^n \xi^n > 0.$$

The group of projective automorphisms of P^n is given by $PGL(n+1; R) = GL(n+1; R)/R^*$, where R^* denotes the center $\{aI_{n+1}; a \in R, a \neq 0\}$ of $GL(n+1; R)$.

Its subgroup $G = O(1, n)/\{\pm I_{n+1}\}$ acts transitively on the domain D . Hence, d_D is invariant by G . In terms of the inhomogeneous coordinates $x^1 = \xi^1/\xi^0, \dots, x^n = \xi^n/\xi^0$, the domain D can be identified with the unit ball $B_n = \{x = (x^1, \dots, x^n) \in \mathbb{R}^n; (x, x) < 1\}$ in \mathbb{R}^n . Let o denote the origin of B_n and the corresponding point of D given by $\xi^1 = \dots = \xi^n = 0$. The isotropy subgroup of G at o is naturally isomorphic to $O(n)$, and B_n may be represented as a symmetric space $G/O(n)$ of rank 1. Later in Example (5.9) we shall describe d_D in terms of its infinitesimal form and show that d_D comes from the Riemannian metric of constant curvature -4 .

4. Affine and projective structures

Let M be an n -dimensional manifold. An *affine structure* on M is given by coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ such that

- (a) $\{U_\alpha\}$ is an open cover of M , i.e., $M = \bigcup U_\alpha$;
- (b) each $\phi_\alpha: U_\alpha \rightarrow \mathbb{R}^n$ is a diffeomorphism onto the open set $\phi_\alpha(U_\alpha)$;
- (c) each coordinate change $\phi_\alpha \circ \phi_\beta^{-1}$ is (the restriction of) an affine transformation.

Similarly, a *projective structure* can be defined by replacing \mathbb{R}^n by P^n and assuming that $\phi_\alpha \circ \phi_\beta^{-1}$ is a projective transformation.

Let M and N be manifolds (not necessarily of the same dimension) with affine structures defined by coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\lambda, \psi_\lambda)\}$, respectively. Then a mapping $F: M \rightarrow N$ is said to be *affine* if it is affine with respect to coordinate charts, i.e., if $\psi_\lambda \circ f \circ \phi_\alpha^{-1}$ is (the restriction of) an affine map. Similarly, a mapping between two manifolds with projective structures is *projective* if it is projective with respect to coordinate charts.

Clearly, every affine structure can be considered as a projective structure, and every affine map as a projective map.

Having defined the concept of projective map, we can extend the definitions of the pseudo-distances d_D and c_D (see (3.4) and (3.7)) from a domain D in P^n to a manifold M with a projective structure. Definition (3.19) of (complete) hyperbolicity is valid for a manifold with a projective structure. Propositions (3.8) through (3.11) and Lemma (3.13) are all valid for manifolds with projective structures.

(4.1) PROPOSITION. *Let M be a manifold with a projective structure. Let \tilde{M} be a covering manifold with the induced projective structure. Let $\pi: \tilde{M} \rightarrow M$ be the projection. Then*

$$d_M(p, q) = \inf_{\tilde{q}} d_{\tilde{M}}(\tilde{p}, \tilde{q}) \quad \text{for } p, q \in M,$$

where \tilde{p} is any point of \tilde{M} such that $\pi(\tilde{p}) = p$ and the infimum is taken over all points \tilde{q} of \tilde{M} such that $\pi(\tilde{q}) = q$.

Since the proof is identical to that of the corresponding result in the complex analytic case (see [12]; p. 48), we shall omit it.

The following result follows from (4.1) also in the same way as in the complex case.

(4.2) PROPOSITION. *Let M and \tilde{M} be as in (4.1). Then \tilde{M} is (complete) hyperbolic if and only if M is.*

One of the basic concepts in the study of affine and projective structures is that of *development*. We shall explain it in the case of affine structure. Let M be a manifold with an affine structure given by coordinate charts $\{(U_\alpha, \phi_\alpha)\}$. We choose one chart (U_0, ϕ_0) . Given a point $p \in M$, we choose a chain of coordinate charts $(U_1, \phi_1), \dots, (U_m, \phi_m)$ such that $U_{i-1} \cap U_i \neq \emptyset$, $i = 1, \dots, m$, and $p \in U_m$. We can then find affine transformations f_1, \dots, f_m of R^n such that

$$f_{i-1} \circ \phi_{i-1} = f_i \circ \phi_i \quad \text{on} \quad U_{i-1} \cap U_i, \quad i = 1, \dots, m,$$

where f_0 is the identity transformation. We set

$$\text{dev}(p) = f_m \circ \phi_m(p).$$

This defines a multivalued map $\text{dev}: M \rightarrow R^n$, called a *development* of M into the affine space R^n ; it is multivalued since $\text{dev}(p)$ depends on the choice of a chain U_1, \dots, U_m (as well as that of U_0). However, the principle of monodromy asserts that dev is a well defined map from M into R^n if M is simply connected. It is clear that dev is an immersion of M into R^n (when M is simply connected).

In the construction above, if $U_m \cap U_0 \neq \emptyset$, then we can compare ϕ_0 with $f_m \circ \phi_m$ on $U_m \cap U_0$. In general, they do not coincide unless M is simply connected, and there is an affine transformation f of R^n such that $f \circ \phi_0 = f_m \circ \phi_m$. We call f the *holonomy transformation* associated with the chain U_0, U_1, \dots, U_m . If we fix a point p_0 in U_0 and consider all chains U_0, U_1, \dots, U_m such that $p_0 \in U_m$, then the set of resulting holonomy transformations forms a subgroup of the affine transformation group of R^n , called the *holonomy group* with the base point p_0 . (This is nothing but the affine holonomy group of the flat affine connection induced by the given affine structure.) If we denote the fundamental group of M with the base point p_0 by π_1 and the holonomy group with the base point p_0 by Φ , then we have a natural homomorphism $\pi_1 \rightarrow \Phi$. If M^* is the (intermediate) covering space of M corresponding to the kernel of this homomorphism, then the development map $\text{dev}: M^* \rightarrow R^n$ is well defined (i.e., single valued). We call M^* the *holonomy covering of M* ; it is a principal bundle over M with structure group Φ . (M^* is nothing but the holonomy bundle of the associated flat affine connection on M .)

Similarly, if M is a manifold with a projective structure, we can define a multivalued map $\text{dev}: M \rightarrow P^n$, which becomes a single valued map on the holonomy covering space M^* and hence on the universal covering space \tilde{M} .

A somewhat related concept is that of *exponential map* for a manifold with an affine structure. Let M be a manifold with an affine structure or, more generally, an affine connection. We fix a point p_0 of M and an isomorphism between the tangent space $T_{p_0}(M)$ and R^n . The exponential map $\exp: R^n \rightarrow M$ is defined only on some domain E in R^n since a geodesic may not extend infinitely. If we take a small neigh-

neighborhood U_0 of p_0 and construct a development map as above starting with the chart (U_0, \exp^{-1}) , then we see that

$$(4.3) \quad \text{dev} \circ \exp(X) = X \quad \text{for} \quad X \in E \subset R^n = T_{p_0}(M).$$

Since dev is multivalued, (4.3) should be actually interpreted as

$$\text{dev} \circ \exp(X) \ni X.$$

If M has trivial holonomy, e.g., if it is simply connected, then the map dev is single valued and (4.3) has no ambiguity.

In general, $\exp: E \rightarrow R^n$ is not surjective, i.e., not every point of M can be joined by geodesic to p_0 . But if there is a geodesic from p_0 to p , we can develop p along this geodesic and obtain $X \in R^n$ such that $\exp(X) = p$. Thus,

$$(4.4) \quad \exp \circ \text{dev}(p) = p \quad \text{if} \quad p = \exp(X) \quad \text{for some} \quad X \in R^n = T_{p_0}(M),$$

where dev is the development along the geodesic $\exp(tX)$, $0 \leq t \leq 1$. Hence we have

(4.5) **PROPOSITION.** *Let M be a simply connected manifold with an affine structure. Suppose there is a point $p_0 \in M$ with the property that any other point p can be joined to p_0 by a geodesic. Then the exponential map \exp at p_0 is an affine isomorphism from a domain $E (= \text{dev}(M))$ of $T_{p_0}(M)$ onto M , and \exp^{-1} is a development map.*

The following result is due to Koszul [19].

(4.6) **PROPOSITION.** *Let M be a manifold with an affine structure, and let $p_0 \in M$. If the domain of definition $E \subset T_{p_0}(M)$ for the exponential map at p_0 is convex, then $\exp: E \rightarrow M$ is surjective and hence is a covering map.*

Proof. There is no loss of generality in assuming that M is simply connected (replace M by its universal covering space). By (4.3), \exp is a map of maximal rank everywhere on E and hence its image $\exp(E)$ is an open subset of M . Let p_1 be a point in the closure of $\exp(E)$, and let $X_1 = \text{dev}(p_1) \in T_{p_0}(M)$. Then X_1 is in the closure of E since dev is locally an affine isomorphism. Since E is convex, $tX_1 \in E$ for $0 \leq t < 1$. As $t \rightarrow 1$, $\exp(tX_1) \rightarrow p_1$, and $\exp(X_1)$ is defined and equal to p_1 . This shows that $\exp(E)$ is a closed subset of M . Being both open and closed, $\exp(E)$ coincides with M . ■

Let M be a manifold with a projective structure, and let $p_0 \in M$. Let $M(p_0)$ denote the set of points $p \in M$ which can be joined by a geodesic to p_0 , i.e., the set of points p for which there is a projective map from the interval I into M passing through p_0 and p .

(4.7) **LEMMA.** *The set $M(p_0)$ defined above is open.*

Proof. Let $p_1 \in M(p_0)$ and let $f: I \rightarrow M$ be a projective map passing through p_0 and p_1 . Going back to the construction of development maps, we cover the geodesic $f(I)$ by a chain of coordinate charts and develop a neighborhood of the geodesic from p_0 to p_1 . The geodesic is developed into a line l from $x_0 = \text{dev}(p_0)$ to $x_1 = \text{dev}(p_1)$ in P^n . A neighborhood of the geodesic is developed onto a neighborhood U of l . If p is a point sufficiently close to p_1 , then $\text{dev}(p)$ is in U and a line

from x_0 to $x = \text{dev}(p)$ is in U . Then the corresponding geodesic goes from p_0 to p . ■

(4.8) THEOREM. *Let M be a manifold with a projective structure. If it is complete hyperbolic, i.e., d_M is a complete distance, then any pair of points can be joined by a geodesic.*

Proof. Let $p_0 \in M$ and let $M(p_0)$ be the set of points $p \in M$ which can be joined to p_0 by a geodesic. By (4.7), $M(p_0)$ is open. It suffices to prove that $M(p_0)$ is closed. We make use of the following general fact on metric spaces and distance-decreasing maps.

(4.9) LEMMA. *Let M and N be manifolds with projective structures, and let $\text{Proj}(N, M)$ denote the set of projective maps from N into M equipped with the compact-open topology. If M is complete hyperbolic, then for any point $a \in N$ and any compact set $K \subset M$ the family*

$$\{f \in \text{Proj}(N, M); f(a) \in K\}$$

is compact.

The proof of this lemma relies only on the fact that every $f \in \text{Proj}(N, M)$ is distance-decreasing with respect to d_M and d_N , see [12]; p. 73.

We apply this lemma to the case where $N = I$ and $K = \{p_0\}$. Let $\{p_i\}$ be a sequence of points in $M(p_0)$ converging to a point $p \in M$. Let $f_i \in \text{Proj}(I, M)$ be a geodesic through p_0 and p_i . Let $a \in I$. Composing f_i with an automorphism of I , we may assume that $f_i(a) = p_0$. By (4.9), there is a converging subsequence $\{f_{i_m}\}$ of $\{f_i\}$. Let $f = \lim f_{i_m}$. Then $f \in \text{Proj}(I, M)$ passes through p_0 and p . ■

(4.10) COROLLARY. *Let M be a manifold with an affine structure, and $p_0 \in M$. Let E be the domain of definition for the exponential map at p_0 . If M is complete hyperbolic, then E is a convex domain in $T_{p_0}(M)$ and $\exp: E \rightarrow M$ is a covering map.*

Proof. By (4.2) the proof can be reduced to the case where M is simply connected. By (4.6) $\exp: E \rightarrow M$ is an affine isomorphism. Since any pair of points of M can be joined by a geodesic, E is convex. ■

We should remark that the idea of the proof for (4.8) is in Vey [26]. In another paper [27] Vey proved also the following result, see also [28].

(4.11) THEOREM. *Let D be a convex domain in R^n containing no lines. If there is a properly discontinuous group G of affine transformations acting on D such that D/G is compact, then D is a cone.*

As pointed out by W. Goldman, (4.11) combined with (4.10) yields the following:

(4.12) COROLLARY. *Let M be a compact manifold with an affine structure. Let E be the domain of definition for the exponential map at $p_0 \in M$. If M is hyperbolic, then E is a convex cone containing no lines in $T_{p_0}(M)$ and $\exp: E \rightarrow M$ is a covering map.*

Proof. Since M is compact, it is complete with respect to d_M and (4.10) applies to M . Let G be the group of covering transformations of E so that $M = E/G$. By (4.2), E is hyperbolic and cannot contain lines. By (4.11), E is a cone. ■

(4.13) *Remark.* If $M = D/G$ is a compact manifold with an affine structure (where D is a domain in R^n and G is a discrete group of affine transformations acting properly discontinuously and freely on D), then D is unbounded. In fact, if D is bounded, M is complete hyperbolic (since D is hyperbolic and M is compact) and, by (4.8), any two points of D can be joined by a geodesic. By (4.11), D must be a cone. This is a contradiction.

We shall now examine a few simple examples.

(4.14) **EXAMPLES.** The only compact surfaces which admit affine structures are a torus and a Klein bottle as shown by Benzécri [4], (see Milnor [22] for stronger results). If M is an Euclidean torus (i.e., if coordinate changes are given by Euclidean motions), then every geodesic extends infinitely in both directions and the universal covering space of M is R^n , implying that $d_M = 0$. Similarly, for an affine torus M in which every geodesic extends infinitely in both directions, we have $d_M = 0$. For the punctured plane $R^2 - \{0\}$, we have also $d_{R^2 - \{0\}} = 0$. Hence, if $M = (R^2 - \{0\})/G$, where G is the infinite cyclic group of homothetic transformations generated by the transformation

$$(x, y) \rightarrow (ax, ay) \quad (a: \text{positive constant} \neq 1),$$

then we have also $d_M = 0$. (M is an affine torus called a *Hopf torus*.)

On the other hand, the quarter-plane $R^+ \times R^+$ is complete hyperbolic. Let G be the discrete abelian group generated by the following two affine transformations:

$$(x, y) \rightarrow (ax, by), \quad (x, y) \rightarrow (cx, dy),$$

where a, b, c, d are positive real numbers such that

$$(\log a)(\log d) - (\log b)(\log c) \neq 0.$$

Then $M = (R^+ \times R^+)/G$ is an affine torus and is complete hyperbolic.

For the upper half-plane $R \times R^+$, the pseudo-distance $d_{R \times R^+}$ degenerates in one direction, i.e., in the direction of the x -axis. Let G be the discrete abelian group generated by the following two affine transformations of $R \times R^+$:

$$(x, y) \rightarrow (x+a, by), \quad (x, y) \rightarrow (x+c, dy),$$

where $a, c \in R$ and $b, d \in R$ such that $a(\log d) - c(\log b) \neq 0$. For the affine torus $M = (R \times R^+)/G$, the pseudo-distance d_M degenerates also in the direction of the x -axis.

These affine tori illustrate how the pseudo-distance d_M may be used to obtain a rough classification of the affine structures on a torus. For a systematic study of the affine structures on a torus, see Nagano-Yagi [23] as well as Kuiper [20].

In [1] and [2] we find several interesting examples of compact affine manifolds whose universal covering spaces are affine spaces. But they all have the trivial pseudo-distance by (3.14) and (4.1).

(4.15) EXAMPLE. In (3.26) we showed that the natural projective structure in the unit ball B_n in R^n is invariant under the transitive group $G = O(1, n)/\{\pm I_{n+1}\}$. Since G leaves the distance d_{B_n} invariant, its action is proper. Hence, if Γ is a discrete subgroup of G acting freely on B_n , then $M = B_n/\Gamma$ is a manifold with a natural projective structure. As we shall see later in Example (5.9), d_{B_n} comes from a G -invariant Riemannian metric of constant negative curvature -4 . Conversely, it is a classical theorem in differential geometry that a Riemannian manifold is projectively flat (i.e., admits a compatible projective structure) if and only if it is of constant sectional curvature. (This goes back to a paper of Beltrami in 1868; see Eisenhart's book on Riemannian Geometry). In particular, every Riemann surface admits a projective structure.

A simple necessary condition for the existence of a projective structure is given by

(4.16) THEOREM. *If a manifold M admits a projective structure, then its Pontryagin classes vanish.*

The proof requires the concept of projective equivalence of affine connections which will be explained in the following section. We cover M with coordinate charts $\{U_\alpha, \phi_\alpha\}$ as explained at the beginning of § 4. Taking U_α sufficiently small, we may assume ϕ_α maps U_α into R^n instead of P^n . We pull back the flat affine connection of R^n to U_α by ϕ_α and obtain a flat affine connection Γ_α on each U_α . In the intersection $U_\alpha \cap U_\beta$, the two connections Γ_α and Γ_β are projectively equivalent in the sense explained in the next section. Then the Pontryagin forms expressed in terms of the curvature of Γ_α coincide with those of Γ_β , [16] (see also [3] where it is shown that the Pontryagin forms can be written in terms of Weyl's projective curvature tensor).

It is still an open question whether the Euler number of a manifold with an affine structure is zero or not. According to Koszul [19], we have

(4.17) THEOREM. *If a compact manifold admits a hyperbolic affine structure, then its first Betti number is nonzero.*

5. Infinitesimal projective metric

Let M be a manifold with a projective structure. We shall now define the infinitesimal pseudo-metric corresponding to the pseudo-distance d_M (see Wu [30]).

Let I be the interval $-1 < u < 1$ with the Poincaré metric $ds_I = 2du/(1-u^2)$ as in § 2. Let V be a tangent vector of I . We set

$$|V| = \text{the Euclidean norm of } V,$$

$$||V|| = \text{the norm defined by } ds_I.$$

Thus, if $V = \lambda(d/du)_a \in T_a(I)$, then

$$(5.1) \quad |V| = |\lambda|,$$

$$(5.2) \quad ||V|| = \frac{2|\lambda|}{(1-a^2)}.$$

In particular, if V is a vector at the origin $0 \in I$, i.e., if $a = 0$, then $||V|| = 2|V|$.

We define a pseudo-metric $F_M: T(M) \rightarrow [0, \infty)$ by

$$(5.3) \quad F_M(p, X) = \inf \|V\| \quad \text{for } X \in T_p(M),$$

where the infimum is taken over all tangent vectors $V \in T(I)$ and all projective maps $f: I \rightarrow M$ such that $f_*(V) = X$.

Since ds_I is invariant under all projective automorphisms of I , we may restrict V in Definition (5.3) to a tangent vector at the origin. Since $\|V\| = 2|V|$ at the origin, we may rewrite (5.3) as follows:

$$(5.4) \quad F_M(p, X) = \inf 2|V| \quad \text{for } X \in T_p(M),$$

where the infimum is taken over all tangent vectors $V \in T_0(I)$ at the origin and all projective maps $f: I \rightarrow M$ such that $f_*(V) = X$.

The following are the infinitesimal versions of (3.8) to (3.10).

(5.5) PROPOSITION. *Let M and M' be manifolds with projective structures. Then*

$$F_{M'}(f(p), f_*(X)) \leq F_M(p, X)$$

for all projective maps $f: M \rightarrow M'$.

(5.6) PROPOSITION. *For the interval I , we have*

$$F_I = ds_I.$$

(5.7) PROPOSITION. *If Φ is a pseudo-metric on a manifold M with a projective structure such that*

$$\Phi(f(a), f_*(V)) \leq ds_I(a, V)$$

for all projective maps $f: I \rightarrow M$, then

$$\Phi \leq F_M.$$

The infinitesimal version of (4.1) is given by the following

(5.8) PROPOSITION. *Let M be a manifold with a projective structure and \tilde{M} a covering manifold with the induced projective structure and the projection $\pi: \tilde{M} \rightarrow M$. Then*

$$\pi^*F_M = F_{\tilde{M}}.$$

(5.9) EXAMPLE. Let $B_n = \{x = (x^1, \dots, x^n) \in R^n; (x, x) < 1\}$ be the open unit ball in R^n . Then

$$F_{B_n} = 2 \frac{\{(1 - (x, x))(dx, dx) + (x, dx)^2\}^{1/2}}{(1 - (x, x))}.$$

Since the group $O(1, n)$ acts transitively not only on B_n but also on the tangent unit sphere bundle of B_n (see (3.26)), it suffices to verify that the metric on the right is invariant by $O(1, n)$ and agrees with F_{B_n} on one nonzero vector at the origin of B_n . The invariance can be verified by identifying B_n with a domain D in P^n as in (3.26) and expressing the metric on the right in terms of the homogeneous coordinates $\xi^0, \xi^1, \dots, \xi^n$.

We shall compare the two metrics above on the vector $\partial/\partial x^1$ at the origin. Restricted to the interval $-1 < x^1 < 1$, $x^2 = \dots = x^n$, the metric on the right

becomes $2dx^1/(1-x^1x^1)$. The mapping $I \rightarrow B_n$ given by $x^1 = u, x^2 = 0, \dots, x^n = 0$ is an isometry with respect to F_I and F_{B_n} . (This follows from (5.5) applied to the map $I \rightarrow B_n$ and to map $B_n \rightarrow I$ given by $u = x^1$.) From (5.6) and (2.2) it follows that, restricted to the interval $-1 < x^1 < 1, x^2 = \dots = x^n = 0$, the metric F_{B_n} becomes also $2dx^1/(1-x^1x^1)$. This completes the proof.

In order to show that d_M is the integrated form of F_M , we establish first the following (proved more generally by Wu [30], see § 6)

(5.10) PROPOSITION. *The projective pseudo-metric $F_M: T(M) \rightarrow [0, \infty)$ is upper semicontinuous.*

Proof. Let $F_M(p_0, X_0) = k$. Given $\varepsilon > 0$, we must find a neighborhood U of (p_0, X_0) in $T(M)$ such that

$$(5.11) \quad F_M(p, X) < k + \varepsilon \quad \text{for} \quad (p, X) \in U.$$

Let $V_0 \in T_0(I)$ and $f: I \rightarrow M$ a projective map such that $f_*(V) = X_0$ and

$$\|V_0\| < k + \frac{1}{2}\varepsilon.$$

Let B_n be the unit ball as in (5.9). We extend f to a projective immersion $\tilde{f}: B_n \rightarrow M$ so that $f(u) = \tilde{f}(u, 0, \dots, 0)$. By the isometric mapping $I \rightarrow B_n$ used in the proof of (5.9), we identify V_0 with a tangent vector of B_n at the origin. Let N be a neighborhood of V_0 in $T(B_n)$ such that

$$F_{B_n}(b, V) < k + \varepsilon \quad \text{for} \quad (b, V) \in N.$$

Such a neighborhood N exists since $F_{B_n}(0, V_0) = \|V_0\| < k + 1/2$ and F_{B_n} is continuous by (5.9). Let $U = \tilde{f}_*(N)$. By (5.5), the inequality (5.11) is satisfied. ■

Since F_M is upper semicontinuous, we can define a pseudo-distance δ on M by the following integral:

$$(5.12) \quad \delta(p, q) = \inf_c \int F(c(t), \dot{c}(t)) dt,$$

where the infimum is taken over all piecewise smooth curves c from p to q and $\dot{c}(t)$ denotes the velocity vector of the curve c .

We claim the following basic result.

(5.13) THEOREM. *We have $d_M = \delta$, i.e.,*

$$d_M(p, q) = \inf_c \int F(c(t), \dot{c}(t)) dt.$$

We shall prove only the inequality in one direction, namely

$$(5.14) \quad d_M \geq \delta.$$

The opposite inequality can be proved in a manner similar to the corresponding result for the complex case, see Royden [25]. The detail can be found in Wu [30].

In order to prove (5.14), we make use of (3.10). It suffices to show that

$$(5.15) \quad \delta(f(a), f(b)) \leq \varrho_I(a, b), \quad a, b \in I$$

for all projective maps $f: I \rightarrow M$. Since ϱ_I is the integrated form of $ds_I = F_I$, inequality (5.15) follows from inequality (5.5) applied to $f: I \rightarrow M$.

In a special case, (5.10) can be improved.

(5.16) PROPOSITION. *If a manifold M with a projective structure is complete hyperbolic, then F_M is a continuous function on $T(M)$.*

We need the following lemma.

(5.17) LEMMA. *If M is complete hyperbolic, then for every $X \in T_p(M)$ there exists a vector $V \in T_0(I)$ together with a projective map $f: I \rightarrow M$ such that $f_*(V) = X$ and $F_M(p, X) = \|V\|$.*

Proof. In other words, we are claiming that the infimum in (5.3) is actually attained if M is complete hyperbolic. To prove this, we consider the family of all projective maps $f: I \rightarrow M$ such that $f(0) = p$. By (4.9) this family is compact. This implies (5.17).

Proof of (5.16). In order to show that F_M is continuous at $(p_0, X_0) \in T(M)$, let $(p_i, X_i) \in T(M)$, $i = 1, 2, \dots$, be a sequence of tangent vectors converging to (p_0, X_0) . For each $i = 0, 1, \dots$, let $V_i \in T_0(I)$ and $f_i \in \text{Proj}(I, M)$ such that $f_{i*}(V_i) = X_i$ and $F_M(p_i, X_i) = \|V_i\|$. Since F_M is upper semicontinuous, we have

$$\lim \|V_i\| = \lim F_M(p_i, X_i) \leq F_M(p_0, X_0) = \|V_0\|.$$

By taking a subsequence we may assume that $\{V_i\}$ converges to a vector $W \in T_0(I)$. By (4.9), a subsequence of $\{f_i\}$, still denoted $\{f_i\}$, will converge to a projective map g . Since $g(W) = \lim f_i(V_i) = \lim X_i = X_0$, we have $F_M(p_0, X_0) \leq \|W\|$. Since $\lim V_i = W$, we have $\|W\| = \lim \|V_i\|$. Hence, $F_M(p_0, X_0) \leq \lim F(p_i, X_i)$. ■

(5.18) EXAMPLE. For the following domain D in R^2 , F_D is not continuous. Let

$$D = \{(x, y) \in R^2; x^2 + y^2 < 1\} - \{(x, 0) \in R^2; 1/2 \leq x\}.$$

Then F_D is not continuous at $(\partial/\partial x)_{(0,0)} \in T_0(D)$.

(5.19) PROPOSITION. *Let M be a manifold with a projective structure. It is hyperbolic if and only if there is a Riemannian metric on M such that*

$$F_M(p, X) \geq \|X\| \quad \text{for } X \in T_p(M),$$

where $\|X\|$ denotes the Riemannian length of X .

Proof. If such a Riemannian metric exists, d_M is bounded below by the Riemannian distance because of (5.13). Hence, M is hyperbolic.

Assume that M is hyperbolic. If we show that in a neighborhood of every point p of M there exists a Riemannian metric with the required property, then we can construct a desired global Riemannian metric by the standard method using a partition of the unity. Let B be an open ball around p . Let r be a small positive number such that $N(2r) = \{q \in M; d_M(p, q) < 2r\}$ is contained in B . Let

$$I_a = \{u \in I; -a < u < a\}, \quad \text{where } r = \log(1+a)/(1-a),$$

so that $I_a = \{u \in I; d_I(0, u) < r\}$. Let $q \in N(r)$ and $X \in T_q(M)$. Let $V \in T_0(I)$ and $f \in \text{Proj}(I, M)$ be such that $f(0) = q$ and $f_*(V) = X$. Since f is distance-decreasing, we have

$$f(I_a) \subset N(r) \subset N(2r) \subset B \subset M.$$

Hence,

$$F_{I_a}(0, V) \geq F_B(q, f_*V) = F_B(q, X).$$

But, $F_I = 2du/(1-u^2)$ and $F_{I_a} = 2adu/(a^2-u^2)$ so that $F_I(0, V) = aF_{I_a}(0, V)$. Hence,

$$F_I(0, V) \geq aF_B(q, X).$$

By taking the infimum over V and f , we obtain

$$F_M(q, X) \geq aF_B(q, X) \quad \text{for } X \in T(N(r)).$$

Since we know the explicit expression of F_B for a ball B (see (5.9)), we know that F_M is bounded below by a Riemannian metric on $N(r)$. ■

We know that if M is hyperbolic, then there is no nonconstant projective map f of R into M . We shall show that the converse holds when M is compact. As in the complex case, for a noncompact manifold the converse is not necessarily true.

(5.20) EXAMPLE. The domain $D = \{(x, y) \in R^2; |x| < 1, |xy| < 1\} - \{(0, y); |y| \geq 1\}$ is not hyperbolic, but there is no nonconstant projective map of R into D .

On the interval $I_r = \{u \in R; -r < u < r\}$ we consider the metric $ds_r = 2r^2 du/(r-u^2)$. In particular, $I = I_1$ and $ds_I = ds_1$. We note that the homothetic map $I \rightarrow I_r$ sending u to ru is an isometry of (I, rds_I) onto (I_r, ds_r) , not of (I, ds_I) onto (I_r, ds_r) . On the other hand, ds_I and ds_r coincide at the origin 0. In fact,

$$(ds_I)_{u=0} = (ds_r)_{u=0} = (2du)_{u=0}.$$

We begin with the proof of the "Reparametrization Lemma" of Brody. The proof is essentially the same as that of the corresponding lemma in the complex case obtained by Brody [6]. (For the projective case, see also Wu [30].)

(5.21) LEMMA. Let M be a manifold with a projective structure. Fix a Riemannian metric ds_M^2 on M . Given $f \in \text{Proj}(I_r, M)$, define a function U on I_r by

$$U = f^*ds_M/ds_r.$$

If $U(0) > c > 0$, then there is a map $g \in \text{Proj}(I_r, M)$ such that

(a) the function g^*ds_M/ds_r is bounded by c on I_r and attains the maximum value c at the origin;

(b) $g = f \circ \mu_a \circ h$, where h is a projective automorphism of I_r and μ_a ($0 < a < 1$) is the multiplication by a (i.e., $\mu_a(u) = au$ for $u \in I_r$).

Proof. For $t \in [0, 1)$, define $f_t \in \text{Proj}(I_r, M)$ by

$$f_t(u) = f \circ \mu_t(u) = f(tu) \quad \text{for } u \in I_r.$$

Set $U_t = f_t^*ds_M/ds_r$. Then

$$U_t = \frac{\mu_t^* f^* ds_M}{\mu_t^* ds_r} \cdot \frac{\mu_t^* ds_r}{ds_r} = \mu_t^* U \cdot \frac{t(r^2 - u^2)}{(r^2 - t^2 u^2)}.$$

Set

$$A(t) = \sup_{u \in I_r} U_t(u) = \sup_{u \in I_r} U(tu) \cdot \frac{t(r^2 - u^2)}{(r^2 - t^2 u^2)}.$$

It is easy to see that $A(t)$ is finite, continuous and monotone-increasing in $[0, 1]$ and that $A(t) > c$ for t sufficiently close to 1 and $A(0) = 0$. Thus

$$c = A(a) = \sup_{u \in I_r} U_a(u) \quad \text{for some } a \in [0, 1].$$

From the explicit expression of $U_t(u)$ given above, we see that $U_a(u)$ approaches zero at the boundary of I_r and hence reaches its maximum in the interior I_r . Let u_0 be a point of I_r , where $U_a(u)$ attains the maximum c . Let h be a projective automorphism of I_r which sends 0 to u_0 . Then $g = f \circ \mu_a \circ h$ possesses all the desired properties. ■

We are now in a position to prove the following projective analogue of Brody's theorem.

(5.22) THEOREM. *Let M be a compact manifold with a projective structure. Fix a Riemannian metric ds_M^2 on M . If M is non-hyperbolic, there is a projective map $h: R \rightarrow M$ such that*

$$h^* ds_M^2 \leq du^2, \quad \text{where } u \text{ is the natural coordinate in } R.$$

Proof. We denote the length of a tangent vector X of M with respect to ds_M^2 by $\|X\|$. By (the trivial half of) (5.19), we can find a sequence of tangent vectors X_m such that $\|X_m\| = 1$ and $F_M(p_m, X_m) < 1/m$. From Definition (5.3) of F_M , we can find projective maps $j_m \in \text{Proj}(I, M)$ and tangent vectors

$$V_m = a_m(d/du)_0 \in T_0(I), \quad a_m > 0,$$

such that $j_m^*(V_m) = X_m$ and $\lim a_m = 0$. (We may further assume that a_m is monotone-decreasing.) Set $r_m = 1/a_m$ and define $f_m \in \text{Proj}(I_{r_m}, M)$ by

$$f_m(u) = j_m(a_m u) \quad \text{for } |u| < r_m.$$

If we set $V_0 = (d/du)_0 \in T_0(R)$, then

$$f_m^*(V_0) = X_m.$$

For each f_m we define a function U_m on I_{r_m} as in (5.21), i.e., $U_m = f_m^* ds_M / ds_{r_m}$. Since the length of V_0 with respect to ds_{r_m} is always 2, we obtain

$$U_m(0) = \|f_m^*(V_0)\|/2 = \|X_m\|/2 = 1/2.$$

Applying (5.21) to each f_m and a positive constant $c < 1/2$, we obtain a map $g_m \in \text{Proj}(I_{r_m}, M)$ such that

$$(a) \quad g_m^* ds_M \leq cds_{r_m} \quad \text{on } I_{r_m},$$

and the equality holds at 0;

$$(b) \quad \text{Image}(g_m) \subset \text{Image}(f_m).$$

By (a) the family $\{g_m\}$ is equicontinuous. To be more precise, since

$$g_m^* ds_M \leq cds_{r_m} \leq cds_r \quad \text{for } m \geq l$$

the family $\{g_m|_{I_{r_l}}; m \geq l\}$ is equicontinuous for each fixed l . Since the family $\{g_m|_{I_{r_1}}\}$ is equicontinuous, the Arzela–Ascoli theorem implies that we can extract a subsequence which converges to a map $h_1 \in \text{Proj}(I_{r_1}, M)$. Applying the same theorem to the corresponding subsequence in the equicontinuous family $\{g_m|_{I_{r_2}}; m \geq 2\}$, we can extract a subsequence which converges to a map $h_2 \in \text{Proj}(I_{r_2}, M)$. Continuing in this way, we obtain maps $h_m \in \text{Proj}(I_{r_m}, M)$. Clearly, h_m is an extension of h_{m-1} . Thus we obtain a map $h \in \text{Proj}(R, M)$ which extends all h_m .

Since $g_m^* ds_M$ at the origin is equal to $(cds_{r_m})_{u=0} = 2c(du)_{u=0}$, it follows that

$$(h^* ds_M)_{u=0} = \lim (g_m^* ds_M)_{u=0} = 2c(du)_{u=0} \neq 0,$$

which shows that h is nonconstant.

Since $g_m^* ds_M \leq cds_{r_m}$, by taking the limit we obtain $h^* ds_M^2 \leq 4c^2 du^2$. By suitably normalizing h , we obtain $h^* ds_M^2 \leq du^2$. ■

(5.23) *Remark.* It is not clear how useful the inequality $h^* ds_M^2 \leq du^2$ is in the projective case. A similar inequality in the complex case is certainly very important.

(5.24) *COROLLARY.* Let M be a compact manifold with a projective structure. Then it is hyperbolic if and only if there is no nonconstant projective map of R into M .

(5.25) *COROLLARY.* Let M be a compact manifold with a projective structure. Then M is hyperbolic if and only if there is no nonzero vector X such that $F_M(p, X) = 0$.

Proof. If M is hyperbolic, $F_M(p, X)$ is positive for every nonzero X whether M is compact or not (see (5.19)). If M is not hyperbolic, there is a nonconstant projective map f of R into M . If X is any vector tangent to the curve $f: R \rightarrow M$, then $F_M(p, X) = 0$ by (5.5) and $F_R = 0$. ■

(5.26) *Remark.* For the second half of (5.25), the compactness assumption is necessary. In fact, for the nonhyperbolic domain D in Example (5.20), $F_D(p, X)$ is positive for every nonzero X .

(5.27) *Remark.* If M is compact and if X is a vector such that $F_M(p, X) = 0$, then there is a projective map f of R into M such that X is tangent to the curve $f: R \rightarrow M$. This follows from the proof of (5.22) as follows. Take a Riemannian metric on M . We may assume that X is a unit vector with respect to the Riemannian metric. In the proof of (5.22), let $X_m = X$ for all m . Then our assertion follows because of (b) and the fact that a geodesic is determined by its tangent vector at one point. The corresponding result in the complex case is unknown since a holomorphic curve is not determined by its tangent vector at one point.

6. d_M and F_M in the nonflat case

We shall now extend the construction of d_M and F_M to a manifold M with an affine connection which is not necessarily flat.

Let $\Gamma = (\Gamma_{jk}^i)$ be an affine connection on M expressed in terms of Christoffel's

symbols with respect to a local coordinate system x^1, \dots, x^n . A geodesic $x(t) = (x^i(t))$ is defined by the system of differential equations

$$(6.1) \quad \frac{d^2 x^i}{dt^2} + \sum \Gamma_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} = \sigma \frac{dx^i}{dt},$$

where σ is a function of t . (The equation means that the acceleration, the left hand side, is proportional to the velocity.) We can choose parameter t to make $\sigma = 0$ in (6.1). (In fact, we have only to replace t by $\int e^{\int \sigma dt} dt$.) A parameter for which $\sigma = 0$ is called an *affine parameter* of the geodesic and is unique up to an affine change $t \rightarrow at + b$.

From (6.1) it is clear that the geodesics remain unchanged if we replace (Γ_{jk}^i) by its symmetric part $(\frac{1}{2}(\Gamma_{jk}^i + \Gamma_{kj}^i))$. Since we are interested mainly in geodesics, we shall assume that $\Gamma = (\Gamma_{jk}^i)$ is torsionfree, i.e.,

$$(6.2) \quad \Gamma_{jk}^i = \Gamma_{kj}^i.$$

Following Weyl we say that two torsion-free affine connections $\Gamma = (\Gamma_{jk}^i)$ and $\bar{\Gamma} = (\bar{\Gamma}_{jk}^i)$ on M are *projectively related* or *projectively equivalent* if there exists a 1-form $\psi = \sum \psi_i dx^i$ on M such that

$$(6.3) \quad \bar{\Gamma}_{jk}^i - \Gamma_{jk}^i = \delta_j^i \psi_k + \delta_k^i \psi_j.$$

Two such connections define the same system of geodesics. In fact, (6.1) is equivalent to

$$(6.4) \quad \frac{d^2 x^i}{dt^2} + \sum \bar{\Gamma}_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} = \bar{\sigma} \frac{dx^i}{dt},$$

where

$$(6.5) \quad \bar{\sigma} = \sigma + \sum 2\psi_k \frac{dx^k}{dt}.$$

It is known that, conversely, two torsion-free affine connections with the same system of geodesics are projectively related if $\dim M \geq 2$.

It is clear from (6.5) that an affine parameter t with respect to the connection Γ needs not be an affine parameter with respect to $\bar{\Gamma}$. Thus, the concept of affine parameter is not a projective invariant. Following J. H. C. Whitehead [29] we introduce the concept of projective parameter. Let $\text{Ric} = (R_{jk})$ denote the Ricci tensor of a connection Γ . Let $x(t) = (x^i(t))$ be a geodesic with an affine parameter t with respect to Γ . Then a *projective parameter* p of $x(t)$ is a solution of the following differential equation:

$$(6.6) \quad \{p, t\} = \frac{2}{n-1} \sum R_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt},$$

where $\{p, t\}$ is the Schwarzian derivative:

$$(6.7) \quad \{p, t\} = \frac{p'''}{p'} - \frac{3}{2} \left(\frac{p''}{p'} \right)^2 = \left(\frac{p''}{p'} \right)' - \frac{1}{2} \left(\frac{p''}{p'} \right)^2,$$

with primes denoting derivatives with respect to t .

As we shall see shortly, a projective parameter p is unique up to a linear fractional transformation:

$$(6.8) \quad p \rightarrow (ap + b)/(cp + d).$$

As Whitehead has shown, p is a projective parameter with respect to any torsion-free affine connection projectively related to Γ . (An outline of the proof of this fact will be given later.) Thus, the concept of projective parameter is a projective invariant.

We say that a mapping f of the interval $I = \{-1 < u < 1\}$ into a manifold M with a torsion-free affine connection Γ is *projective* if the curve $f(u)$ is a geodesic and u is a projective parameter. Clearly, this concept depends only on the projective equivalence class of a torsion-free affine connection Γ . If M and M' are manifolds with torsion-free affine connections, a mapping $f: M \rightarrow M'$ is said to be *projective* if, for every projective map $h: I \rightarrow M$, the composed map $f \circ h: I \rightarrow M'$ is projective, i.e., if f maps every geodesic of M into a geodesic of M' preserving a projective parameter.

Once the concept of projective map $f: I \rightarrow M$ is established, we can extend Definition (3.4) of d_D and Definition (5.3) of F_M to a manifold with a torsion-free affine connection. Many of the results proved in the preceding sections extend immediately to manifolds with torsion-free affine connection. In particular, the following are valid in the case of torsion-free affine connection: (3.8), (3.9), (3.10), (3.11), (3.13), (4.1), (4.2), (4.9), (5.5), (5.7), (5.8), (5.10), (5.13), (5.16), (5.17), (5.21), (5.22), (5.24), (5.25). It should be noted, however, that our proof of (5.10) that F_M is upper semicontinuous makes use of the flatness of the structure. A proof valid in the nonflat case was given by Wu [30]. The trivial half of (5.19) is valid in the nonflat case. The other half is probably true also in the nonflat case. The proof of (5.22) makes use of (5.19). But we can avoid its use in the compact case arguing as in (5.27).

In order to prove the uniqueness (up to a linear fractional transformation) and the projective invariance of a projective parameter p , we need the following lemma whose proof is straightforward.

(6.9) LEMMA. *Formula for Schwarzian derivatives of composite functions:*

$$\{s, u\} = (\{s, t\} - \{u, t\}) \left(\frac{dt}{du} \right)^2.$$

Setting $s = t$, we obtain

$$(6.10) \quad \{s, u\} = -\{u, s\} \left(\frac{ds}{du} \right)^2.$$

In order to show that a solution p of (6.6) is unique up to a transformation (6.8), suppose that q is another solution of (6.6). Then

$$\{p, t\} - \{q, t\} = 0.$$

By (6.9), $\{p, q\} = 0$. From Definition (6.7) it is easy to solve the equation $\{p, q\} = 0$ explicitly to obtain $p = (aq + b)/(cq + d)$.

In order to prove the projective invariance of p , we need also the following formula whose proof is again straightforward. (It goes back to Weyl.)

(6.11) LEMMA. If R_{jkh}^i and \bar{R}_{jkh}^i denote the curvature components of two torsion-free affine connections Γ and $\bar{\Gamma}$ which are projectively related by (6.3), then

$$\bar{R}_{jkh}^i - R_{jkh}^i = \delta_j^i(\psi_{h;k} - \psi_{k;h}) + \delta_h^i(\psi_{j;k} - \psi_j \psi_k) - \delta_k^i(\psi_{j;h} - \psi_j \psi_h),$$

where $\psi_{h;k}$ denotes the covariant derivative of ψ_h with respect to the connection.

Consequently, for Ricci tensors $R_{jh} (= \sum R_{jih}^i)$ and \bar{R}_{jh} we have

$$(6.12) \quad \bar{R}_{jh} - R_{jh} = -n(\psi_{j;h} - \psi_j \psi_h) + \psi_{h;j} - \psi_j \psi_h.$$

If \bar{t} and \bar{p} are affine and projective parameters with respect to the connection $\bar{\Gamma}$, then (6.6) and (6.12) imply

$$(6.13) \quad \{\bar{p}, \bar{t}\} = \frac{2}{n-1} \sum \bar{R}_{jh} \frac{dx^j}{d\bar{t}} \frac{dx^h}{d\bar{t}} = \{p, t\} - \sum 2(\psi_{j;h} - \psi_j \psi_h) \frac{dx^j}{dt} \frac{dx^h}{dt} \left(\frac{dt}{d\bar{t}} \right)^2.$$

On the other hand, from

$$\frac{d^2 x^i}{dt^2} + \sum \bar{\Gamma}_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} = \bar{\sigma} \frac{dx^i}{dt} \quad \text{with} \quad \bar{\sigma} = 2 \sum \psi_j \frac{dx^j}{dt}$$

it follows that

$$\bar{t} = \int e^{\int \bar{\sigma} dt} dt.$$

Hence,

$$(6.14) \quad \frac{\bar{t}''}{\bar{t}'} = \bar{\sigma} = 2 \sum \psi_j \frac{dx^j}{dt}, \quad \frac{\bar{t}'''}{\bar{t}'} = 2 \sum \psi_{j;h} \frac{dx^j}{dt} \frac{dx^h}{dt}.$$

From (6.13) and (6.14) we obtain

$$(6.15) \quad \{\bar{p}, \bar{t}\} = (\{p, t\} - \{\bar{t}, t\}) \left(\frac{dt}{d\bar{t}} \right)^2.$$

But (6.9) used twice yields the following formula:

$$(6.16) \quad \{s, u\} = \left(\{s, t\} - (\{u, r\} - \{t, r\}) \left(\frac{dr}{dt} \right)^2 \right) \left(\frac{dt}{du} \right)^2.$$

Setting $s = \bar{p}$, $u = p$, $t = \bar{t}$, $r = t$ in (6.16) and using (6.15), we obtain $\{\bar{p}, p\} = 0$. This shows that $\bar{p} = (ap + b)/(cp + d)$, thus establishing the invariance of the projective parameter under the projective change of connection $\Gamma \rightarrow \bar{\Gamma}$.

Now we shall state some of the results in [14], [15], [17], [30] without proof. The Ricci tensor of an affine connection is not necessarily symmetric. So we say that it is positive or negative definite when its symmetric part ($\frac{1}{2}(R_{ij} + R_{ji})$) is positive or negative definite. From (6.6) it is expected that only the symmetric part matters.

(6.17) THEOREM. *Let M be a manifold with a torsion-free affine connection.*

(a) *If its Ricci tensor is negative definite everywhere, then M is hyperbolic, i.e., d_M is a distance.*

(b) *If the affine connection is complete and its Ricci tensor is positive semi-definite, then d_M vanishes identically.*

Actually, Wu proves (a) under a slightly weaker condition (see Theorem 1 in [30]). In the proof he uses the infinitesimal metric F_M . Both (a) and (b) rely on the following classical result.

(6.18) LEMMA. *If y_1, y_2 are linearly independent solutions of the differential equation*

$$y''(t) + Q(t)y(t) = 0,$$

then the general solution of the differential equation

$$\{u, t\} = 2Q(t)$$

is given by

$$u(t) = (ay_1 + by_2)/(cy_1 + dy_2) \quad \text{with} \quad ad - bc \neq 0.$$

Sturm's comparison theorems allow us to estimate the range of a projective parameter.

In the Riemannian case, F_M can be compared with the Riemannian metric when the Ricci tensor is strongly negative.

(6.19) THEOREM. *Let M be a Riemannian manifold with metric $ds_M^2 = \sum g_{jk} dx^j dx^k$ and Ricci tensor R_{jk} such that*

$$(R_{jk}) \leq -c^2(g_{jk}) \quad (c > 0).$$

Then

$$F_M^2 \geq \frac{4c^2}{n-1} ds_M^2.$$

In particular, if ds_M^2 is a complete metric, so is F_M .

If the metric ds_M^2 is Einstein with $R_{jk} = -c^2 g_{jk}$, then $F_M^2 = ds_M^2$.

(6.20) Remark. The last part of (6.19) can be applied to a torsion-free affine connection Γ with negative definite parallel Ricci tensor R_{jk} . If we set $g_{jk} = -\frac{1}{2}(R_{jk} + R_{kj})$, then $ds_M^2 = \sum g_{jk} dx^j dx^k$ is a Riemannian metric parallel with respect to Γ so that Γ is the Levi-Civita connection of ds_M^2 and $R_{jk} = -g_{jk}$.

(6.21) COROLLARY. *If M is a compact Riemannian manifold with negative Ricci tensor, then its group of projective automorphisms is finite.*

Proof. By (6.19) the group leaves the metric F_M invariant and hence is compact. On the other hand, the vanishing theorem à la Bochner for infinitesimal projective transformations shows that the group is discrete (see Couty [9]).

Koszul [18], [19] studied affine manifolds admitting a 1-form whose covariant derivative is positive definite. His work can be understood in terms of projective equivalence defined by (6.3). We shall state here one related result.

(6.22) **THEOREM.** *Let M be a compact manifold with a torsion-free affine connection. If its Ricci tensor is negative semi-definite and if there is a 1-form $\psi = \sum \psi_j dx^j$ such that its covariant derivative $D\psi = \sum \psi_{j;k} dx^j \otimes dx^k$ (or more precisely, the symmetric part of $D\psi$) is positive definite, then M is hyperbolic, i.e., d_M is a distance.*

Proof. From the given connection Γ , we construct a new connection $\bar{\Gamma}$ by (6.3). Replacing ψ by $c\psi$ with a small positive constant c if necessary, we see from (6.12) that the Ricci tensor \bar{R}_{jk} of $\bar{\Gamma}$ is negative definite. By (6.17), M is hyperbolic. ■

(6.23) **Remark.** The special case of (6.22) where M is locally affine and ψ is closed (so that $D\psi$ is symmetric) is the second half of the main theorem in Koszul [19].

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Added in prof. The following papers, though not quoted in the text, are related to the subject discussed here. In particular, [31], [32], [36] are on projectively invariant Riemannian metrics on convex domains. My short paper [33] may be read as an introduction to this paper.

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Presented to the Semester
Differential Geometry
(September 17–December 15, 1979)
