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CONVERGENCE RATE OF THE SOLUTIONS OF SINGULARLY PERTURBED TIME-OPTIMAL CONTROL PROBLEMS

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The order reduction procedure for time-optimal control problems is investigated. Estimates of the convergence rate of the minimal time are obtained for a general set of admissible controls. As applications we consider problems with measurable constrained controls, differentiable constrained controls, controls with integral constraints and state constrained problems.

1. Introduction

Consider a control system described by the following equations:

$$\begin{aligned} \dot{x} &= A_1x+A_2y+B_1u, \quad x(0) = x_0, \\ \lambda\dot{y} &= A_3x+A_4y+B_2u, \quad y(0) = y_0, \end{aligned} \tag{1}$$

where $x \in \boldsymbol{R}^m$, $y \in \boldsymbol{R}^n$, $(x(t), y(t))$ is the state variable, $u(t) \in \boldsymbol{R}^r$ is the control, A_i, B_j, $i = 1, \ldots, 4$, $j = 1, 2$ are constant matrices with appropriate dimensions, λ is a small positive parameter which represents a singular perturbation. We shall denote by $\mathscr{U}(t)$ the set of admissible controls on the interval $[0, t]$, specified in Section 2. We shall study the time-optimal problem (P_λ) for system (1) with a target—the origin $(0_m, 0_n) \in \boldsymbol{R}^{m+n}$. Assuming that A_4^{-1} exists, for $\lambda = 0$ system (1) becomes

$$\dot{x} = A_0x+B_0u, \quad x(0) = x_0, \tag{2}$$

where $A_0 = A_1-A_2A_4^{-1}A_3$, $B_0 = B_1-A_2A_4^{-1}B_2$. Denote by (P_0) the reduced problem, which is the time-optimal problem for system (2) with a target – 0_m. In this paper we study the convergence of the optimal time T_λ for problem (P_λ) to the optimal time T_0 for problem (P_0), when λ tends to zero. Section 2 contains our main theorem, which gives an estimate of the difference $|T_\lambda-T_0|$, depending on the properties of the admissible controls. In the next sections we specialize this estimate in the case of measurable constrained controls, differentiable constrained controls and controls with integral constraints. Section 5 shows that our analysis can be extended to problems with state constraints.

For related results see papers [4], [6]. Work [6] gives an approximate solution of the full order problem on the basis of a separation of the slow and fast modes. [4] contains a detailed proof of the convergence of the optimal controls for λ tending to zero, when $\mathscr{U}(t)$ is a set of measurable functions. The approach presented here is different from those in [4], [6], and provides a basis for estimating the convergence rate of the optimal time in the case of measurable constrained controls as well as in the case of other important classes of admissible controls.

2. Main theorem

We shall define the set of admissible controls. Let $\mathscr{U}_k(t)$ be the set of all $(k-1)$-times continuously differentiable functions $u(\cdot)$: $[0,t]\to \boldsymbol{R}^r$ with absolutely continuous $(k-1)$st derivative, such that

$$u^{(i)}(0) = u^{(i)}(t) = 0_r, \quad i = 0, \dots, k-1,$$
$$u^{(i)}(\tau) \in U_i, \; i = 1, \dots, k \quad \text{for a.e. } \tau \in [0, t].$$

Here U_i, $i = 0, \dots, k$ are convex closed sets in $\boldsymbol{R}^r$, containing the origin in their interior, and U_k is bounded. Let the set of admissible controls on the interval $[0,t]$ be

$$\mathscr{U}(t) = \Big\{u(\cdot) \in \mathscr{U}_k(t),\, a_i \int_0^t |[u(s)]^i|^p ds \leqslant 1,\; i = 1, \dots, r\Big\}, \qquad p \geqslant 1,\; a_i \geqslant 0,$$

where $[u]^i$ is the ith component of u. In the sequel we assume that

A1. The eigenvalues of the matrix A_4 have negative real parts, i.e., $\operatorname{Re}\sigma(A_4) \leqslant -\varrho < 0$;

A2. $\operatorname{rank}[B_0, A_0B_0, \dots, A_0^{m-1}B_0] = m$;

A3. $\operatorname{rank}[B_2, A_4B_2, \dots, A_4^{n-1}B_2] = n$;

A4. Problem (P_0) has a solution.

Let us denote by T_0 the optimal time.

A5. There exists $a_0 > 0$ such that for each $a \in (-a_0, a_0)$ problem (P_0) has a solution in the set

$$\bigcup_{t>0} \Big\{u(\cdot) \in \mathscr{U}_k(t),\, a_i \int_0^t |[u(s)]^i|^p ds \leqslant 1 - a,\; i = 1, \dots, r\Big\}.$$

If T^a is the optimal time, then $\lim\limits_{a\to 0} w(a) = \lim\limits_{a\to 0} |T^a - T_0| = 0$.

The function $w(\cdot)$ can be interpreted as a sensitivity measure for problem (P_0) with respect to perturbations in the constraints.

Let us denote by σ the index of controllability of the pair (A_0, B_0), i.e., the smallest integer such that $\operatorname{rank}[B_0, A_0B_0, \ldots, A_0^{\sigma-1}B_0] = m$. In this section we prove the following result:

THEOREM 1. *There exist constants $\lambda_0 > 0$ and c such that for every $\lambda \in (0, \lambda_0)$ problem (P_λ) has a solution, and if T_λ is the optimal time, then*

$$|T_\lambda - T_0| \leqslant c\lambda^{1/(\sigma+k)} + w(c\lambda^{(kp+1)/(\sigma+k)}).$$

The proof will be presented after a sequence of lemmas.

In paper [6] it is proved that there exists a linear transformation

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = S(\lambda)\begin{bmatrix} x \\ y \end{bmatrix},$$

which reduces system (1) to the system

$$\begin{aligned} \dot{\xi} &= \tilde{A}_0(\lambda)\xi + \tilde{B}_0(\lambda)u, \\ \lambda\dot{\eta} &= \tilde{A}_4(\lambda)\eta + \tilde{B}_2(\lambda)u, \end{aligned} \qquad \begin{bmatrix} \xi(0) \\ \eta(0) \end{bmatrix} = S(\lambda)\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \xi_0^\lambda \\ \eta_0^\lambda \end{bmatrix}, \tag{3}$$

where $\tilde{A}_0(\lambda) = A_0 + O(\lambda)$, $\tilde{B}_0(\lambda) = B_0 + O(\lambda)$, $\tilde{A}_4(\lambda) = A_4 + O(\lambda)$ and $\tilde{B}_2(\lambda) = B_2 + O(\lambda)$. Here and further in the paper we denote by $O(\varepsilon)$ any vector or matrix function $F(\cdot)$ which satisfies the inequality $|F(\varepsilon)| \leqslant c\varepsilon$ for some constant c and for all sufficiently small $\varepsilon > 0$. It is proved in [6] that the matrix $S(\lambda)$ is invertible, its norm is uniformly bounded in λ an $\xi_0^\lambda - x_0 = O(\lambda)$. For each $\lambda > 0$ problem (P_λ) is equivalent to problem $(\tilde{\mathrm{P}}_\lambda)$, which is: to steer the initial point $(\xi_0^\lambda, \eta_0^\lambda)$ to the origin according to (3) in a minimal time.

Denote by $D_\lambda(t)$ $(D_0(t))$ the set of all initial points (ξ, η) (resp. x) which can be steered to the origin according to (3) (resp. (2)) by means of controls from $\mathcal{U}_k(t)$.

LEMMA 1. *Let q be an arbitrary integer. There exist constants $c_1 > 0$ and $t_1 > 0$ such that for every $t \in (0, t_1)$ and for any selection of vectors $v_1, \ldots, v_q \in \mathbf{R}^r$, $|v_i| \leqslant c_1 t^{i+k}$, $i = 1, \ldots, q$, the problem of moments*

$$\int_0^t s^{i-1}u(s)\,ds = v_i, \quad i = 1, \ldots, q \tag{4}$$

has a solution in $\mathcal{U}_k(t)$.

Proof. Let $0 = a_0 < a_1 < \ldots < a_{q+k+1} = 1$ and let $\tau_j = a_j t$. Define the function $w(\cdot)$: $[0, t] \to \mathbf{R}^r$ as $w(\tau) = w_j$ for $\tau \in [\tau_{j-1}, \tau_j)$, where the vectors w_j will be specified below, so that $w_j \in U_k$, $j = 1, \ldots, q+k+1$.

Since $0_r \in \operatorname{Int} U_i$, $i = 0, \ldots, k$ and U_k is bounded, there exists a constant $t_1 > 0$ such that if $t \in (0, t_1)$, then the function

$$u(s) = \int_0^s \int_0^{s_1} \ldots \int_0^{s_{k-1}} w(s_k)\, ds_k \ldots ds_1$$

satisfies the inclusion $u^{(i)}(s) \in U_i$ for each $s \in [0, t]$, $i = 0, \ldots, k$. By induction we obtain that $u^{(k-i)}(t) = 0_r$, $i = 1, \ldots, k$, exactly when

$$\sum_{j=1}^{q+k+1} \left(\sum_{l=1}^{i-1} \frac{1}{(i-l)!\, l!} (a_j - a_{j-1})^{i-l} (1 - a_j)^l \right) w_j = 0_r, \qquad i = 1, \ldots, k. \tag{5}$$

If (5) holds, then integrating by parts we can rewrite equation (4) in the form

$$\frac{(i-1)!}{(i+k-1)!} \int_0^t s^{k+i-1} u^{(k)}(s)\, ds = \frac{(i-1)!\, t^{k+i}}{(i+k)!} \sum_{j=1}^{q+k+1} (a_j^{k+i} - a_{j-1}^{k+i}) w_j = v_i. \tag{6}$$

It is easy to verify that the numbers $a_1, \ldots, a_{q+k}$ can be chosen in such a way that the linear algebraic system consisting of equations (5) and (6) has a solution with respect to $w_1, \ldots, w_{q+k+1}$. If the inequality $|v_i| \leqslant c_1 t^{i+k}$ holds for $i = 1, \ldots, q$ and for a sufficiently small constant c_1, then $w_j \in U_k$, $j = 1, \ldots, q+k+1$ and $u(\cdot) \in \mathscr{U}_k(t)$. Since $u(\cdot)$ satisfies also equalities (4), this completes the proof.

LEMMA 2. *There exist constants $t_2 > 0$ and $c_2 > 0$ such that*

$$\{x \in \boldsymbol{R}^m, |x| \leqslant c_2 t^{\sigma+k}\} \subset D_0(t) \quad \textit{for every } t \in (0, t_2).$$

Proof. From condition A2 and from the definition of σ it follows that there exist unit vectors $w_1, \ldots, w_m$ and integers $s_1, \ldots, s_m \in \{1, \ldots, \sigma\}$ such that the vectors $A_0^{s_j-1} B_0 w_j$, $j = 1, \ldots, m$, are linearly independent. Denote

$$v_i^j = \begin{cases} c_1 t^{s_j+k} w_j & \text{for} \quad i = s_j, \\ 0_m & \text{for} \quad i \neq s_j,\ j = 1, \ldots, m. \end{cases}$$

Let $v_i^j = -v_i^{j-m}$, $w_j = -w_{j-m}$ for $i = 1, \ldots, m$, $j = m+1, \ldots, 2m$. According to Lemma 1 for each $j = 1, \ldots, 2m$ there exists $u_j(\cdot) \in \mathscr{U}_k(t)$ which satisfies equations (4) for $v_i = v_i^j$ and $q = m$. Applying the Taylor formula we obtain that

$$\int_0^t \exp(-A_0 s) B_0 u_j(s)\, ds = \frac{c_1}{(s_j-1)!} t^{s_j+k} (-A_0)^{s_j-1} B_0 w_j + O(t^{m+k+1}).$$

Since $\sigma \leqslant m$, we conclude that there exist constants $\gamma > 0$ and $t_2 > 0$ so that

$$\mathrm{co}\left\{\pm \frac{\gamma c_1}{(s_j-1)!} t^{s_j+k}(-A_0)^{s_j-1} B_0 w_j,\ j = 1, \ldots, m\right\} \subset D_0(t).$$

From the choice of the vectors w_j we get the desired result.

LEMMA 3. *There exist constants $a_3 > 0$ and $\lambda_3 > 0$ such that for each $a \in (0, a_3)$ one can choose a constant $c(a) > 0$ such that the inclusion*

$$\left\{\frac{1}{\lambda}\int_0^{a\lambda} \exp(-\tilde{A}_4 s/\lambda)\tilde{B}_2 u(s)\,ds,\ u(\cdot) \in \mathscr{U}_k(a\lambda)\right\} \supset \{\eta \in \boldsymbol{R}^n,\ |\eta| \leqslant c(a)\lambda^k\}$$

holds for every $\lambda \in (0, \lambda_3)$.

Proof. Let s_j, $j = 1, \ldots, n$, w_j and $u_j(\cdot)$, $j = 1, \ldots, 2n$, be chosen as in the proof of Lemma 2, but for the matrices A_4 and B_2 instead of A_0 and B_0 and for $t = a\lambda$ (a is a positive number). From the choice of $u_j(\cdot)$ and from the estimate

$$\sup\{|u(\tau)|,\ \tau \in [0, \varepsilon]\} = O(\varepsilon^k), \tag{7}$$

applying the Taylor formula we obtain successively

$$\begin{aligned}
\eta_j &= \frac{1}{\lambda}\int_0^{a\lambda} \exp(-\tilde{A}_4 s/\lambda)\tilde{B}_2 u_j(s)\,ds \\
&= \sum_{q=1}^{n} \frac{\lambda^{-1}}{(q-1)!}(-\tilde{A}_4/\lambda)^{q-1}\tilde{B}_2 \int_0^{a\lambda} s^{q-1}u_j(s)\,ds + O(\lambda^k a^{n+k+1}) \\
&= c_1 a^{s_j+k}\lambda^k(-A_4)^{s_j-1}B_2 w_j + \sum_{q=1}^{n}\frac{\lambda^{-q}}{(q-1)!}\Big((-\tilde{A}_4)^{q-1}\tilde{B}_2 - \\
&\qquad\qquad - (-A_4)^{q-1}B_2\Big)\int_0^{a\lambda} s^{q-1}u_j(s)\,ds + O(\lambda^k a^{n+k+1}) \\
&= c_1 a^{s_j+k}\lambda^k(-A_4)^{s_j-1}B_2 w_j + O(\lambda^{k+1}a^{k+1}) + O(\lambda^k a^{n+k+1}).
\end{aligned} \tag{8}$$

Since $s_j \leqslant n$, there exists a constant $\gamma > 0$ such that

$$\begin{aligned}
&\mathrm{co}\{\pm c_1 a^{s_j+k}\lambda^k(-A_4)^{s_j-1}B_2 w_j,\ j = 1, \ldots, n\} \\
&\qquad\qquad \supset \{\eta \in \boldsymbol{R}^n,\ |\eta| \leqslant \gamma a^{n+k}\lambda^k\}.
\end{aligned} \tag{9}$$

From (8) and (9) we obtain the statement of the lemma for $c(a) = \gamma a^{n+k}/2$.

LEMMA 4. *For any $N > 0$ there exist constants $c(N)$ and $\lambda_4 > 0$ such that*

$$\{\xi \in \mathbf{R}^m,\ |\xi| \leqslant N\lambda\} \times \{\eta \in \mathbf{R}^n,\ |\eta| \leqslant N\} \subset D_\lambda\big(c(N)\lambda^{1/(\sigma+k)}\big)$$

for every $\lambda \in (0, \lambda_4)$.

Proof. Let for each $\lambda \in (0, \min\{\lambda_2, \lambda_3\})$ the point $(\xi^\lambda, \eta^\lambda)$ satisfy the inequalities

$$|\xi^\lambda| \leqslant N\lambda, \qquad |\eta^\lambda| \leqslant N. \tag{10}$$

Write $\varepsilon = \lambda^{1/(\sigma+k)}$. Taking the number $q \geqslant (N/c_2)^{1/(\sigma+k)}$, we have from Lemma 2 that $\xi^\lambda \in D_0(\varepsilon q)$. Each point from the set $-\exp(A_0\varepsilon q)D_0(\varepsilon q)$ can be reached from the origin according to (2) by means of a control from $\mathscr{U}_k(\varepsilon q)$. For all sufficiently small $\lambda > 0$ this set contains a ball with centre 0_m and radius $N\lambda/2$. Hence there exist unique vectors $x_0, \ldots, x_m \in \mathbf{R}^m$ and a constant $\beta > 0$ such that the point $\xi_i^\lambda = \frac{1}{2}N\lambda x_i$ can be reached from ξ^λ according to (2) by means of a control $u_i^\lambda(\cdot) \in \mathscr{U}_k(2\varepsilon q)$ $i = 0, \ldots, m$, and if

$$|\bar{\xi}_i^\lambda - \xi_i^\lambda| \leqslant \beta N\lambda, \tag{11}$$

then $0_m \in \mathrm{co}\{\bar{\xi}_i^\lambda, i = 0, \ldots, m\}$.

Let $a \in (0, a_3)$ and let $M \geqslant a$ be an arbitrary number. We can find unique vectors $y_0, \ldots, y_n \in \mathbf{R}^n$ and a constant $\gamma > 0$ so that if

$$|\bar{\eta}_j^\lambda - \lambda^k c(a) y_j| \leqslant \gamma\lambda^k c(a), \tag{12}$$

then $0_n \in \mathrm{co}\{\bar{\eta}_j^\lambda,\ j = 0, \ldots, n\}$. According to Lemma 3, for each $j \in \{0, \ldots, n\}$ there exists a control $v_j^\lambda(\cdot) \in \mathscr{U}_k(a\lambda)$ such that

$$\frac{1}{\lambda}\int_0^{a\lambda} \exp(-\tilde{A}_4 s/\lambda)\tilde{B}_2 v_j^\lambda(s)\,ds = \lambda^k c(a) y_j = \eta_j^\lambda.$$

Define the control

$$u_{ij}^\lambda(\tau) = \begin{cases} u_i^\lambda(\tau) & \text{for} \quad \tau \in [0, 2\varepsilon q], \\ 0_r & \text{for} \quad \tau \in \big(2\varepsilon q, (2q+M)\varepsilon - a\lambda\big), \\ v_j^\lambda\big(\tau - (2q+M)\varepsilon + a\lambda\big) & \text{for} \quad \tau \in [(2q+M)\varepsilon - a\lambda, (2q+M)\varepsilon]. \end{cases}$$

Let $\big(\xi_{ij}^\lambda(\cdot), \eta_{ij}^\lambda(\cdot)\big)$ be the solution of (3) for $u(\cdot) = u_{ij}^\lambda(\cdot)$ and an initial condition $(\xi^\lambda, \eta^\lambda)$. Taking into account (7) and applying the Cauchy formula to the first equation of (3) we obtain

$$\big|\xi_{ij}^\lambda\big((2q+M)\varepsilon\big) - \xi_i^\lambda\big| \leqslant |\xi_{ij}^\lambda(2\varepsilon q) - \xi_i^\lambda| + \big|\xi_{ij}^\lambda\big((2q+M)\varepsilon\big) - \xi_{ij}^\lambda(2\varepsilon q)\big|$$

$$\leqslant \|\exp(\tilde{A}_0 2\varepsilon q) - \exp(A_0 2\varepsilon q)\|\ |\xi^\lambda| +$$

$$+\int_0^{2\varepsilon q}\left\|\exp\left(\tilde A_0(2\varepsilon q-s)\right)\tilde B_0-\exp\left(A_0(2\varepsilon q-s)\right)B_0\right\|\,|u_i^\lambda(r)|\,ds+$$
$$+\|\exp(A_0\varepsilon M)-I_m\|\,|\xi_{ij}^\lambda(2\varepsilon q)|+$$
$$+\int_{(2q+M)\varepsilon-a\lambda}^{(2q+M)\varepsilon}\left\|\exp\left(\tilde A_0(M\varepsilon-s)\right)\tilde B_0\right\|\,|u_{ij}^\lambda(s)|\,ds$$
$$\leqslant O(\lambda)N\lambda+O(\lambda)O(2\varepsilon q)+O(\varepsilon M)\left(|\xi_i^\lambda|+\right.$$
$$\left.+O(\lambda)N\lambda+O(\lambda)O(2\varepsilon q)\right)+O(a\lambda)O(a^k\lambda^k)$$
$$\leqslant p_0(M\varepsilon+a)\lambda,$$

where I_m is the $m\times m$ identity matrix, p_0 is an appropriate constant. Further we estimate

$$\left|\eta_{ij}^\lambda\left((2q+M)\varepsilon\right)-\eta_j^\lambda\right|\leqslant\left\|\exp\left(\tilde A_4(2q+M)\varepsilon/\lambda\right)\right\|\,|\eta^\lambda|+$$
$$+\left|\frac{1}{\lambda}\int_0^{2\varepsilon q}\exp\left(\tilde A_4\left((2q+M)\varepsilon-s\right)/\lambda\right)\tilde B_2u_i^\lambda(s)\,ds\right|+$$
$$+\left|\frac{1}{\lambda}\int_{2\varepsilon q}^{(2q+M)\varepsilon}\exp\left(\tilde A_4\left((2q+M)\varepsilon-s\right)/\lambda\right)\tilde B_2u_{ij}^\lambda(s)\,ds-\eta_j^\lambda\right|$$
$$\leqslant p_1\exp\left(-\varrho(2q+M)\varepsilon/2\lambda\right)+p_2\exp(-\varrho M\varepsilon/2\lambda)+$$
$$+\left|\frac{1}{\lambda}\int_0^{a\lambda}\exp\left(\tilde A_4(a\lambda-\tau)/\lambda\right)\tilde B_2v_j(\tau)\,d\tau-\eta_j^\lambda\right|,$$

where p_1 and p_2 are such that

$$\left\|\exp\left(\tilde A_4(2q+M)\varepsilon/\lambda\right)\right\|N\leqslant p_1\exp\left(-\varrho(2q+M)\varepsilon/2\lambda\right),$$
$$\left|\frac{1}{\lambda}\int_0^{2\varepsilon q}\exp\left(\tilde A_4\left((2q+M)\varepsilon-s\right)/\lambda\right)\tilde B_2u_i^\lambda(s)\,ds\right|\leqslant p_2\exp(-\varrho M\varepsilon/2\lambda).$$

Hence we obtain that for appropriate constants p_3 and p_4,

$$\left|\eta_{ij}^\lambda\left((2q+M)\varepsilon\right)-\eta_j^\lambda\right|\leqslant p_3\exp\left(-\varrho M\varepsilon/2\lambda\right)+\|\exp(\tilde A_4a)-I_n\|\,|\eta_j^\lambda|$$
$$=p_3\exp\left(-\tfrac{1}{2}\varrho M\lambda^{-(\sigma+k-1)/(\sigma+k)}\right)+p_4a\lambda^kc(a).$$

We can choose $a>0$ such that $p_0a\leqslant\frac{1}{4}\beta N$ and $p_4a\leqslant\gamma/2$. If $\sigma+k-1>0$, then taking $M=a$ we can find $\bar\lambda>0$ such that

$$p_3\exp\left(-\tfrac{1}{2}\varrho M\lambda^{-(\sigma+k-1)/(\sigma+k)}\right)<\gamma\lambda^kc(a)/2$$

for every $\lambda\in(0,\bar\lambda)$. If $\sigma+k-1=0$, then $k=0$ and we can choose $M\geqslant a$ such that

$$p_3\exp\left(-\tfrac{1}{2}\varrho M\right)<\tfrac{1}{2}\gamma c(a).$$

In both cases we can take the numbers $a > 0$, $M \geqslant a$ and $\bar{\lambda} > 0$ such that (12) holds for $\bar{\eta}_j^\lambda = \eta_{ij}^\lambda\big((2q+M)\varepsilon\big)$ and $\lambda \in (0, \bar{\lambda})$. If $\bar{\lambda}$ is sufficiently small, then (11) is also fulfilled for $\lambda \in (0, \bar{\lambda})$. Since the set $\mathscr{U}_k\big((2q+M)\varepsilon\big)$ is convex, we obtain the statement of the lemma.

Proof of Theorem 1. Let $u_0(\cdot) \in \mathscr{U}_k(T_0)$ be an optimal control for the problem (P_0). For each $a \in (0, a_0)$ there exists a control $u_0^a(\cdot) \in \mathscr{U}_k(T^a)$ which steers the initial point x_0 to 0_m according to (2) and

$$a_i \int_0^{T^a} |[u_0^a(s)]^i|\, ds \leqslant 1-a, \quad i = 1, \ldots, r \qquad \text{(see A5)}.$$

Let $\big(\xi_\lambda^a(\cdot), \eta_\lambda^a(\cdot)\big)$ be the solution of (3) for $u(\cdot) = u_0^a(\cdot)$. Then there exists a constant N independent of a such that

$$|\xi_\lambda^a(T^a)| \leqslant N\lambda, \qquad |\eta_\lambda^a(T^a)| \leqslant N$$

for all sufficiently small $\lambda > 0$. From Lemma 4 we conclude that there exists a control $v_\lambda(\cdot) \in \mathscr{U}_k\big(c(N)\lambda^{1/(\sigma+k)}\big)$ which steers the point $\big(\xi_\lambda^a(T^a), \eta_\lambda^a(T^a)\big)$ to $(0_m, 0_n)$ according to (3). Thus the control

$$u_\lambda^a(\tau) = \begin{cases} u_0^a(\tau) & \text{for} \quad \tau \in [0, T^a], \\ v_\lambda(\tau - T^a) & \text{for} \quad \tau \in (T^a, T^a + c(N)\lambda^{1/(\sigma+k)}] \end{cases}$$

steers the point $(\xi_0^\lambda, \eta_0^\lambda)$ to $(0_m, 0_n)$ according to (3). It follows from (7) that, for appropriate constants $\bar{c}$ and c,

$$a_i \int_0^{T^a + c(N)\lambda^{1/(\sigma+k)}} |[u_\lambda^a(s)]^i|^p\, ds \leqslant 1 - a + \bar{c}\big(c(N)\lambda^{1/(\sigma+k)}\big)^{kp+1} = 1 - a + c\lambda^{(kp+1)/(\sigma+k)}.$$

Let us take $a = c\lambda^{(kp+1)/(\sigma+k)}$. We obtain that problem $(\tilde{\mathrm{P}}_\lambda)$, and thus (P_λ), has a solution and the optimal time T_λ satisfies the inequality

$$T_\lambda \leqslant T^a + c(N)\lambda^{1/(\sigma+k)} = T_0 + w(c\lambda^{(kp+1)/(\sigma+k)}) + c(N)\lambda^{1/(\sigma+k)}. \tag{13}$$

Now let $u_\lambda(\cdot) \in \mathscr{U}_k(T_\lambda)$ be an optimal control for problem (P_λ) and let $x_0^\lambda(\cdot)$ be the corresponding solution of (2). Since T_λ is bounded in λ, applying the Cauchy formula to the first equation of (3) and to (2) we obtain that for an appropriate constant L, $|x_0^\lambda(T_\lambda)| \leqslant L\lambda$. According to Lemma 2, x_0 can be steered to 0_m by means of a control $w_\lambda(\cdot) \in \mathscr{U}_\lambda(T_\lambda + d\lambda^{1/(\sigma+k)})$, where $d = (L/c_2)^{1/(\sigma+k)}$. For some constant M, we have

$$a_i \int_0^{T_\lambda + d\lambda^{1/(\sigma+k)}} |[w_\lambda(s)]^i|^p\, ds \leqslant 1 + M\lambda^{(kp+1)/(\sigma+k)}, \quad i = 1, \ldots, r.$$

Hence $T^a \leqslant T_\lambda + d\lambda^{1/(\sigma+k)}$ for $a = -M\lambda^{(kp+1)/(\sigma+k)}$, and from A5 we obtain

$$T_0 \leqslant T_\lambda + d\lambda^{1/(\sigma+k)} + w(M\lambda^{(kp+1)/(\sigma+k)}). \tag{14}$$

Combining (13) and (14) we complete the proof of the theorem.

3. Measurable and differentiable constrained controls

In this section we shall apply Theorem 1 in the case $\mathscr{U}(t) = \mathscr{U}_k(t)$, i.e., $a_i = 0$, $i = 1, \ldots, r$. Time-optimal control problems with differentiable constrained controls are studied in [1], [5], [7]–[9]. Conditions for optimality and some properties of the optimal controls are established in [1], [7], [8]. The relationship between the problem with differentiable controls and the problem with measurable controls (the case $k = 0$) is studied in [5], [9].

Taking into account that condition A5 is fulfilled for $w(\cdot) = 0$ and applying Theorem 1, we obtain the estimate

$$|T_\lambda - T_0| \leqslant c\lambda^{1/(\sigma+k)} \tag{15}$$

for $\lambda \in (0, \lambda_0)$.

Remark 1. The convergence $\lim\limits_{\lambda \to 0} |T_\lambda - T_0| = 0$ has been proved in [4] in the "classical" case $k = 0$.

Remark 2. The accuracy of the obtained estimate remains an open question. We note only that there are examples with $\sigma = 1$, $k = 0$, such that $T_\lambda - T_0 = c\lambda$, and there are examples with $\sigma = 1$, $k = 1$, such that $|T_\lambda - T_0| > O(\lambda)$.

4. Controls with L_p constraints

The estimate obtained in Theorem 1 depends on the sensitivity measure $w(\cdot)$ of the reduced problem (P_0). In this section we shall estimate the function $w(\cdot)$ in case $p > 1$. In order to simplify the exposition we shall consider the case $k = 0$, i.e., the set of admissible controls on the interval $[0, t]$ consists of all measurable functions $u(\cdot)$ such that $u(s) \in U_0$ for a.e. $s \in [0, t]$ and $\int\limits_0^t |[u(s)]^i|^p ds \leqslant 1$, $i = 1, \ldots, r$. Time-optimal problems with such constraints are investigated in [3].

LEMMA 5. *There exist constants $a_0 > 0$ and L such that*

$$w(a) \leqslant L|a|^{(p-1)/(p\sigma-1)}$$

for every $a \in (-a_0, a_0)$.

Proof. Let $a \in (0, 1)$ and let $u_0(\cdot) \in \mathscr{U}(T_0)$ steer the point x_0 to 0_m according to equation (2). For the control $u_a(\cdot) = (1-a)u_0(\cdot) \in \mathscr{U}_0(T_0)$ we have

$$\int\limits_0^{T_0} |[u_a(s)]^i|^p ds \leqslant (1-a)^p \leqslant 1-a, \quad i = 1, \ldots, r.$$

Let $x_a(\cdot)$ be the solution of (2) for $u(\cdot) = u_a(\cdot)$. Then for some constant d, $|x_a(T_0)| \leqslant da$. From Lemma 2 we obtain that, using controls from the set $\varepsilon\mathscr{U}_0(t)$ ($\varepsilon \in (0,1)$, $t \in (0, t_2)$), we can steer to 0_m each point from the ball $\{x \in \boldsymbol{R}^m,\ |x| \leqslant \varepsilon c_2 t^\sigma\}$. Furthermore, there exists a constant M such that if $u(\cdot) \in \varepsilon\mathscr{U}_0(t)$, then

$$\int_0^t |[u(s)]^i|^p ds \leqslant M\varepsilon^p t, \qquad i = 1, \ldots, r.$$

Taking $t = (dM^{1/p}/c_2)^{p/(p\sigma-1)} a^{(p-1)/(p\sigma-1)} = La^{(p-1)/(p\sigma-1)}$ and $\varepsilon = (a/Mt)^{1/p}$ we obtain the inequalities $M\varepsilon^p t \leqslant a$ and $da \leqslant \varepsilon c_2 t^\sigma$. Hence $T^a \leqslant T_0 + La^{(p-1)/(p\sigma-1)}$ for $a \in (0,1)$.

Now let $v_a(\cdot) \in \mathscr{U}_0(T^{-a})$ steer x_0 to 0_m according to (2) and let

$$\int_0^{T^{-a}} |[v_a(s)]^i|^p \leqslant 1 + a, \qquad i = 1, \ldots, r.$$

Taking $u_a(\cdot) = (1-3a)v_a(\cdot)$, we have

$$\int_0^{T^{-a}} |[u_a(s)]^i|^p ds \leqslant (1-3a)^p(1+a) \leqslant 1 - a$$

for all sufficiently small $a > 0$ and for $i = 1, \ldots, r$. Repeating the above arguments we estimate

$$T_0 \leqslant T^{-a} + La^{(p-1)/(p\sigma-1)}.$$

Since, obviously, $T^a \geqslant T_0$ and $T_0 \geqslant T^{-a}$ for $a > 0$, we get the desired result.

Applying Lemma 5 and Theorem 1 we obtain the estimate

$$|T_\lambda - T_0| \leqslant c(\lambda^{1/\sigma} + \lambda^{(p-1)/\sigma(p\sigma-1)}) \leqslant 2c\lambda^{(p-1)/\sigma(p\sigma-1)}. \tag{16}$$

Estimates of this type can be obtained similarly in the case of integral constraints on the derivatives of the controls (if $k \geqslant 1$),

Estimate (16) has no sense in the case $p = 1$. In this case it is possible that condition A5 is not fulfilled. There exist examples in which problem (P_0) has no solution for $a > 0$ as well as examples in which $T^a < +\infty$ but $\lim\limits_{\substack{a\to 0\\ a>0}} T^a > T_0$.

EXAMPLE 1.

$$\dot{x}^1 = x^2, \qquad x^1(0) = x_0^1 = \sqrt{2}/2 - 1,$$
$$\dot{x}^2 = x^1 + u, \qquad x^1(0) = x_0^2 = -3\sqrt{2}/2,$$

$|u| \leqslant 1$, $\int_0^{+\infty} |u(s)|ds \leqslant 3\pi/4$. It can be seen that $T_0 \leqslant 5\pi/4$ but $T^a \in [3\pi/2, 3\pi]$ for all sufficiently small $a > 0$. Observe that the initial point (x_0^1, x_0^2) does not belong to a flat part of the set $\bigcup_{t\geqslant 0} D_0(t)$ (see [3]).

The following example shows that condition A5 is essential for the convergence $\lim_{\lambda\to 0} T_\lambda = T_0$.

EXAMPLE 2.

$$\begin{aligned} \dot{x}^1 &= x^2, & x_0^1 &= \sqrt{2}/2-1,\\ \dot{x}^2 &= x^1+y, & x_0^2 &= -3\sqrt{2}/2,\\ \lambda\dot{y} &= -y+u, & y_0 &= 0. \end{aligned}$$

The constraints on the control are the same as in Example 1. The reduced problem is exactly the problem considered in the previous example. It can be seen that $\lim_{\lambda\to 0} T_\lambda > T_0$. However, for some other initial points condition A5 holds and so we have the convergence $\lim_{\lambda\to 0} T_\lambda = T_0$.

5. State constrained problems

In this section we extend the result obtained in Section 3 to the problems with state constraints for the slow phenomena. Namely, we show that estimate (15) remains true under an additional condition of Slater's type. We limit our consideration to the case of measurable controls ($k = 0$) and time invariant state constraints.

Admissible controls are all measurable functions with values in a convex compact set U_0, $0_r \in \operatorname{Int} U_0$. We consider the problems (P_λ) and (P_0) defined in Section 1 but with the additional constraint $x \in X$, where X is a convex closed set in $\boldsymbol{R}^m$ such that 0_m, $x_0 \in \operatorname{Int} X$. We assume that conditions A1–A4 hold as well as the following condition:

B. There exist a constant $\varepsilon_0 > 0$ and an admissible control $u_s(\cdot)$ such that if $x_s(\cdot)$ is the solution of (2) for $u(\cdot) = u_s(\cdot)$, then $x_s(t) \in \operatorname{Int} X$ for every $t \in [0, T_0+\varepsilon_0]$.

THEOREM 2. *There exist constants c and $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0)$ problem* (P_λ) *has a solution and if T_λ is the optimal time, then*

$$|T_\lambda - T_0| \leqslant c\lambda^{1/\sigma}.$$

Proof. The proof of Theorem 6.1 [2] contains the following result:

LEMMA 6. *There exists a constant L such that if $u_\lambda(\cdot) \in \mathscr{U}_0(t_\lambda)$, $\lambda \in (0, 1)$, $t_\lambda \leqslant t_0$, then*

$$|x_\lambda(s) - x_0^\lambda(s)| \leqslant L\lambda, \qquad |y_\lambda(t_\lambda)| \leqslant L$$

for every $s \in [0, t_\lambda]$, $\lambda \in (0, 1)$, where $\big(x_\lambda(\cdot), y_\lambda(\cdot)\big)$ is the solution of (1) *and $x_0^\lambda(\cdot)$ is the solution of* (2) *for $u(\cdot) = u_\lambda(\cdot)$.*

Without loss of generality we suppose that X is compact. Then there exists a constant $\gamma > 0$ so that for each $a \in (0, 1)$, $t \in [0, T_0 + \varepsilon_0]$ and $x \in X$ the inequality

$$\operatorname{dist}\big(ax_s(t) + (1-a)x,\ \partial X\big) \geqslant \gamma a \tag{17}$$

holds, where $\operatorname{dist}(z, \partial X) = \min\{|z - \bar{x}|, \bar{x} \in \partial X\}$ and ∂X is the boundary of X.

Let $u_0(\cdot)$ be an optimal control for problem (P_0). Then $u_0^\lambda(\cdot) = (1 - L\lambda/\gamma)u_0(\cdot) + (L\lambda/\gamma)u_s(\cdot)$ is an admissible control for each sufficiently small λ. Let $\big(x_\lambda(\cdot), y_\lambda(\cdot)\big)$ be the solution of (1) and let $x_0^\lambda(\cdot)$ be the solution of (2) for $u(\cdot) = u_0^\lambda(\cdot)$. From Lemma 6 and (17) we obtain successively

$$x_0^\lambda(t) \in X, \qquad \operatorname{dist}\big(x_0^\lambda(t), \partial X\big) \geqslant \gamma L\lambda/\gamma = L\lambda, \qquad x_\lambda(t) \in X$$

for all $t \in [0, T_0]$. Since $0_m \in \operatorname{Int} X$, Lemma 4 holds also in the case of state constraints. From Lemma 6 and Lemma 4 we conclude that problem (P_λ) has a solution and

$$T_\lambda \leqslant T_0 + c(L)\lambda^{1/\sigma}. \tag{18}$$

Now let $u_\lambda(\cdot)$ be an optimal control for problem (P_λ) and let $\big(x_\lambda(\cdot), y_\lambda(\cdot)\big)$ be the corresponding solution of (1). Define $\bar{u}_0^\lambda(\cdot) = (1 - L\lambda/\gamma)u_\lambda(\cdot) + (L\lambda/\gamma)u_s(\cdot)$. As above, $\bar{u}_0^\lambda(\cdot)$ is an admissible control for all sufficiently small λ. Let $x_0^\lambda(\cdot)$ and $\bar{x}_0^\lambda(\cdot)$ be the solutions of (2) for $u(\cdot) = u_\lambda(\cdot)$ and for $u(\cdot) = \bar{u}_0^\lambda(\cdot)$, respectively. Then

$$\begin{aligned}\bar{x}_0^\lambda(t) &= (1 - L\lambda/\gamma)x_0^\lambda(t) + (L\lambda/\gamma)x_s(t)\\ &= (1 - L\lambda/\gamma)x_\lambda(t) + (L\lambda/\gamma)x_s(t) + (1 - L\lambda/\gamma)\big(x_0^\lambda(t) - x_\lambda(t)\big).\end{aligned}$$

Applying Lemma 6, (17) and Lemma 2 we obtain the inequality $T_0 \leqslant T_\lambda + c_2 L^{1/\sigma}\lambda^{1/\sigma}$, which combined with (18) completes the proof.

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