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ON GRADED BUNDLES AND THEIR GEOMETRY

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0. Introduction

In nature there is a perfect equilibrium between particles of two sorts, namely bosons (which have integer-valued spins) and fermions (having half-integer-valued spins). Roughly speaking if we put a boson into a set of identical bosons then our boson will probably "choose" for itself the same state as the remaining bosons possess. Meanwhile, any fermion, when placed among identical fermions, will probably "choose" a state completely different from states of other particles. The bosonic behaviour is crucial in order to explain such phenomena as superconductivity and superfluidity while the fermionic nature of electrons is necessary for the explanation, e.g. the Mendeleev table. So the duality between bosons and fermions is a fundamental feature of the physical image of the world which was emphasized in the Feynman's famous book [13].

In quantum mechanical models, an anti-symmetric relation of commutativity is assigned to any system of bosons and a symmetric relation of anti-commutativity is associated with any system of fermions. Thus any mathematical theory, in order to be useful in physics of systems of elementary particles, should include both commutativity and anti-commutativity relations. This postulate completely fails in the case of differential geometry because for two vector fields defined by means of the Leibnitz rule

$$X(\alpha\beta) = X(\alpha)\beta + \alpha X(\beta)$$

the commutator is well-defined but anti-commutator does not make any sense (as a vector field).

Therefore new geometries for needs of elementary particles and quantum physics have been pursued since the early seventies. The so called Lie superalgebras elaborated by V. G. Kac, see [19], were an interesting result. Kac developed a theory of "Lie algebras" with a partially anti-symmetric and partially symmetric bracket and obtained many results concerning their classifications and representations.

Nevertheless his results belong rather to algebra than geometry and new geometric properties of considered spaces were not obtained in this way.

But geometric nature of anti-commuting object occupied physicists since these objects were closely connected with a "spin-geometry" (so called supersymmetries, see [5], [16], [1], [28] and [23]) and a curved Minkowski space (so called supergravity, see [2], [9], [10] and [24]). K. Gawędzki has proved in [14] that the only "reasonable" model of a supermanifold M, i.e., a manifold admitting both commuting and anti-commuting vector fields looks as follows: M as a space is identical to an "ordinary" manifold but the algebra of smooth local functions $C^{\infty}_{\mathrm{loc}}(M)$ (or, in other words, sections of the trivial bundle $M\times K$, where $K = R$ or $K = C$) must be replaced with an algebra $\Gamma^{\infty}_{\mathrm{loc}}(\Lambda E)$ of sections of an auxiliary Grassmann bundle ΛE which, contrary to $M\times K$, may be endowed with an ample geometry. (The idea of using a Grassmann algebra in order to join commutators and anti-commutators in a common algebra is due to F. A. Berezin, cf. [5]). Thus supermanifolds provide us with a generalization of the structure of a tangent bundle which is more adequate for phenomena concerning elementary particles.

Thus the natural problem is to determine a class of vector bundles-like objects which would be as generalizations of supermanifolds as vector bundles are generalizations of tangent bundles. The next problems are what is an effect of the geometry of an "auxiliary" bundle ΛE in the geometry of supermanifolds and their generalizations. The present paper is devoted to both these problems.

In Section 1 we define a vector bundle-like object called a graded bundle in which the Grassmann bundle ΛE is involved in transition functions. Then we state that each smooth graded bundle is trivial, i.e., it may be transformed into a tensor product of vector bundles, but there exist non-trivial holomorphic graded bundles. Certain estimations for dimensions of spaces of some graded bundles having a fixed auxiliary bundle are given but we are still far from a precise computation of these dimensions. We present graded bundles associated with instanton solutions of gauge fields.

In Section 2 we form a differential geometry for graded bundles based on a superspace tensor calculus, see [14]. In particular we define curvature for graded bundles and then we show that the geometric procedure for Chern classes works only in special cases. Thus we see that both a local geometry and a global nature of vector bundles may be extended onto a domain of graded structures but the method for Chern classes, which is a link spanning them, is getting to be broken down.

About thirty years ago, when tangent bundles had been well-known and general constructions of vector bundles have been just appearing, mathematicians and physicists asked why such a complicated structure would be needed. Who does listen to such a question today?

I believe that the above situation will have something in common with the development of the graded bundles.

1. Graded bundles. Triviality and non-triviality

Throughout the paper we deal with two cases: either the considered manifolds (real or compiex) and mappings will be of C^∞-class (this case will not usually be mentioned) or they will be complex and holomorphic (then manifolds will be called complex and mappings holomorphic).

In order to obtain "graded effects" in the theory of fibre bundles the linear group $\mathrm{GL}(n, \boldsymbol{K})$ should be replaced by a group of all invertible $n \times n$ matrices of which elements belong to a Grassmann algebra ΛV. These matrices naturally decompose

$$A = A^{(0)} + A^{(1)} + \ldots + A^{(k)},$$

where $a_{ij}^{(p)} \in \Lambda^p V$, $\dim_K V = k$, $p = 1, \ldots, k$, $i, j = 1, \ldots, n$, $\det A^{(0)} \neq 0$.

Denote this group by $\mathrm{GL}(n, \Lambda V)$.

The equality $A^{(0)} = I_n$ determines its normal subgroup $\mathrm{HL}(n, \Lambda V)$ which is nilpotent.

The quotient space $\mathrm{GL}(n, \Lambda V)/\mathrm{GL}(n, \boldsymbol{K})$ is diffeomorphic to $\boldsymbol{K}^{n^2(2^k-1)}$. Then the Steenrod theorem determines a natural reduction of every smooth $\mathrm{GL}(n, \Lambda V)$-bundle to a linear bundle.

Thus we propose a more general approach to "graded bundles". Let the structural group $\mathrm{GL}(n, \boldsymbol{K})$ of a bundle over a base manifold M be replaced by a family of groups $\mathrm{GL}(n, \Lambda E_x)$, $x \in M$, where E_x are fibres of a certain auxiliary vector bundle E over M. Thus from now on the structural group will transform (together with fibres) when passing from one point of M to another one. More precisely no isomorphism between groups of transformations of all fibres is given.

A set of all non-equivalent (vector or principal) $\mathrm{GL}(n, \boldsymbol{K})$-bundles is isomorphic to $H^1(M, \mathrm{GL}(n, \boldsymbol{K}))$, where the considered cohomology is given in a sheaf of sections of the trivial bundle $M \times \mathrm{GL}(n, \boldsymbol{K})$. Thus a set of all non-equivalent "graded bundles" should be isomorphic to $H^1(M, \mathrm{GL}(n, \Lambda E))$, where the cohomology is considered in the sheaf of sections of the bundle $\pi_1\colon \mathrm{GL}(n, \Lambda E) \to M$, $\pi_1^{-1}(x) := \mathrm{GL}(n, \Lambda E_x)$, which is not so simple as $M \times \mathrm{GL}(n, \boldsymbol{K})$.

The step by step construction of a "graded bundle" is presented in the following schema

space in a given step	the base manifold M	$\to$	the auxiliary vector bundle E over M	$\to$
its dimension if $\boldsymbol{K} = \boldsymbol{R} \mid \boldsymbol{K} = \boldsymbol{C}$	m		$m+k \mid m+2k$	

$\to$	the Grassmann bundle ΛE over M	$\to$	the bundle $\mathrm{GL}(n, \Lambda E)$ of invertible matrices with elements in the bundle ΛE over M	$\to$	the graded bundle, i.e. a class of cocycles in the sheaf of sections of $\mathrm{GL}(n, \Lambda E)$
	$m+2^k \mid m+2^{k+1}$		$m+n^2 2^k \mid m+n^2 2^{k+1}$		$m+n2^k \mid m+n2^{k+1}$

Let us observe that ΛE_x-module structures in the fibres make of graded bundles more subtle objects than ordinary vector bundles.

DEFINITION. We say that we have a *graded bundle* if the following system is given: $\mathscr{G}(E) = \langle \mathscr{G}, \pi_g, M, \langle E, \pi, M\rangle, (\Omega_\alpha), (d_\alpha), (g_{\alpha\beta})\rangle$, where

(1) $\langle E, \pi, M\rangle$ is a vector bundle of rank k over the base manifold M. We call it the *auxiliary bundle*.

(2) $\mathscr{G}$ is a manifold and $\pi_g\colon \mathscr{G} \to M$ is a projection on the base M (base is the same as previously) such that each fibre $\pi_g^{-1}(x)$ is isomorphic (algebraically and topologically) to $\overset{n}{\oplus} \Lambda E_x$.

(3) (Ω_α) is a covering of M and d_α

$$d_\alpha\colon \pi_g^{-1}(\Omega_\alpha) \to \overset{n}{\oplus} \Lambda E_\alpha, \quad \text{where} \quad E_\alpha := E|_{\Omega_\alpha}$$

are isomorphisms such that

$$d_\alpha d_\beta^{-1}(e_x) = g_{\alpha\beta}(x) e_x, \quad e_x \in \overset{n}{\oplus} \Lambda E_x, \quad x \in \Omega_{\alpha\beta} := \Omega_\alpha \cap \Omega_\beta,$$

and $g_{\alpha\beta}$ is a section of $\mathrm{GL}(n, \Lambda E_{\alpha\beta})$ over $\Omega_{\alpha\beta}$, i.e.,

$$\Omega_{\alpha\beta} \ni x \to g_{\alpha\beta}(x) \in \mathrm{GL}(n, \Lambda E_x).$$

The set (d_α) is called an *atlas* of the graded bundle $\mathscr{G}(E)$. All the sections $g_{\alpha\beta}$ form a cocycle, i.e.,

$$g_{\beta\alpha} = (g_{\alpha\beta})^{-1} \quad \text{on} \quad \Omega_{\alpha\beta},$$
$$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma} \quad \text{on} \quad \Omega_{\alpha\beta\gamma}.$$

The graded bundle is holomorphic if all the above manifolds and mappings are complex and holomorphic, resp.

This definition in fact is due to "graded vector bundles" only. But there is no serious problem if we want to define "principal graded bundles".

We say that graded bundles $\mathscr{G}(E^1)$ and $\mathscr{G}(E^2)$ over the same M are *equivalent* if the vector bundles E^1 and E^2 are isomorphic, $E^1 \equiv E^2 \equiv E$, and there exists a family of sections (s_α) of $\mathrm{GL}(n, \Lambda E_\alpha)$ over Ω_α such that

$$g^2_{\alpha\beta} = s_\alpha g^1_{\alpha\beta} s_\beta^{-1}$$

(the multiplication is the same as in sets of Grassmann-valued matrices).

Thus a set of all equivalence classes of graded bundles is in a natural 1-1 correspondence with the cohomology space $H^1((\Omega_\alpha), \mathrm{GL}(n, \Lambda E))$ associated with the covering (Ω_α). Exactly in the same way as in the definition of a fibre bundle we may get rid of the dependence on a particular choice of the covering (Ω_α) by means of the relation of compatibility and then by passing to the inductive limit. We can identify the set of all non-equivalent graded bundles with the Čech cohomology space $H^1(M, \mathrm{GL}(n, \Lambda E))$, cf. [18].

The graded bundle $\mathscr{G}(E)$ is said to be *trivial* if there exists a graded bundle $\mathscr{G}'(E)$ such that $\mathscr{G}'(E) \sim \mathscr{G}(E)$ and for a covering (Ω_α) all the transition functions

$g'_{\alpha\beta}$ take values in $\mathrm{GL}(n, K) \subset \mathrm{GL}(n, \Lambda E_x)$. Therefore the trivial graded bundles are of the type $W \otimes \Lambda E$, where W is the vector bundle determined by the cocycle $g'_{\alpha\beta}$. So that the existence of non-trivial graded bundles is a question "to be or not to be" for the idea of graded bundles.

THEOREM. (1) *If $\mathscr{G}(E)$ is a smooth graded bundle over a paracompact base manifold M then it is trivial.*

(2) *There are non-trivial holomorphic graded bundles and the following inequalities are satisfied*

$$2\varepsilon_k h^1_w(M, E) \leqslant \dim H^1_w(M, \mathrm{HL}(n, \Lambda E)) \leqslant 2n^2 \sum_{i=1}^{k} h^1_w(M, \Lambda^i E),$$

$$\varepsilon_k := \begin{cases} n^2, & k = 1, \\ [n^2/4], & k > 1, \end{cases}$$

where $h^1_w(M, E) = \dim_C H^1_w(M, E)$, "$w$" denotes cohomology in a complex, holomorphic sheaf, k is rank of E and "dim" *in the middle is meant topologically.*

The proof, based on the technics of sheaves is given in [7].

Remark 1. The space $H^1_w(M, \mathrm{GL}(n, \Lambda E))$ may be regarded as a set of data for a generalized-multi-dimensional-mixed (additive and multiplicative)-skew symmetric Cousin problem. The right side of equalities of the equivalence relation for matrix elements $f^l_{\alpha\beta}$ of $h^{(l)}_{\alpha\beta}$ are sums of terms of the type t^l_α, t^j_β, $t^l_\alpha t^j_\beta$, $t^l_\alpha f^1_{\alpha\beta} t^j_\beta$, where t^l_α are matrix elements of $s^{(i)}_\alpha$. We recognize a huge number of particular realizations of the equivalence relation so that dimensions $h^1(M, \mathrm{HL}(n, \Lambda E))$ seem to be much lower than $2n^2 \sum_i h^1_w(M, \Lambda^i E)$.

Remark 2. The necessary condition for the existence of non-trivial HL-graded bundles is $H^1(M, \Lambda^i E) \neq 0$ for some i. Thus we face a surprising phenomenon, namely graded bundles hardly coexist with global meromorphic sections of their auxiliary bundles.

The space $H^1_w(M, \mathrm{GL}(n, \Lambda E))$ may be considered as a space of all non-equivalent holomorphic graded bundles $\mathscr{G}(E)$ having the identical auxiliary bundle E. Observe that the multiplication of cocycles by complex numbers

$$(zg_{\alpha\beta})^{(i)} := z^i g^{(i)}_{\alpha\beta}, \qquad z \in C$$

determines multiplication of graded bundles by any complex number as a continuous mapping in $H^1_w(M, \mathrm{GL}(n, \Lambda E))$.

For a graded bundle $\mathscr{G}(E)$ corresponding to a cocycle $(g_{\alpha\beta})$ the graded bundle $0 \cdot \mathscr{G}(E)$ corresponds to the cocycle $(G_{\alpha\beta})$ where $G_{\alpha\beta} := g^{(0)}_{\alpha\beta}$, which cocycle defines up to equivalence the vector bundle $\pi_0(\mathscr{G}(E))$. Thus we get the following surjection

$$\pi_0\colon H^1_w(M, \mathrm{GL}(n, \Lambda E)) \to H^1_w(M, \mathrm{GL}(n, C)).$$

Since $\mathscr{G}(E)$ and $0 \cdot \mathscr{G}(E)$ can be always joined by line $t \cdot \mathscr{G}(E)$, $t \in [0, 1]$, then π_0 gives us a 1-1 correspondence between connected components of $H^1_w(M, \mathrm{GL}(n, \Lambda E))$

and $H^1_w(M, \mathrm{GL}(n, C))$. Furthermore both these spaces of cohomologies are of the same type of homotopy because all fibres $\pi_0^{-1}(p)$ are contractible.

Apart from complex lines $z\cdot\mathscr{G}(E)$ planes of the type $I_n+A\oplus H^1_w(M, E)$, where A is $n\times n$ such that $A^2=0$, may be distinguished in $H^1_w(M, \mathrm{HL}(n, \Lambda E))$.

Furthermore, $H^1_w(M, \mathrm{GL}(n, \Lambda E))$ admits a stratification according to the degree of bundles. This degree is equal to the degree of Grassmann algebra elements in transition functions, which is maximal in a given cocycle and simultaneously minimal in the class of all equivalent cocycles. The construction of a 2nd degree graded bundle can be done over P^1C.

If a holomorphic graded bundle $\mathscr{G}(E)$ is given then we can subordinate to it a holomorphic vector bundle $\mathscr{G}_v(E)$ of rank $n2^k$ by forgetting all the E_x-moduli structures and remaining the linear structures of the fibres. Note, that not all vector bundles belonging to the equivalence class correspond to a graded bundle in such a way if one of them corresponds. If $\mathscr{G}(E)$ is a trivial graded bundle then $\mathscr{G}_v(E)=\pi_0(\mathscr{G}(E))\otimes\Lambda E$.

There are "inverse" problems: when a given vector bundle W does admit a non-trivial graded bundle $\mathscr{G}(E)$ such that $W=\mathscr{G}_v(E)$ and what is a method of reproduction of the auxiliary bundle E if the vector bundle $\mathscr{G}_v(E)$ is given. The partial answer is

Assertion. (1) Each holomorphic vector bundle $\mathscr{G}_v(E)$ associated with a holomorphic graded bundle $\mathscr{G}(E)$ admits a sequence of subbundles: $\mathscr{G}_v(E):=W_0 \supset W_1\supset\ldots\supset W_k$ such that rank of W_i is $n\left(\binom{k}{i}+\binom{k}{i+1}+\ldots+\binom{k}{k}\right)$ and $W_k \equiv\pi_0(\mathscr{G}(E))\otimes\Lambda^kE$. These subbundles can be obtained by acting of the cocycles on $\bigoplus_{j=i}^{k}\Lambda^jE$ and then by forgetting of the ΛE_x-module structures.

(2) If vector bundles $\pi_0(\mathscr{G}(E))$ and $\pi_0(\mathscr{G}^1(E))$ belong to the same connected component of $H^1_w(M, \mathrm{GL}(n, C))$ then graded bundles $\mathscr{G}(E)$ and $\mathscr{G}^1(E)$ belong to the same component of $H^1_w(M, \mathrm{GL}(n, \Lambda E))$ and $\mathscr{G}_v(E)$ and $\mathscr{G}^1_v(E)$ belong to the same component of $H^1_w(M, \mathrm{GL}(n\times 2^k, C))$. In particular the first Chern class holds

$$c_1(\mathscr{G}_v(E))=c_1\big(\pi_0(\mathscr{G}(E))\otimes\Lambda E\big).$$

Examples. A. Let $\pi\colon E\to M$ be a spin structure over a Riemannian 4-manifold M, see [24]. Then any graded bundle $\mathscr{G}(E)$ is trivial. It means the non-existence of any global collective bundle-like structure consisting of a number of isomorphic spin-structures which would be locally separable into a direct sum of the spin structures but globally would not. In particular all spin structures considered in the extended supergravity, see [8], [9], [10], are separable in the sense of graded bundles. This fact can simplify the equations of the extended supergravity.

B. The holomorphic vector bundles E admitting non-trivial graded bundles $\mathscr{G}(E)$ are very popular in complex analysis and mathematical physics. Here below a few examples.

B1. Let E_t be a trivial holomorphic line bundle. Then $H^1_w(M, E_t) \neq 0$ means that the additive Cousin problem for functions cannot be solved for every data. If D is an open domain in C^2 then $H^1_w(D, E_t) \neq 0$ iff D is not holomorphically convex, e.g. if $C^2 - D$ contains compact components.

B2. Let $M = P^1C$ (Riemann sphere) and $E = H^k$ be a line bundle such that $c_1(H^p) = p < -1$. Then $h^1_w(P^1C, E) = -p-1 \neq 0$.

In the general case when $E = \bigoplus_{l=1}^{k} H^{p_l}$, $\pi_0(\mathscr{G}(E)) = \bigoplus_{j=1}^{n} H^{r_j}$ the space of all graded bundles contains points of a complex plane C^d, where

$$d = \sum_{i,j=1}^{n} \sum_{l=1}^{k} H(r_j - r_i - p_l - 1), \quad \text{where} \quad H(a) = \begin{cases} a, & a \geqslant 0, \\ 0, & a \leqslant 0. \end{cases}$$

Thus the condition $H^1(M, E) \neq 0$ is not necessary for the existence of (general) non-trivial graded bundles.

B3. Let M be a compact Riemann surface and put $E = TM$. Then

$$h^1_w(M, E) = \begin{cases} 0, & \text{genus } (M) = 0, \\ 1, & \text{genus } (M) = 1, \\ 3(g-1) & \text{genus } (M) = g \geqslant 2. \end{cases}$$

B4. M is a Stein manifold. Then the vanishing theorem makes each HL-graded bundle trivial. Furthermore, by adopting the Grauert theorem we obtain $H^1_w(M, \mathrm{GL}(n, \Lambda E)) \equiv H^1_w(M, \mathrm{GL}(n, \Lambda E))$, which isomorphism implies the triviality of all holomorphic graded bundles over M.

B5. Graded bundles over instanton solutions. The Penrose transformation allows us to associate uniquely with any instanton (which we mean as a self-dual solution of Euclidean Yang–Mills equations over S^4 up to gauge transformations) a holomorphic bundle E over P^3C of rank $k > 1$ together with an anti-involutive map I: $E \to E$ so that

(1) $c_1(E) = 0$, $c_2(E) = p > 0$, $c_3(E) = 0$.

(2) E is stable, i.e., $H^0_w(P^3C, E) = 0$.

(3) Let the projection of P^3C on the quaternion projective space P^1H given by

$$[z_1, z_2, z_3, z_4] \to [z_1 + jz_2, z_3 + jz_4]$$

determine the bundle ϱ: $P^3C \to S^4$. Then all bundles $E|_{\varrho^{-1}(x)}$, $x \in S^4$, are trivial.

(4) I: $E \to E$ is a conjugate linear map such that $I^2 = -\mathrm{id}$ and $I(E_x) = E_{\tau(x)}$, where τ: $P^3C \to P^3C$ is the anti-podal mapping which exchanges anti-podal points in all fibres $\varrho^{-1}(x)$.

Instantons for wich $\mathrm{rank}(E) = k$, $c_2(E) = p$ form a real manifold of dimension $4kp - k^2 + 1$, see [3], [17]. Note that for the numbers k, p hold $k \leqslant 2p$.

The dimension of the first cohomology with coefficients in the instanton bundle E is

$$h^1_w(P^3C, E) = 2p - k.$$

This means that non-trivial HL-graded bundle having the instanton bundle E as an auxiliary bundle exist iff the number p is greater than $k/2$, where $\mathrm{rank}(E) = k$. Then such a graded bundle of rank n can be viewed as a system of n interacting p-instantons. Recall that each p-instanton looks like a system of p interacting so called pseudoparticles. Thus the considered graded bundles can be interpreted as systems of $p \times n$ interacting pseudoparticles.

B6. Massless particles. By means of the Penrose transformation one obtains a 1-1 correspondence between a set of wave functions of helicity s of massless particles and the cohomology $H^1_w(P^3C^+, H^{-2s-2})$, where $P^3C^+ \subset P^3C$ is a space of projective positive twistors (in the sense of some (2, 2)-hermitian form). Thus the procedure of graded bundles enables us to superpose the wave functions.

2. Supergeometry of graded bundles

In this section underlying manifolds and mappings can be both smooth and holomorphic unless no other assumption is made.

K. Gawędzki in [14], cf. also [20] and [21], worked out a superspace tensor calculus which is adequate for graded bundles too. As a superspace we mean the base manifold M together with an algebra $\Gamma_{\mathrm{loc}}(\Lambda E)$ of local sections of the Grassmann bundle ΛE (this algebra replaces the algebra $C^\infty_{\mathrm{loc}}(M)$ or $C^w_{\mathrm{loc}}(M)$ from differential geometry). The algebra $\Gamma_{\mathrm{loc}}(\Lambda E)$ possesses a natural Z_2 gradation

$$\Lambda E = \Lambda_0 E \oplus \Lambda_1 E,$$

where

$$\Lambda_0 E := \sum_{p=0}^{[k/2]} \Lambda^{2p} E, \quad \Lambda_1 E := \sum_{p=0}^{[(k-1)/2]} \Lambda^{2p+1} E, \quad k = \dim_K E_x, \quad x \in M.$$

We consider as a (graded) vector any K-linear mapping X of $\Gamma_{\mathrm{loc}}(\Lambda E)$ which can be decomposed $X = X_e + X_o$, where X_e (the even part) and X_o (the odd part) satisfy

$$\begin{aligned}
&X_e(\Gamma_{\mathrm{loc}}(\Lambda_i E)) \subset \Gamma_{\mathrm{loc}}(\Lambda_i E),\\
&X_e(\alpha\beta) = X_e(\alpha)\beta + \alpha X_e(\beta), \qquad \alpha, \beta \in \Gamma_{\mathrm{loc}}(\Lambda E),\\
&X_o(\Gamma_{\mathrm{loc}}(\Lambda_i E)) \subset \Gamma_{\mathrm{loc}}(\Lambda_{i\oplus 1} E),\\
&X_o(\alpha\beta) = X_o(\alpha)\beta + I(\alpha) X_o(\beta), \qquad \text{where} \quad I|_{\Lambda^p E} = (-1)^p.
\end{aligned}$$

One can easily check that for even vector fields a commutator is well defined but anti-commutator makes no sense but for odd vector fields it is just anti-commutator that makes sense whereas commutator does not.

Any vector X has the following expansion, of [11],

$$X = \sum_{i=1}^{m} \alpha^i \frac{\partial}{\partial x^i} + \sum_{j=1}^{k} \beta^j \frac{\partial}{\partial \theta^j}, \qquad m = \dim M,$$

where coefficients $\alpha^i(x)$, $\beta^j(x)$ belongs to the Grassmann algebras ΛE_x, $\frac{\partial}{\partial x^i}$ are "ordinary" vectors which constitute basises of tangent spaces $T_x M$ and $\frac{\partial}{\partial \theta^j}$ are contractions in the Grassmann algebras ΛE_x with local sections θ^j of E forming basises in the fibres E_x. The vectors of type $\alpha \frac{\partial}{\partial x}$, $\alpha(x) \in \Lambda_0 E_x$ (resp. $\alpha(x) \in \Lambda_1 E_x$) and $\beta \frac{\partial}{\partial \theta}$, $\beta(x) \in \Lambda_1 E_x$ (resp. $\beta(x) \in \Lambda_0 E_x$) are even (resp. odd).

Using the above notion of vector K. Gawędzki built in [14] a complete graded tensor calculus. Let us note that both symmetric and anti-symmetric graded tensor consists of two parts and one of them is symmetric while the second one is anti-symmetric in the usual sense. The exterior derivative of anti-symmetric graded tensor fields holds the equality $d^2 = 0$ which determines cohomologies of superspaces, see [20].

Our main purpose is to verify whether the geometric Weil procedure for Chern classes, see [6], works in the case of graded bundles or not. Thus we will need special differential graded forms, namely

$$\varphi = \alpha \frac{\partial}{\partial x^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{i_r}}, \qquad \psi = \beta \frac{\partial}{\partial x^{j_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{j_s}}.$$

Let us observe that if $\alpha(x) \in \Lambda_i E_x$, $\beta(x) \in \Lambda_j E_x$ then

$$\varphi \wedge \psi = (-1)^{rs+ij} \psi \wedge \varphi.$$

As a connection in the graded bundle $\mathscr{G}(E)$ we mean a mapping $\nabla\colon \Gamma_{\mathrm{loc}}(\mathscr{G}) \to \Gamma_{\mathrm{loc}}(\mathscr{G} \oplus T^*M)$ such that

$$\nabla(s_1+s_2) = \nabla s_1 + \nabla s_2,$$
$$\nabla(\alpha s) = d\alpha \oplus s + \alpha \nabla s, \quad s \in \Gamma_{\mathrm{loc}}(\mathscr{G}), \quad a \in \Gamma_{\mathrm{loc}}(\Lambda E).$$

For a given system $s = (s_1, \ldots, s_n)$, $s_i \in \Gamma_{\mathrm{loc}}(\mathscr{G})$, of local sections which form a basis of the ΛE-module $\Gamma_{\mathrm{loc}}(\mathscr{G})$ one gets a connection matrix

$$\nabla s_i = w_i^j s_j.$$

Observe that w_i^j are graded one-forms

$$w_i^j = \sum_{l=1}^{m} \sum_{p=0}^{k} a_{il}^{jp}\, dx_l, \qquad a_{il}^{jp}(x) \in \Lambda^p E_x.$$

If $s_i' = g_i^j s_j$ is another basis then w transforms as follows

$$w'g = \nabla g + gw,$$

where ∇ is a connection in $\Lambda E \otimes K^{n^2}$. Note that in order to define a connection in $\mathscr{G}(E)$ a connection in the auxiliary bundle must be given.

If any connection in $\mathscr{G}(E)$ is given then we can define horizontal sections of the graded bundle in a traditional way.

Similarly as in geometry of fibre bundles we may introduce a connection by means of a connection graded form γ on the space of the principal graded bundle which fulfills $w = s^*\gamma$.

It is natural to define a curvature matrix Ω by

$$\Omega := \nabla w - w \wedge w$$

(however we should remember that the skew-product of $\alpha_1 dx_i$, $\beta_1 dx_j$, $\alpha_1(x)$, $\beta_1(x) \in \Lambda_1 E_x$ is commutative). The transformation rule is

$$\Omega' = g\Omega g^{-1}.$$

There is no problem with the Bianchi identity

$$\nabla\Omega + \Omega\Lambda w - w\Lambda\Omega = 0.$$

Nevertheless apart from the perfect accordance with classical geometry certain formulae of geometry of fibre bundles fail in geometry of graded bundles. For instance instead of $d\,\mathrm{tr}\,\Omega = 0$ the weaker equations hold

$$d\,\mathrm{tr}\,\Omega_1 = 0, \quad d\,\mathrm{tr}\,\Omega_0 = d\,\mathrm{tr}\,w_1 \wedge w_1 \tag{0}$$

($\Omega_1 := \Omega|_{\Lambda_1 E}$, where the cutting is made in the set of values) so that $d\,\mathrm{tr}\,\Omega$ is a graded 3-form depending on w_1.

From now on all bundles and graded bundles will be complex. Let us recall the Weil construction of Chern classes, see [6]. It makes sense since: there exist j-homogeneous polynomial functions in matrix elements defined in the set of all complex $n \times n$ matrices p_j such that

$$p_j(GAG^{-1}) = p_j(A), \quad G \in \mathrm{GL}(n, C), \quad j = \deg p_j \leqslant n, \tag{1}$$

$$dp_j(\Omega) = 0. \tag{2}$$

Let us observe that none of the above properties can be maintained in the case of graded bundles.

In order to prove the non-existence of any polynomial-function satisfying (1) it is sufficient to check for a certain matrix A with elements in ΛV that eigenvalues of matrices gAg^{-1}, $g \in \mathrm{GL}(n, \Lambda V)$ may change independently each from another one.

Since $p_1(\Omega) = \mathrm{tr}\,\Omega$ then formula (0) makes (2) not correct for graded bundles.

But the above obstructions do not exclude any possibility of defining "graded Chern classes" in this way for special graded bundles. For instance if $\mathscr{G}(E)$ is a graded $K(n, \Lambda E)$-bundle such that the group $K(n, \Lambda E)|_{\Lambda^0 E}$ is Abelian and if we assume that $d\,\mathrm{tr}\,w_1 \wedge w_1 = 0$ then

$$c_g^1(\Omega) := \mathrm{tr}\,\Omega|_{(\Lambda^0 + \Lambda^1)E}$$

defines an element of $H^1(M, Z \oplus E)$ which looks like a first Chern class of such graded bundles.

The impossibility of the direct extension of the Weil construction of Chern classes seems to be relevant to the impossibility of founding any natural integration of odd graded forms, see [8].

Now we will point out a substitute of the first Chern class for projective holomorphic graded bundles.

By a projective graded bundle up to the equivalence we understand an element of the cohomology $H^1_w(M, \mathrm{GL}(n, \Lambda E)/N(E))$, where $N(E_x)$ is a normal subgroup of $\mathrm{GL}(n, \Lambda E_x)$ such that all quotient groups are mutually isomorphic and holomorphic.

Assume that $N(E_x)$ are Abelian. Then we have the following exact sequence

$$\to H^1_w(M, N(E)) \to H^1_w(M, \mathrm{GL}(n, \Lambda E))$$
$$\to H^1_w(M, \mathrm{GL}(n, \Lambda E)/N(E)) \xrightarrow{\delta} H^2_w(M, N(E)),$$

where extreme terms are topological groups and terms in the middle are topological spaces with distinguished zero-elements.

If $N(E) = I_n \oplus \Lambda^j E \oplus I_n$, where $j = k$ or j is even and $k/2 < j < k$ then δ takes values in $H^2_w(M, \Lambda^j E)$ and is much similar to the first Chern class of line bundles.

3. Comment

In graded bundles the great group $\mathrm{GL}(n, \Lambda V)$ is involved. In this theory we have no reduction of this group to a compact or an Abelian part as we have usually in the theory of fibre bundles. I expect to see in graded bundles a lot of effects of the lack of such reductions.

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