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# SOME PARAMETER ESTIMATION PROBLEMS FOR HYPOELLIPTIC HOMOGENEOUS GAUSSIAN DIFFUSIONS

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This paper is concerned with the statistical problem of parameter estimation for hypoelliptic homogeneous Gaussian diffusions. Since quadratic forms of the processes under study play a central role, some of their properties are proved first. Then the maximum likelihood method is used to derive ordinary and sequential plans for parameter estimation and characteristics of these plans are studied.

## 1. Introduction

Stochastic linear models for dynamical systems in continuous time have been intensively studied both in the literature concerning systems theory ([3], [18]) and in that about the statistics of random processes ([1], [10])

In the present paper we are concerned with a multidimensional model which is defined by an autonomous linear stochastic differential equation of the following form:

$$dX_t^x = AX_t^x\,dt + G\,dW_t, \qquad t \geqslant 0,\ X_0^x = x \tag{1.1}$$

where

$W = (W^1, \ldots, W^r)'$ is some standard Brownian motion in $\boldsymbol{R}^r$ defined on some basic probability space $(\Omega, \mathscr{A}, (\mathscr{A}_t)_{t \geqslant 0}, P)$,

$A$ and $G$ are $n \times n$ and $n \times r$ constant matrices,

$x$ stands for any initial state of the system in $\boldsymbol{R}^n$.

Let us recall that (cf. [10]), for every $x \in \boldsymbol{R}^n$, the process $X^x = (X_t^x, t \geqslant 0)$ is a Gaussian Markov process with mean function

$$m_t^x = EX_t^x = e^{At}x, \qquad t \geqslant 0,$$

and covariance function (not depending on $x$)

$$K_{t,s} = E\{(X_t^x - EX_t^x)(X_s^x - EX_s^x)'\} = \begin{cases} e^{A(t-s)}K_s & \text{if } t \geqslant s, \\ K_t e^{A'(s-t)} & \text{if } t \leqslant s, \end{cases}$$

where the variance function $(K_t, t \geqslant 0)$ is given by

$$\dot{K}_t = AK_t + K_t A' + GG', \qquad t \geqslant 0,\ K_0 = 0$$

or equivalently

$$K_t = e^{At} \int_0^t e^{-As} GG' e^{-A's} ds\, e^{A't},\ t \geqslant 0.$$

Moreover, $(X^x, x \in \boldsymbol{R}^n)$ is a homogeneous Gaussian diffusion corresponding to the differential generator

$$L = \frac{1}{2} \sum_{i,j=1}^{n} b_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j=1}^{n} a_{ij} x_j \frac{\partial}{\partial x_i},$$

where $A = ((a_{ij}))$ and $B = ((b_{ij})) = G \cdot G'$. It is known (cf. [6], [7]) that if the differential generator is hypoelliptic, i.e., equivalently, if the pair $[A, G]$ is controllable or

$$\operatorname{rank}[G, AG, \ldots, A^{n-1}G] = n, \tag{H1}$$

then, for every $0 \leqslant s < t$, the integral $\int_s^t e^{-Au} GG' c^{-A'u} du$ is a positive definite matrix and in particular, for every $t > 0$, the covariance matrix $K_t$ is regular. Obviously this occurs for example when $G$ is an $n \times n$ regular matrix.

It is also known (cf. [7], [19]) that if (H1) is satisfied together with the assumption

$$A \text{ is a stable matrix} \tag{H2}$$

then there exists a unique invariant probability measure for the diffusion $(X^x, x \in \boldsymbol{R}^n)$; this measure is Gaussian with mean zero and nonsingular covariance matrix

$$K_{A,B} = \lim_{t \to +\infty} K_t = \int_0^{+\infty} e^{As} GG' e^{A's} ds, \tag{1.2}$$

which is the unique nonnegative definite symmetric matrix satisfying the Lyapunov equation

$$AK + KA' + GG' = 0.$$

Moreover, from [11] it follows that under (H1)–(H2) the diffusion $(X^x, x \in \boldsymbol{R}^n)$ is ergodic.

Here we are interested in the statistical problem of estimating the parameters $A$ and $B = GG'$ of the diffusion when they are unknown, in view of the observation on some time interval $[0, T]$ of one trajectory. This problem has been studied in [1] and [8] under the assumption that the differential generator is elliptic, i.e., $\det B \neq 0$, and sequential procedures have been investigated for the one-dimensional case (cf. [10]) and the two-dimensional case (cf. [13])

In the present work we show that the methods used in these papers may be extended in some ways under the weaker assumption (H1). Since preliminary results, which are of interest on their own merits, are needed in statistical considerations, the paper is organized as follows:

Section 2 is devoted to the computation of some Radon–Nikodym derivatives and to the study of quadratic forms of the observed process.

Section 3 is concerned with parameter estimation by the maximum likelihood method.

In Section 4 sequential schemes for parameter estimation are studied.

## 2. Preliminary results

Let $C = C(\boldsymbol{R}_+; \boldsymbol{R}^n)$ be the space of all continuous functions from $\boldsymbol{R}_+$ into $\boldsymbol{R}^n$ and, for every $T > 0$, let $\mathscr{C}_T$ be the $\sigma$-algebra generated on $C$ by the coordinates $\pi_t$ for $0 \leqslant t \leqslant T$. If $Z = (Z_t; t \geqslant 0)$ is some random process with continuous sample paths, $\mu_Z^T$ will stand for the restriction to $\mathscr{C}_T$ of the probability distribution induced by $Z$ on $C$.

**2.1. Radon–Nikodym derivatives.** The following lemma will be important in our future considerations:

LEMMA 2.1.1. *Let $(\Gamma(t); t \geqslant 0)$ be some $n \times n$ matrix-valued function which is absolutely continuous with derivative $\dot{\Gamma}(t)$ and such that for every $T > 0$*

$$\int_0^T |\Gamma_{kl}(t)|\, dt < +\infty, \qquad k, l = 1, \ldots, n,$$

*where $\Gamma(t) = ((\Gamma_{kl}(t)))$. Let $Y^x = (Y_t^x, t \geqslant 0)$ be the solution process of the stochastic differential equation*

$$dY_t^x = [A + GG'\Gamma(t)]\, Y_t^x\, dt + G\, d\eta_t, \qquad t \geqslant 0,\ Y_0^x = x, \tag{2.1}$$

*where $(\eta_t, t \geqslant 0)$ is some $r$-dimensional Brownian motion. Then the measure $\mu_{Xx}^T$ is absolutely continuous with respect to the measure $\mu_{Yx}^T$ with the following Radon–Nikodym derivative:*

$$\frac{d\mu_{Xx}^T}{d\mu_{Yx}^T} = \exp\Big\{-\int_0^T \pi_t' \Gamma'(t)\, GG^+\, d\pi_t + \tfrac{1}{2}\int_0^T \pi_t' \Gamma'(t)\, GG^+\, [2A + GG'\Gamma(t)]\, \pi_t\, dt\Big\}.$$

*If, moreover, $\Gamma(t)$ is a symmetric matrix for $t \geqslant 0$, then the Radon–Nikodym derivative can be written*

$$\frac{d\mu_{Xx}^T}{d\mu_{Yx}^T} = \exp\left\{\tfrac{1}{2}\int_0^T \mathrm{Tr}\big(G'\Gamma(t)G\big)dt + \tfrac{1}{2}x'\Gamma(0)x - \tfrac{1}{2}\pi_T'\Gamma(T)\pi_T\right\} \times$$

$$\times \exp\left\{\tfrac{1}{2}\int_0^T \pi_t'[\dot{\Gamma}(t) + A'\Gamma(t) + \Gamma(t)A + \Gamma(t)GG'\Gamma(t)]\pi_t\,dt\right\}.$$

*Proof.* Note that the equation

$$\dot{A}x - \big(A + GG'\Gamma(t)\big)x = G\alpha_t(x)$$

admits the solution

$$\alpha_t(x) = -G'\Gamma(t)x.$$

Then (cf. [10]) one has $\mu_{Xx}^T \ll \mu_{Yx}^T$ and, moreover,

$$\frac{d\mu_{Xx}^T}{d\mu_{Yx}^T} = \exp\left\{\int_0^T [A\pi_t - \{A + GG'\Gamma(t)\}\pi_t]'[GG']^+ d\pi_t - \right.$$

$$\left. -\tfrac{1}{2}\int_0^T [A\pi_t - \{A + GG'\Gamma(t)\}\pi_t]'[GG']^+[A\pi_t + \{A + GG'\Gamma(t)\}\pi_t]dt\right\}.$$

On using the properties of pseudoinverses, namely $[GG'][GG']^+ = GG^+$ (cf. [10]), the first assertion in the lemma immediately follows. Now, if $\Gamma(t)$ is symmetric, the stochastic integral can be computed as follows:

$$\begin{aligned}\int_0^T \pi_t'\Gamma'(t)GG^+ d\pi_t &= \int_0^T \pi_t'\Gamma(t)GG^+ d\pi_t \\ &= \int_0^T \pi_t'\Gamma(t)d\pi_t - \int_0^T \pi_t'\Gamma(t)(E - GG^+)d\pi_t \\ &= \int_0^T \pi_t'\Gamma(t)d\pi_t - \int_0^T \pi_t'\Gamma(t)(E - GG^+)A\pi_t\,dt,\end{aligned}$$

where $E$ stands for the $n \times n$ identity matrix. By the Itô formula one gets:

$$\begin{aligned}\int_0^T \pi_t'\Gamma(t)d\pi_t &\doteq \tfrac{1}{2}\left\{\int_0^T \pi_t'\Gamma(t)d\pi_t + \int_0^T d\pi_t'\Gamma(t)\pi_t\right\} \\ &= \tfrac{1}{2}\left\{\pi_T'\Gamma(T)\pi_T - x'\Gamma(0)x - \int_0^T \mathrm{Tr}\big(G'\Gamma(t)G\big)dt - \int_0^T \pi_t'\dot{\Gamma}(t)\pi_t\,dt\right\}.\end{aligned}$$

Then one has

$$\frac{d\mu_{Xx}^T}{d\mu_{Yx}^T} = \exp\{\tfrac{1}{2}\int_0^T \mathrm{Tr}(G'\Gamma(t)G)\,dt + \tfrac{1}{2}x'\Gamma(0)x - \tfrac{1}{2}\pi_T'\Gamma(T)\pi_T\} \times$$
$$\times \exp\{\tfrac{1}{2}\int_0^T \pi_t'\dot{\Gamma}(t)\pi_t\,dt\} \times$$
$$\times \exp\{\int_0^T \pi_t'\Gamma(t)(E-GG^+)A\pi_t\,dt + \tfrac{1}{2}\int_0^T \pi_t'\Gamma(t)GG^+[2A+GG'\Gamma(t)]\pi_t\,dt\}.$$

The last term within brackets can be written:

$$\tfrac{1}{2}\int_0^T \pi_t'\Gamma(t)(E-GG^+)A\pi_t\,dt + \tfrac{1}{2}\int_0^T \pi_t'A'(E-GG^+)\Gamma(t)\pi_t\,dt$$
$$+\tfrac{1}{2}\int_0^T \pi_t'\Gamma(t)GG^+A\pi_t\,dt + \tfrac{1}{2}\int_0^T \pi_t'A'GG^+\Gamma(t)\pi_t\,dt + \tfrac{1}{2}\int_0^T \pi_t'\Gamma(t)GG'\Gamma(t)\pi_t\,dt,$$

that is,

$$\tfrac{1}{2}\int_0^T \pi_t'[A'\Gamma(t) + \Gamma(t)A + \Gamma(t)GG'\Gamma(t)]\pi_t\,dt,$$

which completes the proof of the last assertion. ∎

Now let us state the following corollary, which will be useful in the maximum likelihood approach to parameter estimation:

COROLLARY 2.1.2. *Let $\tilde{X}^x = (\tilde{X}_t^x, t \geqslant 0)$ be the solution process of the stochastic differential equation*

$$d\tilde{X}_t^x = (E-GG^+)A\tilde{X}_t^x\,dt + G\,d\eta_t, \qquad t \geqslant 0,\ \tilde{X}_0^x = x \tag{2.2}$$

*where $(\eta_t, t \geqslant 0)$ is as in Lemma 1.1.1. Then $\mu_{Xx}^T \ll \mu_{\tilde{X}x}^T$ with the following Radon–Nikodym derivative*

$$\frac{d\mu_{Xx}^T}{d\mu_{\tilde{X}x}^T} = \exp\{\mathrm{Tr}[(GG')^+\{\int_0^T d\pi_t\pi_t'A' - \tfrac{1}{2}A\int_0^T \pi_t\pi_t'\,dt\,A'\}]\}.$$

*Proof.* Applying the first part of Lemma 1.1.1. with $\Gamma(t) = -(GG')^+A$, $t \geqslant 0$, one gets

$$\frac{d\mu_{Xx}^T}{d\mu_{\tilde{X}x}^T} = \exp\{\int_0^T \pi_t'A'(GG')^+(GG')(GG')^+\,d\pi_t -$$
$$-\tfrac{1}{2}\int_0^T \pi_t'A'(GG')^+(GG')(GG')^+A\pi_t\,dt -$$
$$-\tfrac{1}{2}\int_0^T \pi_t'A(GG')^+[E-(GG')(GG')^+]A\pi_t\,dt\}$$
$$= \exp\{\int_0^T \pi_t'A'(GG')^+\,d\pi_t - \tfrac{1}{2}\int_0^T \pi_t'A'(GG')^+A\pi_t\,dt\},$$

which can be written as in the corollary. ∎

**2.2. Quadratic forms of the process.** Since the quadratic functional $\int_0^T X_t^x X_t^{x\prime}\,dt$ will play a central role in what follows, we now investigate some of its properties.

LEMMA 2.2.1. *Under assumption* (H1) *the matrix* $\int_0^T X_t^x X_t^{x\prime}\,dt$ *is almost surely positive definite. If, moreover,* (H2) *is satisfied then,* $K_{A,B}$ *being defined by* (1.2), *one has*:

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T X_t^x X_t^{x\prime}\,dt = K_{A,B} \qquad \textit{almost surely.}$$

*Proof.* In order to prove the first assertion one has to show that the set

$$\Omega_0 = \bigcup_{h\in\mathbf{R}^n\setminus\{0\}} \Big[h'\Big(\int_0^T X_t^x X_t^{x\prime}\,dt\Big)h = 0\Big]$$

is negligible. It is clear that since $X^x$ has continuous sample paths it is sufficient to prove that

$$\Omega_1 = \bigcup_{h\in\mathbf{R}^n\setminus\{0\}}\ \bigcap_{s\in[0,T]} [\langle X_s^x, h\rangle = 0]$$

is negligible. Let $0 = t_0 < t_1 < \ldots < t_n \leqslant T$; since

$$\Omega_1 \subset \Omega_1^* = \{\det[X_{t_1}^x, \ldots, X_{t_n}^x] = 0\}$$

and the subset of $(\mathbf{R}^n)^n$ $\{(v_1, \ldots, v_n)\in(\mathbf{R}^n)^n\colon \det[v_1, \ldots, v_n] = 0\}$ has Lebesgue measure zero, the assertion of the lemma will be proved if one asserts that the Gaussian random vector $(X_{t_1}^{x\prime}, \ldots, X_{t_n}^{x\prime})'$ is nonsingular. But one can write

$$\begin{bmatrix} X_{t_1}^x \\ \vdots \\ X_{t_n}^x \end{bmatrix} = \begin{bmatrix} e^{At_1} & & 0 \\ & \ddots & \\ 0 & & e^{At_n} \end{bmatrix} \begin{bmatrix} E & & 0 \\ \vdots & \ddots & \\ E & \ldots & E \end{bmatrix} \begin{bmatrix} x + \int_0^{t_1} e^{-As} G\,dW_s \\ \int_{t_1}^{t_2} e^{-As} G\,dW_s \\ \cdots\cdots \\ \int_{t_{n-1}}^{t_n} e^{-As} G\,dW_s \end{bmatrix},$$

where the random vectors $\int_{t_i}^{t_{i+1}} e^{-As} G\,dW_s$, $i = 0, \ldots, n$ are independent with the respective covariance matrices $\int_{t_i}^{t_{i+1}} e^{-As} GG' e^{-A's}\,ds$. Since it has been noticed (cf. Section 1) that these matrices are positive definite when (H1) is satisfied, the proof of the first statement is completed. The second assertion is a simple consequence of the ergodicity of the diffusion $(X^x, x \in \mathbb{R}^n)$, which has been stated (cf. Section 1) under (H1)–(H2), since, for the invariant measure $\mu$, one has

$$\int_{\mathbb{R}^n} yy'\,d\mu(y) = K_{A,B} = \int_0^{+\infty} e^{As}GG'e^{A's}\,ds. \qquad \blacksquare$$

From the second part of Lemma 2.2.1. it follows that under (H1)–(H2) for every symmetric nonnegative matrix $S \neq 0$

$$\lim_{T\to+\infty} \int_0^T X_t^{x\prime} S X_t^x\,dt = +\infty \qquad \text{a.s..}$$

Now we shall show that, in order that this limit should hold, assumption (H2) is not essential. We look at the Laplace transform of $\int_0^T X_t^x X_t^{x\prime}\,dt$, i.e., the functional $\varphi_x^T$ defined on the set $\mathscr{S}$ of symmetric nonnegative definite matrices $S$ by

$$\varphi_x^T(S) = E\{\exp[-\operatorname{Tr} S \int_0^T X_t^x X_t^{x\prime}\,dt]\} = E\{\exp[-\int_0^T X_t^{x\prime} S X_t^x\,dt]\}. \quad (2.3)$$

LEMMA 2.2.2. *Under assumption* (H1) *the functional* $\varphi_x^T$ *is given by*:

$$\begin{aligned}\varphi_x^T(S) = \exp\{\tfrac{1}{2}\int_0^T \operatorname{Tr}(G'\gamma_t^{-1}(S)G)\,dt + \tfrac{1}{2}x'x\} \times \\ \times \exp\{\tfrac{1}{2}x'\phi_T'(S)(E+\gamma_T^{-1}(S)\Delta_T(S))^{-1}\gamma_T^{-1}(S)\phi_T(S)x\} \times \\ \times[\det\{E+\gamma_T^{-1}(S)\Delta_T(S)\}]^{-\frac{1}{2}}, \qquad S\in\mathscr{S},\end{aligned}$$

*where* $(\gamma_t(S),\, t \geqslant 0)$ *is a positive definite matrix-valued function which is defined by the equation*

$$\dot\gamma_t = A\gamma_t + \gamma_t A' + GG' - 2\gamma_t S\gamma_t, \qquad t \geqslant 0,\ \gamma_0 = E, \quad (2.4)$$

*and*

$$\Delta_t(S) = \phi_T(S)\int_0^T \phi_t^{-1}(S)GG'\phi_t'^{-1}(S)\,dt\,\phi_T'(S)$$

*with*

$$\dot\phi_t(S) = (A + GG'\gamma_t^{-1}(S))\phi_t(S), \qquad t \geqslant 0,\ \phi_0(S) = E.$$

*Proof.* The existence of a unique solution to (2.4) is established in [10] (see also [14]); the fact that $\gamma_t$ is invertible comes from (H1) (cf. [10]). It is easy to see that, setting $\Gamma(t) = \gamma_t^{-1}(S)$, one has

$$\dot\Gamma(t) + A'\Gamma(t) + \Gamma(t)A + \Gamma(t)GG'\Gamma(t) - 2S = 0, \qquad t \geqslant 0,\ \Gamma(0) = E.$$

Then, taking into account the second part of Lemma 2.1.1, one obtains

$$\varphi_x^T(S) = E\left\{\exp\left[-\int_0^T Y_t^{x'} S Y_t^x \, dt\right] \frac{d\mu_{Xx}^T}{d\mu_{Yx}^T}(Y^x)\right\}$$

$$= E\left\{\exp\left\{-\int_0^T Y_t^{x'} S Y_t^x \, dt + \tfrac{1}{2}\int_0^T \operatorname{Tr}(G'\gamma_t^{-1}(S)G)\,dt + \tfrac{1}{2}x'x - \right.\right.$$

$$\left.\left. - \tfrac{1}{2} Y_T^{x'} \gamma_T^{-1}(S) Y_T^x + \int_0^T Y_t^{x'} S Y_t^x \, dt \right\}\right\}$$

$$= \exp\left\{\tfrac{1}{2}\int_0^T \operatorname{Tr}(G'\gamma_t^{-1}(S)G)\,dt + \tfrac{1}{2}x'x\right\} \times E\exp\{-\tfrac{1}{2} Y_T^{x'}\gamma_T^{-1}(S)Y_t^x\}.$$

Moreover, from (2.1) we know that $Y_T^x$ is a Gaussian random vector with the mean

$$m_x^T(S) = \phi_T(S)x$$

and the covariance matrix

$$\Delta_T(S) = \phi_T(S)\int_0^T \phi_t^{-1}(S)GG'\phi_t'^{-1}(S)\,dt\,\phi_T'(S)$$

with $(\phi_t(S),\ t \geqslant 0)$ as in the statement. Then (cf. Lemma 11.6 in the Russian version of [10]) one gets

$$\begin{aligned} &E(\exp\{-\tfrac{1}{2} Y_T^{x'}\gamma_T^{-1}(S)Y_T^x\}) \\ &\quad = \exp\{-\tfrac{1}{2} m_x^{T'}(S)(E+\gamma_T^{-1}(S)\Delta_T(S))^{-1}\gamma_T^{-1}(S)m_x^T(S)\} \times \\ &\qquad\qquad \times \{\det[E+\gamma_T^{-1}(S)\Delta_T(S)]\}^{-\frac{1}{2}} \end{aligned}$$

and the proof is complete. ∎

Now we are able to prove the following statement:

COROLLARY 2.2.3. *If* (H1) *is satisfied, then, for every* $x \in \boldsymbol{R}^n$ *and* $S \in \mathscr{S}$, $S \neq 0$, *there exist strictly positive constants* $\alpha_x(S)$ and $\beta(S)$ *such that*

$$\varphi_x^T(S) \leqslant \alpha_x(S)e^{-\beta(S)T}, \qquad T \geqslant 0.$$

*Proof.* First, since $\gamma_T^{-1}(S) \in \mathscr{S}$ and $\Delta_T(S) \in \mathscr{S}$, it is clear that

$$\exp\{\tfrac{1}{2}x'x\} \times \exp\{-\tfrac{1}{2}x'\phi_T'(S)(E+\gamma_T^{-1}(S)\Delta_t(S))^{-1}\gamma_T^{-1}(S)\phi_T(S)x\} \leqslant \exp\{\tfrac{1}{2}x'x\}$$

So we only have to prove that

$$\exp\{\tfrac{1}{2}\int_0^T \operatorname{Tr}(G'\gamma_t^{-1}(S)G)\,dt\}\times$$
$$\times(\det[E+\gamma_T^{-1}(S)\Delta_T(S)])^{-\frac{1}{2}} \leqslant K(S)\,e^{-\beta(S)T} \tag{2.5}$$

for some positive constants $K(S)$ and $\beta(S)$.

But, setting

$$\Lambda_T(S) = \int_0^T \phi_t^{-1}(S)\,GG'\,\phi_t'^{-1}(S)\,dt,$$

we have

$$\begin{aligned}\det[E+\gamma_T^{-1}(S)\Delta_T(S)] &= \det[E+\gamma_T^{-1}(S)\phi_T(S)\Lambda_T(S)\phi_T'(S)]\\ &\geqslant 1+\det[\gamma_T^{-1}(S)\phi_T(S)\Lambda_T(S)\phi_T'(S)]\\ &= [\det\phi_T(S)]^2\frac{\det\Lambda_T(S)+\dfrac{\det\gamma_T(S)}{[\det\phi_T(S)]^2}}{\det\gamma_T(S)}.\end{aligned}$$

Then the first member of (2.5) is bounded by

$$\exp\{\tfrac{1}{2}\int_0^T \operatorname{Tr}(G'\gamma_t^{-1}(S)G)\,dt\}\times$$
$$\times(\det\phi_T(S))^{-1}\times\left[\frac{\det\gamma_T(S)}{\det\Lambda_T(S)+\dfrac{\det\gamma_T(S)}{(\det\phi_T(S))^2}}\right]^{\frac{1}{2}}. \tag{2.6}$$

But, since

$$\det\phi_T(S) = \exp\{\int_0^T \operatorname{Tr}(A+GG'\gamma_t^{-1}(S))\,dt\},$$

one gets for the product of the first two terms in (2.6)

$$\exp\{-\int_0^T \operatorname{Tr}(A+GG'\gamma_t^{-1}(S))\,dt\}$$

or, by using (2.4),

$$\exp\{-\int_0^T \operatorname{Tr}\gamma_t(S)S\,dt-\tfrac{1}{2}\int_0^T \operatorname{Tr}\dot\gamma_t(S)\gamma_t^{-1}(S)\,dt\},$$

i.e., taking into account the fact that

$$\frac{d}{dt}(\gamma_t^{-1}(S)) = -\gamma_t^{-1}(S)\dot\gamma_t(S)\gamma_t^{-1}(S), \qquad \gamma_0^{-1}(S) = E,$$

the expression

$$\exp\left\{-\int_0^T \operatorname{Tr}\gamma_t(S)\,S\,dt\right\}[\det\gamma_t(S)]^{-\frac{1}{2}}.$$

Now (2.6) can be written

$$\left[\det\Lambda_T(S)+\frac{\det\gamma_T(S)}{(\det\phi_T(S))^2}\right]^{-\frac{1}{2}}\times\exp\left\{-\int_0^T \operatorname{Tr}\gamma_t(S)\,S\,dt\right\}$$

and in order to get (2.5) it remains to prove that

$$\inf_{t\geqslant 0}\left\{\det\Lambda_t(S)+\frac{\det\gamma_t(S)}{(\det\phi_t(S))^2}\right\}>0$$

and

$$\inf_{t\geqslant 0}\operatorname{Tr}\{\gamma_t(S)\,S\}>0.$$

The first inequality follows from the fact that $\det\Lambda_t(S)$ increases with $t$ and is zero if and only if $t=0$, and that $\det\gamma_t(S)/(\det\phi_t(S))^2$ is positive with value 1 for $t=0$. In order to prove the second inequality let us consider $(\sigma_t(S), t\geqslant 0)$, the solution of

$$\dot{\sigma}_t = A\sigma_t+\sigma_t A'+GG'-2\sigma_t S\sigma_t, \qquad t\geqslant 0,\ \sigma_0=0.$$

It is known (cf. [17]) that, since (H1) is satisfied, $\sigma_t(S)$ is monotone nondecreasing and, moreover, for every $t>0$, $\sigma_t(S)$ is positive definite. It is easy to see that, setting $\delta_t(S)=\gamma_{T-t}(S)-\sigma_{T-t}(S)$, one has $\delta_T(S)=E$ and for $t\geqslant 0$:

$$\dot{\delta}_t(S)+(A-2\sigma_{T-t}(S)S)\delta_t(S)+\delta_t(S)(A-2\sigma_{T-t}(S)S)'-2\delta_t(S)S\delta_t(S)=0.$$

So (cf. [17]), for every $t\in[0, T]$, $\delta_t(S)$ is nonnegative definite and therefore for every $t\geqslant 0$ the matrix $\gamma_t(S)-\sigma_t(S)$ is also nonnegative definite. All these properties lead to what is needed. ∎

## 3. Maximum likelihood estimation of the drift parameter

Here we investigate the problem of estimating the unknown parameters $A$ and $B=GG'$ of the diffusion process under consideration in view of the observation of $[0, T]$ of one trajectory starting from zero at time zero, since in view of the preliminary results this is not really a restriction. So we start with a process $X=(X_t, t\geqslant 0)$ satisfying (1.1) with $x=0$.

An estimation of matrix $B$ can be obtained by using the quadratic variation $[X]$ of process $X$. Precisely, $B$ can be computed with probability one on every finite time interval $[0, T]$ by $B=\frac{1}{T}[X]_T$. So we can consider the problem of estimating the matrix $A$ when $B$ is assumed known without any restriction, $B$ having been previously computed. First we investigate the case where the matrix $A$ is completely unknown (for which the results have been announced in [9]) and then the case where it is known up to a multiplicative constant.

**3.1. The case where the drift parameter is completely unknown.** Since from (1.1) one has

$$\int_0^T dX_t X_t' = A \int_0^T X_t X_t' dt + G \int_0^T dW_t X_t',$$

one also has

$$\int_0^T \{d(E-BB^+)X_t\} X_t' = (E-BB^+)A \int_0^T X_t X_t' dt$$

because $BB^+ = GG^+$ is nothing but the matrix of the orthogonal projection on the subspace of $\mathbb{R}^n$ generated by the columns of $G$. Then, by using Lemma 2.2.1, under (H1) one gets

$$(E-BB^+)A = \{\int_0^T [d(E-BB^+)X_t X_t']\} \{\int_0^T X_t X_t' dt\}^{-1}. \tag{3.1}$$

Let us notice that the stochastic integral in the second member of (3.1) is in fact an ordinary integral since the process $((E-BB^+)X_t, t \geqslant 0)$ has locally bounded variation. Finally, as $B$, the matrix $(E-BB^+)$ $A$ can be computed with probability one on some finite time interval; this allows us to assume that this matrix is known too.

Now, from Corollary 2.1.2, the measure $\mu_{\tilde{X}0}^T$, which only depends on known matrices $B$ and $(E-BB^+)$ $A$, can be considered as a dominating measure for the statistical space associated with the estimation problem concerned; the log-likelihood function can be written in the following form:

$$\operatorname{Tr}\{B^+ [\int_0^T dX_t X_t' A' - \tfrac{1}{2} A \int_0^T X_t X_t' dt\, A']\}.$$

Then the next result is an immediate consequence of Lemma 2.2.1:

Proposition 3.1.1. *Under* (H1) *a maximum likelihood estimator of the matrix $A$ is given by*

$$\hat{A}_T = [\int_0^T dX_t X_t'] [\int_0^T X_t X_t' dt]^{-1}. \tag{3.2}$$

*Remark* 3.1.2. From (1.1) one can write

$$\hat{A}_T = A + G[\int_0^T dW_t X_t'] [\int_0^T X_t X_t' dt]^{-1}. \tag{3.3}$$

This ensures, because of the controllability of the pair $[A, G]$, that the pair $[\hat{A}_T, G]$ is itself controllable; so the estimator $\left[\hat{A}_T, \left(\frac{1}{T}[X]_T\right)^{\frac{1}{2}}\right]$ takes its values in the parameter space. Moreover, let us notice that the matrix $(E-BB^+)\hat{A}_T$ provides again the matrix $(E-BB^+)A$ defined by (3.1).

Now we state the asymptotic properties of the estimator:

PROPOSITION 3.1.3. *Under* (H1) *and* (H2) *the estimator* $\hat{A}_T$ *defined by* (3.2) *is strongly consistent*, i.e.,

$$\lim_{T\to+\infty} \hat{A}_T = A \qquad a.s..$$

*Moreover, the vector* $T^{1/2}\operatorname{vec}(\hat{A}_T-A)$ *is asymptotically normally distributed with mean zero and covariance matrix* $K_{A,B}^{-1}\otimes B$ *where* $K_{A,B}$ *is given by* (1.2).

*Proof.* We start from the decomposition of the estimator given in equation (3.3). By Lemma 2.2.1 one gets

$$\lim_{T\to+\infty}\left[\frac{1}{T}\int_0^T X_t X_t'\,dt\right]^{-1} = K_{A,B}^{-1} \qquad \text{a.s..}$$

Moreover, since for every coordinate $(X_t^i, t\geqslant 0)$ of the observed process $X$ one has

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T (X_t^i)^2\,dt = K_{A,B}^{i,i} > 0 \qquad \text{a.s.,}$$

by Theorem 4.1, Appendix 1, [4], one obtains

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T dW_t X_t' = 0 \qquad \text{a.s..}$$

Then the first assertion is proved.

Now, still from (3.3), one can write

$$\sqrt{T}\operatorname{vec}(\hat{A}_T-A) = \left\{\left[\frac{1}{T}\int_0^T X_t X_t'\,dt\right]^{-1}\otimes G\right\}\operatorname{vec}\left\{\frac{1}{\sqrt{T}}\cdot\int_0^T dW_t X_t'\right\}$$

where

$$\operatorname{vec}\left\{\frac{1}{\sqrt{T}}\int_0^T dW_t X_t'\right\} = \frac{1}{\sqrt{T}}\int_0^T (X_t\otimes E)\,dW_t.$$

From Lemma 2.2.1 and

$$\int_0^T (X_t \otimes E)(X_t \otimes E)' \, dt = \int_0^T (X_t \otimes E)(X_t' \otimes E) \, dt = \int_0^T (X_t X_t') \otimes E \, dt$$

one gets

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T (X_t \otimes E)(X_t \otimes E)' \, dt = K_{A,B} \otimes E.$$

Then, by using the results of [16] concerning the asymptotic normality of stochastic integrals, one deduces that

$$\lim_{T \to +\infty} \operatorname{vec}\left( \frac{1}{\sqrt{T}} \int_0^T dW_t X_t' \right) = N(0, K_{A,B} \otimes E),$$

where the limit stands in the sense of convergence in probability distribution.

It follows that in the same sense $\sqrt{T} \operatorname{vec}(\hat{A}_T - A)$ converges to the Gaussian distribution with mean zero and covariance matrix

$$(K_{A,B}^{-1} \otimes G)(K_{A,B} \otimes E)(K_{A,B}^{-1} \otimes E)(K_{A,B}^{-1} \otimes G)' = ((K_{A,B}^{-1} K_{A,B}) \otimes (GE))(K_{A,B}^{-1} \otimes G')$$

$$= (E \otimes G)(K_{A,B}^{-1} \otimes G') = K_{A,B}^{-1} \otimes (GG') = K_{A,B}^{-1} \otimes B. \quad \blacksquare$$

*Remark* 3.1.4. Note that if one assumes that the parameter space is that of pairs $[A, B]$ such that, on the one hand, $[A, G]$ is controllable and, on the other hand, $A$ is a stable matrix, then the estimator $\hat{A}_T$ defined by (3.2) is not a maximum likelihood estimator; in fact, under these conditions such an estimator does not exist. The matrix $\hat{A}_T$ is not stable in general but the probability that it is tends to one when $T$ increases to infinity.

In the case where $B$ is nonsingular it is possible to modify $\hat{A}_T$ in order to obtain an estimator of $A$ which is stable: one has

$$\int_0^T dX_t X_t' + \int_0^T X_t \, dX_t' = X_T X_T' - [X]_T;$$

so

$$\left(\int_0^T dX_t X_t' - \tfrac{1}{2} X_T X_T'\right)\left(\int_0^T X_t X_t' \, dt\right)^{-1} \frac{1}{T} \int_0^T X_t X_t' \, dt +$$

$$+ \frac{1}{T} \int_0^T X_t X_t' \, dt \left(\int_0^T X_t X_t' \, dt\right)^{-1} \left(\int_0^T X_t \, dX_t' - \tfrac{1}{2} X_T X_T'\right) = -B.$$

This shows that the Lyapunov equation

$$\tilde{A}_T Q + Q \tilde{A}_T' = -B,$$

where

$$\tilde{A}_T = \left(\int_0^T dX_t X_t' - \tfrac{1}{2} X_T X_T'\right)\left(\int_0^T X_t X_t' dt\right)^{-1},$$

admits the positive definite matrix $\frac{1}{T}\int_0^T X_t X_t' dt$ as a solution, which implies that $\tilde{A}_T$ is stable (see [15]). Moreover, it is clear that the estimator $\tilde{A}_T$ has the same asymptotic properties as $\hat{A}_T$.

So, in some sense, under asumptions (H1)–(H2) the problem of parameter estimation is completely solved. The question is: does the maximum likelihood estimator given by (3.2) still converge if one drops assumption (H2)? We have no answer to this in general, but the problem has been positively solved in the one-dimensional case (cf. [10]).

We are now going to look at the problem of parameter estimation in the case where the drift coefficient is known up to an unknown multiplicative constant; the results will include those cited before concerning the one-dimensional case.

**3.2. The case where the drift matrix is known up to a multiplicative constant.** Here we assume that $A$ belongs to the set $\{\theta A_0;\ \theta \in \boldsymbol{R}\}$, where $A_0$ is some known $n \times n$ matrix and $\theta$ is an unknown parameter which one has to estimate. Since if $BB^+ A_0 = 0$ then $\theta$ can be computed from (3.1), we shall also assume that $BB^+ A_0 \neq 0$.

Because the log-likelihood function is equal to

$$\theta \int_0^T X_t' A_0' B^+ dX_t - \tfrac{1}{2}\theta^2 \int_0^T X_t' A_0' B^+ A_0 X_t dt,$$

the maximum likelihood estimator of $\theta$ is given by

$$\hat{\theta}_T = \frac{\int_0^T X_t' A_0' B^+ dX_t}{\int_0^T X_t' A_0' B^+ A_0 X_t dt} \tag{3.4}$$

PROPOSITION 3.2.1. *Under* (H1) *the estimator* $\hat{\theta}_T$ *defined by* (3.4) *is strongly consistent*, i.e.,

$$\lim_{T\to\infty} \hat{\theta}_T = \theta \qquad a.s..$$

*Proof.* As before, from (1.1) one can write

$$\hat{\theta}_T = \theta + \frac{\int_0^T X_t' A_0' B^+ G\, dW_t}{\int_0^T X_t' A_0' B^+ A_0 X_t dt}.$$

But $\left(\int_0^t X_s' A_0' B^+ A_0 X_s\,ds,\ t \geq 0\right)$ is the quadratic variation process of the martingale $\left(\int_0^t X_s' A_0' B^+ G\,dW_s,\ t \geq 0\right)$ and

$$\lim_{T\to\infty} \int_0^T X_t' A_0' B^+ A_0 X_t\,dt = \infty \qquad \text{a.s.}$$

(see Corollary 2.2.3). So, by the analogue for continuous time martingales of the strong law of large numbers (see [4], Theorem 4.1, p. 394) the assertion follows. ∎

*Remark* 3.2.2. Note that

$$\int_0^T X_t' A_0' B^+\,dX_t = \tfrac{1}{2}(X_T' B^+ A_0 X_T - T\,\mathrm{Tr}\,BB^+ A_0) +$$
$$+\tfrac{1}{4}\mathrm{Tr}(A_0' B^+ - B^+ A_0)\left(\int_0^T dX_t X_t' - \int_0^T X_t\,dX_t'\right).$$

So, if $B^+ A_0$ is symmetric, then

$$\hat{\theta}_T = \frac{X_T' B^+ A_0 X_T - T\,\mathrm{Tr}\,BB^+ A_0}{2\int_0^T X_t' A_0' B^+ A_0 X_t\,dt}.$$

Moreover, if $B^+ A_0$ is skew symmetric, then

$$\hat{\theta}_T = \frac{\mathrm{Tr}\,A' B^+ \left(\int_0^T dX_t X_t' - \int_0^T X_t\,dX_t'\right)}{2\int_0^T X_t' A_0' B^+ A_0 X_t\,dt}.$$

*Remark* 3.2.3. The results of this section (under (H1)) may easily be generalized to the following case. Let $A = \sum_{i=1}^{p} \theta_i A_i$, where $A_i$, $i = 1, \ldots, p$, are known matrices such that $BB^+ A_i \neq 0$, $i = 1, \ldots, p$, and

$$A_i' B^+ A_j + A_j' B^+ A_i = 0, \qquad i \neq j.$$

In this case, the maximum likelihood estimator of $\theta_i$, $i = 1, \ldots, p$, is given by

$$\hat{\theta}_{i,T} = \frac{\int_0^T X_t' A_i' B^+\,dX_t}{\int_0^T X_t' A_i' B^+ A_i X_t\,dt},$$

and it is clear that for all $i = 1, \ldots, p$

$$\lim_{T \to \infty} \hat{\theta}_{i,T} = \theta_i \qquad \text{a.s.}.$$

Now we give an example of a 2-dimensional process which is a model for the so-called geophysical problem.

EXAMPLE 3.2.4. It is known (see [2], [13]) that the instantaneous axis of rotation of the earth is displaced with respect to the minor axis of the terrestrial ellipsoid. This displacement consists of a periodic part and a fluctuating part. The latter can be assumed to be a solution of a system of stochastic differential equations of the form

$$dX_1(t) = \theta_1 X_1(t)\,dt - \theta_2 X_2(t)\,dt + g\,dw_1(t),$$
$$dX_2(t) = \theta_2 X_1(t)\,dt + \theta_1 X_2(t)\,dt + g\,dw_2(t),$$

where $(w_1(t),\, t \geqslant 0)$ and $(w_2(t),\, t \geqslant 0)$ are two independent Wiener processes, $\theta_1$ and $\theta_2$ are unknown and $g^2$ is known. It is clear that the above system of equations can be written in the form (1.1) with

$$A = \begin{pmatrix} \theta_1 & -\theta_2 \\ \theta_2 & \theta_1 \end{pmatrix}, \qquad G = g\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover, because $A = \theta_1 A_1 + \theta_2 A_2$, where

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $A_1' A_2 + A_2' A_1 = 0$ the maximum likelihood estimators of $\theta_1$ and $\theta_2$ are given by

$$\hat{\theta}_{1,T} = \frac{\int_0^T X_1(t)\,dX_1(t) + \int_0^T X_2(t)\,dX_2(t)}{\int_0^T [X_1^2(t) + X_2^2(t)]\,dt},$$

$$\hat{\theta}_{2,T} = \frac{\int_0^T X_1(t)\,dX_2(t) - \int_0^T X_2(t)\,dX_1(t)}{\int_0^T [X_1^2(t) + X_2^2(t)]\,dt}.$$

It has been shown (see [4]) that these estimators are consistent when the solution process

$$X_t = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$$

is assumed to be stationary. Taking into account Remark 3.2.3, we can assert that these estimators are still strongly consistent when one drops the assumption of stationarity.

## 4. Sequential estimation

As in Section 3.2, we assume that $A$ belongs to the set $\{\theta A_0;\, \theta \in \boldsymbol{R}\}$, where $A_0$ is some known $n \times n$ matrix such that $BB^+ A_0 \neq 0$. Here we deal with the problem of sequential estimation of unknown parameter $\theta$. A mathematical statement of the problem and some details concerning the case of continuous time observations may be found for example in [10], [12].

Let $H$ be a nonnegative number. Define the stopping time

$$\tau(H) = \inf\{t\colon \int_0^t X_s' A_0' B^+ A_0 X_s\, ds = H\}. \tag{4.1}$$

In the case of a one-dimensional process $X$ it has been shown (cf. [10]) that $E\tau^n(H) < \infty$ for all $n$. We shall prove the following stronger result:

LEMMA 4.1. *Under* (H1) *there exists a* $\delta > 0$ *such that*

$$E\exp\{\delta\tau(H)\} < \infty.$$

*Proof.* From Corollary 2.2.3 it follows that there exist positive constants $\alpha$ and $\beta$ such that

$$E\exp\{-\int_0^T X_t' A_0' B^+ A_0 X_t\, dt\} \leqslant \alpha e^{-\beta T}.$$

So

$$P(\tau(H) \geqslant T) = P(\int_0^T X_t' A_0' B^+ A_0 X_t\, dt \leqslant H)$$
$$\leqslant e^H E\exp\{-\int_0^T X_t' A_0' B^+ A_0 X_t\, dt\} \leqslant \alpha e^H e^{-\beta T}.$$

Now let $0 < \delta < \beta$. Since

$$e^{\delta T} P(\tau(H) \geqslant T) \leqslant e^H \alpha e^{-(\beta-\delta)T},$$

we have also

$$E\exp\{\delta\tau(H)\} = \delta \int_0^\infty e^{\delta t} P(\tau(H) \geqslant t)\, dt \leqslant \delta\alpha e^H \int_0^\infty e^{-(\beta-\delta)t}\, dt = \frac{\delta\alpha\varepsilon^H}{\beta-\delta} < \infty,$$

which completes the proof. ∎

Define the sequential plan $(\tau(H), \hat{\theta}_{\tau(H)})$, where $\tau(H)$ is given by (4.1) and

$$\hat{\theta}_{\tau(H)} = \frac{1}{H} \int_0^{\tau(H)} X_t' A_0' B^+ \, dX_t.$$

Note that $\hat{\theta}_{\tau(H)}$ is in fact the maximum likelihood estimator of $\theta$ based on the observation of the process $X$ on the random time interval $[0, \tau(H)]$. It is also easy to see that the case studied here covers those considered in [10]. We shall now prove:

PROPOSITION 4.2. *Under* (H1) *the sequential plan* $(\tau(H), \hat{\theta}_{\tau(H)})$ *has the following properties for all* $\theta \in \boldsymbol{R}$:

(i) $\hat{\theta}_{\tau(H)}$ *is normally distributed with* $E\hat{\theta}_{\tau(H)} = \theta$ *and* $\operatorname{var} \hat{\theta}_{\tau(H)} = \frac{1}{H}$;

(ii) *in the class of sequential plans* $(\tau, \tilde{\theta})$ *such that*

$$E\tilde{\theta}^2 < \infty, \qquad E \int_0^{\tau} X_t' A_0' B^+ A_0 X_t \, dt \leqslant H$$

*the sequential plan* $(\tau(H), \hat{\theta}_{\tau(H)})$ *is admissible and minimax with respect to the quadratic loss function.*

*Proof.* Part (i) follows from the equality

$$\hat{\theta}_{\tau(H)} = \theta + \frac{1}{H} \int_0^{\tau(H)} X_t' A_0' B^+ G \, dW_t$$

and the fact that $\left( \int_0^{\tau(H)} X_t' A_0' B^+ G \, dW_t, \; H \geqslant 0 \right)$ is a Wiener process.

To prove admissibility note that for all sequential plans $(\tau, \tilde{\theta})$ such that $E\tilde{\theta}^2 < \infty$ and $E \int_0^{\tau} X_t' A_0' B^+ A_0 X_t \, dt \leqslant H$:

$$E(\tilde{\theta} - \theta)^2 \geqslant \frac{(1 + b'(\theta))^2}{H} + b^2(\theta),$$

where $b(\theta) = E(\tilde{\theta} - \theta)$. This is a simple generalization of the Cramér–Rao inequality (see [10]). Next, suppose that $(\tau(H), \hat{\theta}_{\tau(H)})$ is inadmissible and show that $b(\theta) \equiv 0$ is the only function satisfying

$$(1 + b'(\theta))^2 + Hb^2(\theta) \leqslant 1.$$

This leads to admissibility. Moreover, $(\tau(H), \hat{\theta}_{\tau(H)})$ has a constant risk, so it is also minimax. ∎

*Remark* 4.3. Assume (H1) and, as in Remark 3.2.3, that $A = \sum_{i=1}^{p} \theta_i A_i$, where $A_i$ are known matrices such that $BB^+ A_i \neq 0$ and $A_i' B^+ A_j + A_j' B^+ A_i = 0$, $i \neq j$, $i, j = 1, \ldots, p$. It is clear that one may use, in order to estimate $\theta_i$, $i = 1, \ldots, p$, the sequential plans $(\tau_i(H), \hat{\theta}_{i,\tau_i(H)})$, where

$$\tau_i(H) = \inf\{t: \int_0^t X_s' A_i' B^+ A_i X_s \, ds = H\}$$

and

$$\hat{\theta}_{i,\tau_i(H)} = \frac{1}{H} \int_0^{\tau_i(H)} X_t' A_i' B^+ \, dX_t.$$

The estimator $\hat{\theta}_{i,\tau_i(H)}$ is normally distributed with $E\hat{\theta}_{i,\tau_i(H)} = \theta_i$ and $\operatorname{var} \hat{\theta}_{i,\tau_i(H)} = 1/H$ (1). Moreover, by the Cramér–Rao inequality, the sequential plan $(\tau_i(H), \hat{\theta}_{i,\tau_i(H)})$ is minimum variance unbiased in the class of unbiased sequential plans $(\tau_i, \tilde{\theta}_i)$ such that

$$E\tilde{\theta}_1^2 < \infty, \qquad E \int_0^{\tau_i} X_t' A_i' B^+ A_i X_t \, dt \leqslant H.$$

By the same argument as in Lemma 4.1, one may also show that for all $i = 1, \ldots, p$ there exists a constant $\delta_i > 0$ such that

$$E \exp\{\delta_i \tau_i(H)\} < \infty.$$

Finally note that in the case of Example 3.2.4 the above sequential plans are nothing but $(\tau(H), \hat{\theta}_{i,\tau(H)})$ where

$$\tau(H) = \inf\{t: \int_0^t [X_1^2(s) + X_2^2(s)] \, ds = H\}$$

and

$$\hat{\theta}_{1,\tau(H)} = \frac{1}{H}\left[\int_0^{\tau(H)} X_1(s) \, dX_1(s) + \int_0^{\tau(H)} X_2(s) \, dX_2(s)\right],$$

$$\hat{\theta}_{2,\tau(H)} = \frac{1}{H}\left[\int_0^{\tau(H)} X_1(s) \, dX_2(s) - \int_0^{\tau(H)} X_2(s) \, dX_2(s)\right],$$

which have been derived in [13].

For some related questions one can also consult [5].

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(1) In fact, $\hat{\theta}_H = (\hat{\theta}_{1,\tau_1(H)}, \ldots, \hat{\theta}_{p,\tau_p(H)})'$ is normally distributed with $E\hat{\theta}_H = \theta$ and $\operatorname{var} \hat{\theta}_H = \frac{1}{H} E$, and then $\hat{\theta}_{1,\tau_1(H)}, \ldots, \hat{\theta}_{p,\tau_p(H)}$ are independent.

## References


[1] M. Arato, *On the parameter estimation of processes satisfying a stochastic differential equation*, Studia Sci. Math. Hungar. **5** (1970), 11–15, in Russian.

[2] –, *On the statistical examination of continuous state Markov processes, I, II, III, IV*, Selected Transl. in Math. Statist. and Probability **4** (1978).

[3] L. Arnold and W. Kliemann, *Qualitative theory of stochastic systems*, in preprint form of A. T. Bharucha-Reid (ed.), *Probabilistic Analysis and Related Topics*, vol. 3, Academic Press, New York 1981.

[4] I. V. Basawa and B. L. S. Prakasa Rao, *Statistical Inference for Stochastic Processes*, Academic Press, London 1980.

[5] S. E. Borobeicikov and W. W. Konev, *Construction of sequential plans for parameters of recurrent type processes*, Mathematical Statistics and Applications, Publ. Tomsk Univ., **6** (1980), 72–81.

[6] M. Chaleyat-Maurel et L. Elie, *Diffusions gaussiennes*, Astérisque **84–85** (1981), 255–279.

[7] K. Ichihara and H. Kunita, *A classification of the second order degenerate elliptic operators and its probabilistic characterization*, Z. Wahrsch. Verw. Gebiete **30** (1974), 235–254.

[8] A. Le Breton, *Parameter estimation in a vector linear stochastic differential equation*, in *Transactions of the Seventh Prague Conference on Information Theory. Statistical Decision Functions and Random Processes*, vol. A (1977), 353–366.

[9] A. Le Breton et M. Musiela, *Estimation des paramètres pour les diffusions homogènes hypoelliptiques*, C.R. Acad. Sci. Paris Sér. I, **294** (1982), 341–344.

[10] R. S. Liptser and A. N. Shiryayev, *Statistics of Random Processes*, vol. I, II, Springer-Verlag, New York 1978.

[11] G. Maruyama and H. Tanaka, *Ergodic properties of n-dimensional recurrent Markov processes*, Mem. Fac. Sc. Kyushu Univ. Ser. A, **13** (1959), 157–172.

[12] M. Musiela, *On sequential estimation of parameters of continuous Gaussian Markov processes*, Probab. Math. Statist. (1982).

[13] A. A. Novikov, *Sequential estimation of the parameters of diffusion processes*, Theory Probab. Appl. **16** (1971), 394–396.

[14] J. Rodrigez-Canabal, *The geometry of the Riccati equation*, Stochastics **1** (1973), 129–149.

[15] J. Snyders and M. Zakai, *On nonnegative solutions of the equation* $AD+DA' = -C$, SIAM J. Appl. Math. **18** (1970), 704–714.

[16] A. F. Taraskin, *Some Limit Theorems for Stochastic Integrals*, in *Theory of Stochastic Processes*, vol. 1, Wiley, New York 1974, 136–151.

[17] W. M. Wonham, *On a matrix Riccati equation of stochastic control*, SIAM J. Control **6** (1968), 681–697.

[18] J. Zabczyk, *Controllability of stochastic linear systems*, Systems Control Lett, **1** (1981), 25–31.

[19] M. Zakai and J. Snyders, *Stationary probabilistic measures for linear differential equations driven by white noise*, J. Differential Equations **8** (1970), 27–33.


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