COMPUTATIONAL MATHEMATICS
BANACH CENTER PUBLICATIONS, VOLUME 13
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1984

ON OPTIMAL METHODS IN NUMERICAL ANALYSIS

A. G. SUKHAREV

Moscow State University, Moscow, USSR

The optimal and sequentially-optimal algorithms (OA and SOA) are defined for a general model of the computation. The algorithm, guaranteeing the best accuracy in the solution of a problem, which is the worst for this algorithm among the problems of a certain class, is called *optimal by accuracy*. It is shown that such a definition does not always reflects the specificity of an effective organization of the real computational processes. The *sequentially-optimal by accuracy* algorithm is defined as an algorithm, which at any stage of the computational process guarantees the best possible accuracy with respect to all the information accumulated at the previous stages of the computation. AO and SOA are constructed for the problems of a global extremum search, optimal recovery and numerical integration of functions. Issues of applications are discussed.

1. General problem and definition of optimality

Let F be a set of a linear space over the real or complex field and let S be a mapping from F to a certain space B with a distance function γ. The problem is to find the best approximation to $S(f) \in B$ for some $f \in F$, i.e. such an approximation element $a(f) \in B$ that

$$\varepsilon(a, f) = \gamma\big(S(f),\ a(f)\big)$$

is minimal. Let us call $\varepsilon(a, f)$ the *accuracy of solution*.

Before defining the general scheme we would like to give a few examples. Let a function f be defined on a set K of n-dimensional Euclidean space E_n. If the maximal value of f over K is sought, then $B = E_1$, $S(f) = \max_{x\in K} f(x)$. If the whole set or a point, at which the maximal value is attained, is sought as well, then $B = E_1 \times 2^K$ with a distance function defined in a suitable way.

For the problem of the optimal recovery of a function one may put $B = \{\varphi \mid \varphi\colon K \to E_1\}$, $S(f) = f$, and for that of numerical integration $B = E_1$, $S(f) = \int_K f(x)dx$.

For choosing a proper method in the problem solution, e.g. one of the enumerated above, it is necessary to have some information of the function f. This information may be obtained proceeding from the physical essence of the mathematical model. Such properties of functions as continuity, smoothness, monotony, convexity, boundness of the change rate may correspond to different physical properties.

The information possessed by the computer (mathematician) is the ground on which he relates f to either set (class) F. If such information is absent or insufficient, there might be a need for a stage of the preliminary research.

After the class F is fixed, all the information consists of the computer's knowledge of the operator S, which is approximated, and the fact that $f \in F$.

A method (algorithm, strategy) α is given by a set of mappings

$$\alpha = (\tilde{x}_1, \ldots, \tilde{x}_N, \tilde{\beta}) \tag{1}$$

with

$$\begin{aligned}
&\tilde{x}_1 \equiv x_1\colon F \to Y_1,\ x_1 \in X_1,\\
&\tilde{x}_2\colon X_1 \times Y_1 \to X_2 \ni x_2\colon F \to Y_2,\\
&\tilde{x}_3\colon X_1 \times Y_1 \times X_2 \times Y_2 \to X_3 \ni x_3 \to Y_3,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\tilde{x}_N\colon X_1 \times Y_1 \times \ldots \times X_{N-1} \times Y_{N-1} \to X_N \ni x_N\colon F \to Y_N,\\
&\tilde{\beta}\colon X_1 \times Y_1 \times \ldots \times X_N \times Y_N \to B \ni \beta.
\end{aligned}$$

The problem solution consists of the successive computations:

$$y_1 = x_1(f),\ x_2 = \tilde{x}_2(x_1, y_1),\ y_2 = x_2(f), \ldots$$
$$\ldots, x_N = \tilde{x}_N(x_1, y_1, \ldots, x_{N-1}, y_{N-1}),\ y_N = x_N(f),$$
$$\beta = \tilde{\beta}(x_1, y_1, \ldots, x_N, y_N).$$

Finally,

$$\alpha(f) = \beta.$$

The operations of computing $x_i(f)$, $i = 1, \ldots, N$, are called *basic*, those of $\tilde{x}_i(x_1, y_1, \ldots, x_{i-1}, y_{i-1})$, $i = 2, \ldots, N$, $\tilde{\beta}(x_1, y_1, \ldots, x_N, y_N)$ *auxiliary* and the last one — auxiliary and *final*.

Computations of various characteristics of a function f at a point $x_i \in K$, such as the values of the function or its derivatives, are often chosen as an operation of a basic computation.

The number of the basic computations N is considered to be fixed and determined by the computer resources. Henceforth, in all the examples of the construction of OA and SOA we shall consider a case:

$$x_i \in K, \quad x_i(f) = f(x_i), \quad X_i = K, \quad Y_i = E_1, \quad i = 1, \ldots, N.$$

Although all definitions are of a general character, it is convenient to bear in mind this very case.

Now, we would like to describe a set of feasible algorithms. Any mappings from $X_1 \times Y_1 \times \ldots \times X_N \times Y_N$ to B are often considered as feasible final operations. This will be the case, if there are no special comments. The sets of feasible mappings $\tilde{x}_2, \ldots, \tilde{x}_N$ depend on the computation model under study. Two extreme cases are of the utmost interest.

The first case corresponds to a situation in which the computer, while performing any basic operation, does not use any information accumulated on the previous steps of the computational process. It might be so, for example, if all the computations are to be performed simultaneously. In this case $\tilde{x}_i = x_i$, $i = 1, \ldots, N$, $a = (x_1, \ldots, x_N, \tilde{\beta})$. Such strategies (algorithms) are called *passive*. Let us denote the set of all of them as A_0^N.

The second case represents a situation where the computer has all the information about the results of the previous basic operations and he possesses of sufficient recources for the storage of this information and its processing. If this is a case, all the algorithms in the form given by (1) are feasible. A set of all such algorithms is called a set of *sequential* algorithms and is denoted as A_1^N.

Intermediate cases are also under consideration ([1], [2]).

The algorithm a^0 is called *optimal* (*minimax*) *by accuracy* in a set A^N of algorithms with N basic operations, if

$$\sup_{f \in F} \varepsilon(a^0, f) = \inf_{a \in A^N} \sup_{f \in F} \varepsilon(a, f).$$

Denote

$$x^i = (x_1, \ldots, x_i), \quad y^i = (y_1, \ldots, y_i), \quad z^i = (x^i, y^i),$$

$$F_{z^i} = \{f \in F \mid x_j(f) = y_j,\ j = 1, \ldots, i\},$$

$$\varepsilon(z^N) = \inf_{\beta \in B} \sup_{f \in F_{z^N}} \gamma\big(S(f), \beta\big), \tag{2}$$

where

$$x_j \in X_j,\ y_j \in Y_j, \quad j = 1, \ldots, i.$$

It is possible to show that

$$(3)\qquad \inf_{\alpha\in A_1^N}\sup_{f\in F}\varepsilon(\alpha,f)=\inf_{x_1\in X_1}\sup_{y_1\in\{y_1|F_{z^1}\neq\emptyset\}}\dots\inf_{x_N\in X_N}\sup_{y_N\in\{y_N|F_{z^N}\neq\emptyset\}}\varepsilon(z^N).$$

After the completion of all the basic computations the computer's information of the problem f is $f\in F_{z^N}$, if the results of the basic computations are

$$x_j(f)=y_j,\quad j=1,\dots,N.$$

In case where all final operations are feasible and the computer aims to achieve the best guaranteed accuracy, he will choose β, at which infimum in (2) is attained, as an optimal final operation. By means of this correspondence

$$\tilde{\beta}^0\colon X_1\times Y_1\times\dots\times X_N\times Y_N\to B$$

is defined. We used to consider in it an element of OA and sometimes call $(\tilde{x}_1,\dots,\tilde{x}_N)$ an algorithm, meaning that the final operation is $\tilde{\beta}^0$.

It is easy to see that

$$\sup_{y^N\in\{y^N|F_{x_1^0,y_1,\dots,x_N^0,y_N}\neq\emptyset\}}\varepsilon(x_1^0,y_1,\dots,x_N^0,y_N)=\inf_{x^N\in X_1\times\dots\times X_N}\sup_{y^N\in\{y^N|F_{z^N}\neq\emptyset\}}\varepsilon(z^N)$$

is the best guaranteed result (accuracy) in the set A_0^N and that the strategy $(x_1^0,\dots,x_N^0,\tilde{\beta}^0)$ is optimal in A_0^N.

2. Construction of optimal algorithms for some concrete problems

2.1. On functional class F. In further examples we dwell on a functional class

$$(4)\qquad F=\{f\colon K\to E_1\mid |f(u)-f(v)|\leqslant M\varrho(u,v);\ u,v\in K\}$$

which is determined by an arbitrary non-negative function ϱ defined on $K\times K$. We assume that ϱ has the properties of pseudometrics:

$$\varrho(u,u)=0,\quad \varrho(u,v)=\varrho(v,u)\geqslant 0,\quad \varrho(u,v)+\varrho(v,w)\geqslant\varrho(u,w),$$
$$u,v,w\in K.$$

If it is not so, one may construct another function with the properties of pseudometrics, which determines the same functional class F. A number of the functional classes may be described in such a form, including the classes of functions satisfying Lipschitz or Hölder condition with a fixed constant or different constants on various parts of K, classes determined by modules of the continuity and even those of discontinuous functions, in case ϱ being discontinuous.

A general class F defined by ϱ suits satisfactorily a number of the practical problems. The matter is that a problem, for which a question of finding an OA deserves a serious consideration, is, as a rule, a problem of a complicated nature. Important examples are given by problems of the computer aided design. The performance of a single basic operation for them may require a solution of a complex problem to calculate the effectiveness of a variant of a technical system which is designed. The information concerning this kind of problem is, as a rule, scarce. In this connection, the constraints on f in (4) are not very restrictive. The inequality in (4) describes the boundness of the change rate in the characteristics of the optimized system as functions of the systems parameters.

2.2. Search for global extremum. Let us suppose that it is necessary to find an approximate value $f(x_0) = \max\limits_{x\in K} f(x)$ and a point of K at which the approximate value is attained; then the accuracy of solution is the difference between the maximum value and its approximation. Then

$$S(f) = \big(f(x_0),\ \operatorname{Arg}\max_{x\in K} f(x)\big), \qquad a(f) = \big(f(x_a),\ x_a\big),$$

$$\gamma\big(S(f),\ a(f)\big) = f(x_0) - f(x_a).$$

Let the basic computations be finished: $y_i = f(x_i)$, $i = 1, \ldots, N$. Clearly, $\tilde{\beta}^0(z^N) = (y_{i_0}, x_{i_0})$, where $y_{i_0} = \max\limits_{i=1,\ldots,N} y_i$. If only the maximum value were sought, it would be necessary to define $\tilde{\beta}^0$ as follows:

$$\tilde{\beta}^0(z^N) = \tfrac{1}{2}\inf_{f\in F_{z^i}} \max_{x\in K} f(x) + \tfrac{1}{2}\sup_{f\in F_{z^i}} \max_{x\in K} f(x).$$

It is shown in [3] that the points $x_1^0, \ldots, x_N^0$ of the optimal strategy are the centers of the covering of K by equal ϱ-spheres of a minimal radius (we call a set $\{x \mid \varrho(x, a) \leqslant r\}$ a ϱ-sphere with a center a and a radius r). Thus

$$\sup_{x\in K} \min_{i=1,\ldots,N} \varrho(x, x_i^0) = \min_{x^N\in K^N} \sup_{x\in K} \min_{i=1,\ldots,N} \varrho(x, x_i).$$

It may be trivial to solve a problem of the optimal covering as it is for

$$K = \{u = (u_1, \ldots, u_n) \mid 0 \leqslant u_i \leqslant 1,\ i = 1, \ldots, n\},$$

$$\varrho(u, v) = \max_{i=1,\ldots,n} |u^i - v^i|, \qquad N = m^n$$

with an integer m. In this case $i_1/2m, \ldots, i_n/2m$ are the centers of the optimal covering, $i_j = 1, 3, \ldots, 2m-1$; $j = 1, \ldots, n$. In other cases, finding of $x_1^0, \ldots, x_N^0$ is a difficult problem of discrete geometry ([4]).

2.3. Optimal recovery. Let us suppose that it is needed to compute the value of f at a number of points of K, but it is not known in advance, which ones will have to be used. The value of f may be computed without an error at every point of the set K, but as the resources of the computer are limited, only N computations may be done. After the selection of points $x_1, \ldots, x_N$ and computation $y_1 = f(x_1), \ldots, y_N = f(x_N)$ the computer constructs a function φ_{z^N} providing the best uniform approximation to f. The computation of values $f(x)$ may be replaced now by the computation of $\varphi_{z^N}(x)$. For this problem

$$B = \{\beta \mid \beta\colon K \to E_1\}, \qquad S(f) = f, \qquad \gamma(f, \beta) = \sup_{x \in K} |f(x) - \beta(x)|,$$

$$\tilde{\beta}^0(z^N) = \varphi_{z^N}.$$

To find an optimal passive strategy one has to find

$$\inf_{x^N \in K^N} \sup_{y^N \in \{y^N \mid F_{z^N} \neq \emptyset\}} \inf_{\beta \in B} \sup_{f \in F_{z^N}} \sup_{x \in K} |f(x) - \beta(x)|$$

and $x_0^N = (x_1^0, \ldots, x_N^0)$, $\tilde{\beta}^0(z^N)$ for which infimums are attained. The strategy $(x_0^N, \tilde{\beta}^0)$ is an optimal by accuracy in the set A_0^N. It appears (see [6]) that x_0^N is the same centers vector of the optimal covering of K as for the problem of the global optimization, and $\tilde{\beta}^0(z^N) = \varphi_{z^N}$ where

$$\varphi_{z^N}(x) = \tfrac{1}{2} \max_{j=1,\ldots,N} \{y_j - M\varrho(x, x_j)\} + \tfrac{1}{2} \min_{j=1,\ldots,N} \{y_j + M\varrho(x, x_j)\}.$$

2.4. Numerical integration. Let $B = E_1$, $S(f) = \int_K f(x)\,dx$. Assume that $K, f \in F$ are measurable, $\varrho(u, v)$ is summable over u for a fixed v.

Now we find optimal coefficients $p_1^0, \ldots, p_N^0$, if points $x_1, \ldots, x_N$ of a quadrature formulae are fixed. The optimal coefficients are defined by an equation

$$\max_{f \in F} \left| \int_K f(x)\,dx - \sum_{i=1}^{N} p_i^0 f(x_i) \right| = \min_{p^N \in E_N} \max_{f \in F} \left| \int_K f(x)\,dx - \sum_{i=1}^{N} p_i^0 f(x_i) \right|$$

with

$$p^N = (p_1, \ldots, p_N).$$

The problem represents a particular case of the general problem, where the computer has to choose only a final operation with a corresponding mapping linear over y_0^N.

Let us define $K_i(x^N)$, $i = 1, \ldots, N$, as any sets, satisfying the following conditions: $\varrho(x, x_i) = \min_{j=1,\ldots,N} \varrho(x, x_j)$ for $x \in K_i(x^N)$, $K_i(x^N)$ is measurable, $\bigcup_{i=1}^{N} K_i(x^N) = K$, $K_i(x^N) \cap K_j(x^N) = \emptyset$ for $i \neq j$. Note that such a partition of K may be non-unique. Let $\mu(A) = \int_A dx$. It may be shown that (see (6)) $p_i^0 = \mu(K_i(x^N))$, $i = 1, \ldots, N$, are the optimal coefficients. Optimal points $x_1^0, \ldots, x_N^0$ may be found as a solution of the problem

$$\int_K \min_{i=1,\ldots,N} \varrho(x, x_i^0)\,dx = \min_{x^N \in K^N} \int_K \min_{i=1,\ldots,N} \varrho(x, x_i)\,dx,$$

which is closely connected with that of the optimal covering. We note further that $\int_K \min_{i=1,\ldots,N} \varrho(x, x_i^0)\,dx$ is the best result (accuracy) guaranteed by quadrature formulas. The best guaranteed result will not be changed, if an arbitrary mapping $\tilde{\beta}$ in the form of (1) is a feasible final operation.

It is even more interesting that the best result guaranteed by the sequential algorithms is equal to that guaranteed by the passive strategies for all problems under discussion, i.e.

$$\inf_{\alpha \in A^N} \sup_{f \in F} \varepsilon(\alpha, f) = \inf_{\alpha \in A_1^N} \sup_{f \in F} \varepsilon(\alpha, f).$$

Thus, the optimal passive strategy is also optimal among all the sequential algorithms. It is corollary of some general results ([7], [8]), although it may be easily proved by a direct way for the functional class we are dealing with.

3. Sequentially-optimal algorithms

3.1. Definition of sequentially-optimal algorithm. Let $K = [0, 1]$, $\varrho(u, v) = M|u-v|$. Then

$$\left(\frac{1}{2N}, \frac{3}{2N}, \ldots, \frac{2N-3}{2N}, \frac{2N-1}{2N}\right)$$

is an optimal passive strategy. As noted above, it is also optimal in the set of all the sequential algorithms A_1^N. Clearly, the same is relevant to the strategy $\left(\frac{1}{2N}, \frac{2N-1}{2N}, \frac{3}{2N}, \ldots, \frac{2N-3}{2N}\right)$.

Suppose that using the latter algorithm the computer has got after two basic computations

$$f\left(\frac{1}{2N}\right) = M\frac{1}{2N}, \quad f\left(\frac{2N-1}{2N}\right) = M\frac{2N-1}{2N}.$$

Then, obviously, $f(x) = x$ for $x \in [1/2N, (2N-1)/2N]$, i.e. there is no need to compute f at the points of $[1/2N, (2N-1)/2N]$. Nevertheless, the OA prescribes to perform all the remaining computations at the points of this very set.

This example shows that the definition of the optimality does not meet fully the requirements of the efficient organization of the real computational processes. The application of the principle of guaranteed results at every stage of a computational process makes necessary to look for an algorithm, which is not only optimal, i.e. guarantees the best a priori possible result, but makes use in an optimal way of all information accumulated during the computational process. A formal definition of such algorithms is the following:

An algorithm $a^0 = (\tilde{x}_1^0, \ldots, \tilde{x}_N^0, \tilde{\beta}^0)$ is called *sequentially-optimal by accuracy* in A_1^N, iff the outer minimum in the expression:

$$\varepsilon_{z^i}(N) = \inf_{x_{i+1}\in X_{i+1}} \sup_{y_{i+1}\in\{y_{i+1}|F_{z^{i+1}}\neq\emptyset\}} \ldots \inf_{x_N\in X_N} \sup_{y_N\in\{y_N|F_{z^N}\neq\emptyset\}} \varepsilon(z^N)$$

is attained at $x_{i+1} = \tilde{x}_{i+1}^0(z^i)$, the minimum in (2) is attained at $\beta = \tilde{\beta}^0(z^N)$ for any possible z^i, $i = 1, \ldots, N$, and the outer minimum in the right-hand side of (3) is attained at $x_1 = \tilde{x}_1^0$.

The notions of the algorithms optimal and sequentially-optimal by the number of the basic computations are also of some interest. Let

$$\varepsilon(N) = \inf_{a\in A_1^N} \sup_{f\in F} \varepsilon(a, f), \qquad N(\varepsilon) = \min\{N|\ \varepsilon(N) \leqslant \varepsilon\}.$$

An algorithm a^0 is called *optimal by the number of the basic computations among the sequential algorithms guaranteeing the accuracy* ε, iff $a^0 \in A_1^{N(\varepsilon)}$ and $\sup_{f\in F} \varepsilon(a^0, f) \leqslant \varepsilon$.

An algorithm a^0 is called *sequentially-optimal by the number of the basic computations among the sequential algorithms guaranteeing the accuracy* ε, iff it is optimal and prescribes to act in any possible situation so that to minimize the number of the remaining basic computations while guaranteeing the achievement of the accuracy ε.

The overall number of the basic computations may prove to be much less than $N(\varepsilon)$.

All definitions of the optimality, that we have given, do not take into account the resources spent on the internal needs of the algorithm, that is — the computations of the functions $\tilde{x}^i, \tilde{\beta}$. This disregarding of the so-called combinatorial cost of the algorithm is justified in case where the problem is so complicated that the overwhelming part of the resources is spent on the basic computations. The applied significance of such problems is out of question and it was discussed in brief above. Other approaches are also investigated ([9]).

Now we shall construct sequentially-optimal algorithms for the problems of a global extremum search, optimal recovery and numerical integration of functions. Here, we confine by the consideration of the functional class

$$F = \{f \mid |f(u)-f(v)| \leqslant M|u-v|,\ u, v \in [0,1]\}, \qquad K = [0,1].$$

3.2. Sequentially-optimal algorithm of global optimization. Let us consider a case, where the accuracy ε is specified and the number of the basic computations is to be minimized ([10], [11]).

The first basic computation (according to the algorithm sequentially optimal by the number of the basic computations) should be effected at any one of the points

$$u_1, \ldots, u_{N(\varepsilon)} \quad \text{where} \quad \bigcup_{j=1}^{N(\varepsilon)} [u_j-\varepsilon, u_j+\varepsilon] \supset [0,1], \qquad N(\varepsilon) = \left]\frac{1}{2\varepsilon}\right[.$$

Here $]t[$ is the minimal integer, which is greater or equal t.

Suppose that the basic computations at the points $x_1, \ldots, x_i$ are performed, $y_1 = f(x_1), \ldots, y_i = f(x_i)$. The upper bound for values of the functions from F_{z^i} at any point $x \in K$ is given by

$$\varphi^+_{z^i}(x) = \min_{j=1,\ldots,i} \{y_j + M|x-x_j|\}.$$

The set

$$K_{z^i} = \{x \mid \varphi^+_{z^i}(x) > \max_{j=1,\ldots,i} y_j + \varepsilon\}$$

consists of the points, at which functions of F_{z^i} may assume values greater than $\max\limits_{j=1,\ldots,i} y_j + \varepsilon$. Suppose that

$$\bigcup_{i=1}^{N_{z^i}(\varepsilon)} [v_j-\varepsilon, v_j+\varepsilon] \supset K_{z^i}$$

and it is impossible to cover K_{z^i} by $N_{z^i}(\varepsilon)-1$ closed intervals with the lengths 2ε. Then $N_{z^i}(\varepsilon)$ is the minimal number of the basic computations, with which the accuracy ε in the situation z^i may be achieved. Any one from the points v_j, $j = 1, \ldots, N_{z^i}(\varepsilon)$, may be chosen as a point x_{i+1} of the $(i+1)$st the basic computation by SOA.

The number of the basic computations by SOA proved to be 4–50 times less than by OA, while the same accuracy was specified.

Now we shall make a few points on the issues of the SOA applications. The problem of a global optimization is taken just as an example, the same points being relevant to other problems. A usual objection to the practical applications of SOA is that the constant M is allegedly unknown for the most practical problems. In this case one may evaluate M, taking

$$M_i = \sigma_i \max_{j,k \leqslant i} |f(x_j)-f(x_k)|/\|x_j - x_k\|$$

with $\sigma_i \geqslant 1$ as an approximation to M after i basic computations. In such manner, a number of new algorithms may be constructed, depending on coefficients σ_i. These algorithms are not optimal and do not even guarantee the finding of a global extremum, however, their high efficiency is confirmed by the practical applications.

3.3. Sequentially-optimal recovery of functions. Let the number N of the basic computations be specified.

Any one of the points $1/2N, 3/2N, \ldots, (2N-1)/2N$ (which are the centers of the optimal covering of $[0, 1]$) may be chosen as a point x_1 of the first basic computation by a SOA. Suppose that the computations at the points $x_1, \ldots, x_i$ have been effected and let $x_{i1} < x_{i2} < \ldots < x_{ii}$, where $x_{i1}, x_{i2}, \ldots, x_{ii}$ is the permutation of the numbers $x_1, \ldots, x_i$. In order to find the point x_{i+1} one has to solve the integer problem for the determination of:

$$\varepsilon_{g}^{i}(N) = \min_{\substack{n_1, \ldots, n_{i+1} \in \{0,1,2,\ldots\} \\ n_1 + \ldots + n_{i+1} = N-i}} \max \left\{ \frac{2x_{i1}}{2n_1+1}, (x_{i2}-x_{i1}) W_{n_2}(l_{i2}), \ldots \right.$$
$$\left. \ldots, (x_{ii}-x_{i,i-1}) W_{n_i}(l_{ii}), \frac{2(1-x_{ii})}{2n_{i+1}+1} \right\},$$

where

$$l_{ij} = \big(-|f(x_{ij})-f(x_{i,j-1})|/M + x_{ij} - x_{i,j-1}\big)/(x_{ij}-x_{i,j-1}),$$

and $W_n(l)$ is the value of the following multistep antagonistic game.

At the first step the minimizing player chooses a point $t \in (0, 1)$ dividing the interval $[0, 1]$ into two, while the maximizing player, knowing t, assigns to these intervals two numbers l_1, l_2 such that $0 \leqslant l_1 \leqslant t$, $0 \leqslant l_2 \leqslant 1-t$, $l_1 + l_2 = l$, where $l \in [0, 1]$ is a previously fixed number known to both players.

At the second step the minimizing player chooses a point within one of the intervals $[0, t]$ and $[t, 1]$ dividing it into two subintervals; the maximizing player, knowing this choice, assigns to them two non-negative numbers, each of which does not exceed the length of its subinterval, with a sum equal to the number that was assigned to the union of these subintervals at the first step, etc. After n steps the maximizing player obtains from the minimizing one the maximum of the numbers assigned to the $n+1$ subintervals, into which the interval $[0, 1]$ has been divided.

Let $T_{n,r}(l)$ be a point of the optimal choice of the minimizing player at the first step of the game. Here r is any integer of the set $\{1, \ldots, r(n, l)\}$. Thus, the optimal choice of the minimizing player is non-unique. The formulas for $W_n(l)$, $T_{n,r}(l)$, $r(n, l)$ are given in [5] and [12].

If $n_1, \ldots, n_{i+1}$ is a set at which the minimum in the right-hand side of (5) is attained, then one may choose for x_{i+1} any one of the points

$$x_{i1}\frac{r_1}{2n_1^i+1}, \qquad r_1 = 1, 3, \ldots, 2n_1^i-1, \quad \text{if} \quad n_1^i > 0;$$

$$x_{i,j-1}+(x_{ij}-x_{i,j-1})T_{n_j^i,r_j}(l_{ij}), \qquad r_j = 1, \ldots, r(n_j^i, l_{ij}),\ j = 2, \ldots, i, \quad \text{if} \quad n_j^i > 0;$$

$$x_{ii}+(1-x_{ii})\frac{r_{i+1}}{2n_{i+1}^i+1}, \qquad r_{i+1} = 2, 4, \ldots, 2n_{i+1}^i, \quad \text{if} \quad n_{i+1}^i > 0.$$

3.4. Sequentially-optimal algorithm of numerical integration. Let the number N of the basic computations be specified. Any one of the points $1/2N, 3/2N, \ldots, (2N-1)/2N$ may be chosen as x_1. Suppose that the computations at the points $x_1, \ldots, x_i$ were effected and let $x_{i1} < x_{i2} < \ldots \ldots < x_{ii}$ where $x_{i1}, x_{i2}, \ldots, x_{ii}$ is the permutation of the numbers $x_1, \ldots, x_i$.

In order to find the point x_{i+1}, one has to solve the integer problem for the determination of

$$(6) \qquad \varepsilon_{z^i}(N) = \min_{\substack{n_1,\ldots,n_{i+1}\in\{0,1,2,\ldots\}\\ n_1+\ldots+n_{i+1}=N-i}} \left[\frac{M}{4}\frac{x_{i1}^2}{n_1+1/2} + \frac{1}{4M}\sum_{j=2}^{i}\frac{(x_{ij}-x_{i,j-1})^2M^2-(y_{ij}-y_{i,j-1})^2}{n_j+1} + \frac{M}{4}\frac{(1-x_{ii})^2}{n_{i+1}+1/2}\right].$$

One may easily get an approximate solution of the problem by rounding off the problem solution with the same objective function as in (6), but without the requirement that $n_1, \ldots, n_i$ are to be integers. The latter problem may be easily solved. The exact solution of the problem (6) may be found, if the Gross' criterion ([13]) is applied.

If $n_1^i, \ldots, n_{i+1}^i$ is a set at which the minimum in the right-hand side of (6) is attained, then, as it is shown in [14], one may choose for x_{i+1} any one of the points

$$x_{i1}\frac{r_1}{2n_1^i+1}, \qquad r_1 = 1, 3, \ldots, 2n_1^i-1, \quad \text{if} \quad n_1^i > 0,$$

$$x_{i,j-1}+(x_{ij}-x_{i,j-1})\frac{r_j}{n_j^i+1}, \qquad r_j = 1, \ldots, n_j^i, \quad \text{if} \quad n_j^i > 0,$$

$$x_{ii}+(1-x_{ii})\frac{r_{i+1}}{2n_{i+1}^i+1}, \qquad r_{i+1} = 2, 4, \ldots, 2n_{i+1}^i, \quad \text{if} \quad n_{i+1}^i > 0.$$

So, one may see that the construction of SOA requires the necessity to solve difficult problems from various mathematical fields: discrete geometry, game theory, integer programming and others. In most cases, the complete solutions are presently obtain for the classes of one-variable functions. Sometimes, however, these results may be applied for the solution of the multidimensional problems ([15], [16]).

References

[1] J. F. Traub, *Iterative methods for the solution of equations*, Prentice-Hall, Englewood Cliffs 1964.

[2] D. J. Wilde and C. S. Beightler, *Foundations of optimization*, Prentice-Hall, Englewood Cliffs 1967.

[3] А. Г. Сухарев, *Об оптимальных стратегиях поиска экстремума*, Ж. вычисл. мат. и мат. физ. 11.4 (1971), 910–924.
A. G. Sukharev, *Optimal strategies of the search for an extremum*, USSR, Computational Mathematics and Mathematical Physics 11.4 (1971), 119–137.

[4] C. A. Rogers, *Packing and covering*, Cambridge University Press, Cambridge 1964.

[5] А. Г. Сухарев, *Оптимальные пассивные и последовательно-оптимальные алгоритмы построения наилучших аппроксимаций для функций, удовлетворяющих условию Липшица*, Доклады АН СССР 231.4 (1976), 814–817.
A. G. Sukharev, *Optimal passive and sequentially-optimal algorithms for constructing best approximations for functions satisfying a Lipschitz condition*, Soviet Mathematics, Dokl. 17.6(1976), 1660–1664.

[6] А. Г. Сухарев, *Оптимальные формулы численного интегрирования для некоторых классов функций многих переменных*, Доклады АН СССР 246.2 (1979), 282–285.
A. G. Sukharev, *Optimal numerical integration formulas for some classes of functions of several variables*, Soviet Mathematics, Dokl. 20.3 (1979), 472–475.

[7] Н. С. Бахвалов, *Об оптимальности линейных методов приближения операторов на выпуклых классах функций*, Ж. вычисл. мат. и мат. физ. 11.4 (1971), 1014–1018.
N. S. Bakhvalov, *On the optimality of linear methods for operator approximation in convex classes of functions*, USSR, Computational Mathematics and Mathematical Physics, 11.4 (1971), 244–249.

[8] Н. Ф. Зализняк и А. А. Лигун, *Об оптимальных стратегиях поиска глобального максимума функции*, Ж. вычисл. мат. и мат. физ. 18.2 (1978), 314–321.

[9] J. F. Traub and H. Wozniakowski, *A general theory of optimal algorithms*, Academic Press, New York 1980.

[10] А. Г. Сухарев, *Наилучшие стратегии последовательного поиска экстремума*, Ж. вычисл. мат. и мат. физ. 12.1 (1972), 35–50.
A. G. Sukharev, *Best sequential search strategies for finding an extremum*, USSR, Computational Mathematics and Mathematical Physics 12.1 (1972), 39–59.

[11] А. Г. Сухарев, *Оптимальный поиск экстремума*, Издательство Московского Университета, Москва 1975.

[12] —, *Решение многошаговой антагонистической игры аппроксимации*, Вестник Московского Университета, Серия вычислительной математики и кибернетики, 2 (1977), 43–57.

[13] O. Gross, *A class of discrete type minimization problems*, Rand Corporation Research Memorandum No. 1644, 1956.

[14] A. G. Sukharev, *A sequentially-optimal algorithm for numerical integration*, J. Optimization Theory Appl. 28.3 (1979), 363–373.

[15] Р. Г. Стронгин, *Численные методы в многоэкстремальных задачах*, „Наука", Москва 1978.

[16] A. G. Sukharev, *Optimal algorithms for iterated numerical integration*, J. Optimization Theory Appl. 28.3 (1979), 375–390.

Presented to the Semester
Computational Mathematics
February 20 – May 30, 1980