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# THE UNIFORM EXPONENTIAL STABILITY OF CERTAIN LINEAR PROCESSES IN A HILBERT PHASE SPACE

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## Introduction

When I was invited to lecture at the Banach Semester on Control Theory, two of my proposed lectures dealt with the uniform exponential stability (u.e.s.) of linear autonomous neutral functional differential equations in a Hilbert phase space [3]. Due to excessive familiarity certain things were second nature to me which were quite new to the audience. In a word I delivered two rather incoherent lectures. I think only professors Olbrot and Zabczyk understood them. This paper is an attempt to rectify the lacunae in those lectures. The work in [3], about which the lectures were centered, basically studies the region of holomorphicity of the Laplace transform of an operator valued function associated with the neutral system. This operator valued function plays the same role as the fundamental transformation of a linear autonomous ordinary differential equation in Euclidean $n$-space. However, it does not generate a semi-group of linear transformations. Unfortunately, I realized this after I gave my lectures. Once one realizes this, a transparently simple criterion can be developed to give necessary and sufficient conditions for a large class of linear autonomous processes which include certain ordinary, partial, functional differential and integral equations and combinations thereof.

The essential ingredient in the systems considered in this paper is that their Laplace transforms are dominated by linear operators which have some of the characteristics of diffusion operators (e.g. the operator $\partial^2/\partial x^2$ of the one dimensional heat equation).

The paper is organized as follows. Section 1 consists of preliminaries. Section 2 develops the main result of this paper, Theorem 2.1. Although the methods used in this section are elementary, it is interesting to note that only in 1974 did Henry [5], using more sophisticated techniques, find necessary and sufficient conditions for the u.e.s. of linear autonomous neutral functional differential equations in $C^n$. Section 3 examines the extent to which stability properties are parametrically dependent. Broadly speaking, it is shown that what one would expect holds. Namely, that if a parametrically dependent family, whose Laplace transforms are continuous in the parameter, is u.e.s. for one value of the parameter, then in some relatively open subset of that parameter all members of the family are uniformly exponentially stable. Section 4 is devoted to examples. The last, Example 4.4, is a system, depending on a parameter, which does not conform to the systems in Section 2. In this example the system is u.e.s. for one value of the parameter, but does not possess a relatively open neighborhood in which every member of the family is uniformly exponentially stable.

I should like to thank Professor Jerzy Zabczyk for the discussions we had concerning the above problem and others related to it. The realization that the semigroup structure is unnecessary for stability considerations of systems involving dissipative type elements is one product of our discussions.

## 1. Preliminaries

$Z$ will denote the complex plane. $H$ will denote a complex Hilbert space, whose zero vector will be denoted by the symbol 0. The space of bounded linear mappings from $H$ into itself will be denoted by $[H]$ and the identity mapping in $[H]$ by $I$. The norms in all Banach spaces will be denoted by $|\cdot|$ and the inner product in any Hilbert space by $(\cdot, \cdot)$. Let $h > 0$. We define a Hilbert function space $\mathscr{H}$ associated with $H$ as follows:

Let $\varphi(0) \in H$ and $\varphi$: $[-h, 0] \to H$ be Bochner square integrable. Then

$$\hat{\varphi} = \langle \varphi(0), \varphi \rangle \in \mathscr{H}. \tag{1.1}$$

The inner product in $\mathscr{H}$ is given by

$$(\hat{\varphi}, \hat{\psi}) = \big(\varphi(0), \psi(0)\big) + \int_{-h}^{0} \big(\varphi(\sigma), \psi(\sigma)\big) d\sigma. \tag{1.2}$$

Another space of interest is the nonreflexive Banach space $C_h$ which consists of the continuous linear mappings $\varphi$: $[-h, 0] \to H$ with norm

$$|\varphi| = \sup\{|\varphi(\sigma)|: \sigma \in [-h, 0]\}.$$

Assumption 1.1. $A_0$ will denote the infinitesimal generator of a $C_0$ semi-group, $T(t)$, defined on $H$, which is compact for $t > 0$ and satisfies the condition that there exist real numbers $M_1$ and $\omega$ such that

$$|(sI - A_0)^{-1}| \leqslant \frac{M_1}{|s-\omega|} \tag{1.3}$$

for $\operatorname{Re} s > \omega$.

The type of process considered in this paper has a Laplace transform of the form

$$\bar{x}(s, \hat{\varphi}) = [sI - A_0 - Q(s)]^{-1}[\varphi(0) + q(s, \varphi)], \tag{1.4a}$$

where $\hat{\varphi} \in \mathscr{H}$, or of the form

$$\bar{x}(s, \varphi) = [sI - A_0 - Q(s)]^{-1}[q_1(s, \varphi) + \varphi(0)], \tag{1.4b}$$

where $\varphi \in C_h$.

In equations (1.4) we impose the following conditions: $q(s, \varphi)$ and $q_1(s, \varphi)$ are finite Laplace transforms, whose inverses $l(t, \varphi)$ and $l_1(t, \varphi)$ have support on some interval $[T_1, T_2]$, $-\infty < T_1 \leqslant T_2 < \infty$, which is independent of $\varphi$, and $l_1$ and $l_2$ are linear mappings which are continuous in $t$, i.e.,

$$l\colon [T_1, T_2] \to [H]$$

and

$$l_1\colon [T_1, T_2] \to [H]$$

are continuous mappings. The operator valued function $Q(s)$ in (1.4) satisfies the assumption:

ASSUMPTION 1.2. In some right half plane $\operatorname{Re} s > -\beta_1$, $\beta_1 > 0$, $Q(s)$ is holomorphic and uniformly bounded in $[H]$, i.e., if $\operatorname{Re} s > -\beta_1$ there is an $M_2$ such that

$$|Q(s)| \leqslant M_2$$

if $\operatorname{Re} s > -\beta_1$.

*Remark* 1.1. It is possible to relax Assumption 1.2 to $Q(s)$ being uniformly bounded in $\operatorname{Re} s \geqslant 0$. We then can obtain conditions for asymptotic stability of the type Miller [7] gave, when $H = R^n$, for the integral differential equation

$$\dot{x}(t) = A_0 x(t) + \int_0^t B(t-\sigma) x(\sigma) d\sigma.$$

One reason for not doing so is that the exposition becomes more complex and camouflages a basically simple idea.

For convenience we shall write

$$\bar{S}(s) = [sI - A_0 - Q(s)]^{-1} \tag{1.5}$$

and denote the inverse Laplace transform of (1.5) by

$$\mathscr{L}^{-1}\big(\bar{S}(s)\big)(t) = S(t). \tag{1.6}$$

Thus (1.4) are the Laplace transforms of the linear processes

$$x(t,\hat{\varphi}) = S(t)\varphi(0)+\int_0^t S(t-\sigma)l(\sigma,\varphi)d\sigma \tag{1.7a}$$

and

$$x(t,\varphi) = S(t)\varphi(0)+\int_0^t S(t-\sigma)l_1(\sigma,\varphi)d\sigma, \tag{1.7b}$$

where of course $t \geqslant 0$.

DEFINITION 1.1. The processes (1.7) will be called *uniformly exponentially stable* (u.e.s.) if there exist $M_3 > 0$ and $\beta_3 > 0$ such that for $\hat{\varphi} \in \mathscr{H}$ or $\varphi \in C_h$, as the case may be,

$$|x(t,\hat{\varphi})| \leqslant M_3 e^{-\beta_3 t}|\hat{\varphi}|$$

and

$$|x(t,\varphi)| \leqslant M_3 e^{-\beta_3 t}|\varphi|$$

if $t \geqslant 0$.

## 2. Some stability properties of (1.7)

*Remark* 2.1. We shall, without loss of generality, from now on assume the spectrum of $A_0$ lies in $\operatorname{Re} s \leqslant -\beta_1$. This is accomplished by observing that $\lambda \geqslant 0$ can be chosen such that

$$(sI-A_0+\lambda I)^{-1} \tag{2.1}$$

is holomorphic in $\operatorname{Re} s > -\beta_1$. Thus we can write (1.5)

$$\begin{aligned} \bar{S}(s) &= \big(sI-A_0+\lambda I-\big(\lambda I+Q(s)\big)\big)^{-1} \\ &= \big[I-(sI-A_0+\lambda I)^{-1}\big(\lambda I+Q(s)\big)\big]\,(sI-A_0+\lambda I)^{-1} \\ &= [I-(sI-\tilde{A}_0)^{-1}\tilde{Q}(s)]\,(sI-\tilde{A}_0)^{-1}, \end{aligned} \tag{2.2}$$

where $\tilde{A}_0$ and $\tilde{Q}(s)$ satisfy Assumptions 1.1 and 1.2. Moreover, (1.3) can be now made to satisfy

$$|(sI-\tilde{A}_0)^{-1}| \leqslant \frac{M}{|s+\beta_1/2|} \tag{2.3}$$

if $\operatorname{Re}\ s > -\beta_1/2$. The reason for these observations is that in the discussion given below extensive discussion will be devoted to the holomorphicity of $\bar{S}(s)$ in $\operatorname{Re} s > -\beta_1$ and inequality (2.3).

Also notice that Assumptions 1.1 and 1.2 guarantee that

$$(sI-A_0+\lambda I)^{-1}\big(\lambda I+Q(s)\big) \tag{2.4}$$

is a compact operator for $\operatorname{Re} s > -\beta_1$, and the values of $s$ in $\operatorname{Re} s > -\beta_1$ for which (2.4) has one as an eigenvalue are isolated (see e.g. [4], Lemma 13, p. 592) and at most finite in number.

The derivative of $\bar{S}(s)$ in (1.5) is

$$(\bar{S}(s))' = \bar{S}(s)\left[-I + (Q(s))'\right]\bar{S}(s). \tag{2.5}$$

By Assumption 1.2 $Q'(s)$ is holomorphic in $\operatorname{Re} s > -\beta_1/2$ and uniformly bounded there. Hence if $\bar{S}(s)$ has no poles in $\operatorname{Re} s > -\beta_1/2$, then for each $x_0 \in H$

$$-tS(t)x_0 = \frac{1}{2\pi i} \int_{-\beta_1/3+i\infty}^{-\beta_1/3+i\infty} e^{st}(\bar{S}(s)x_0)'\,ds. \tag{2.6}$$

Making use of Remark 2.1 and Assumptions 1.1 and 1.2 we easily deduce the existence of $M_4 > 0$, independent of $x_0$ in $H$, such that

$$|tS(t)x_0| \leqslant M_4 e^{-\beta_1 t/3}|x_0|. \tag{2.7}$$

Consequently we can state the following lemma.

LEMMA 2.1. *Let Assumptions* 1.1 *and* 1.2 *hold. If* $\bar{S}(s)$ *has no poles in* $\operatorname{Re} s > -\beta_1/2$, *then systems* (1.7) *are uniformly exponentially stable* (u.e.s.).

*Proof.* The proof is a consequence of inequality (2.7) and the fact that $l$ and $l_1$ in (1.7) have compact support.

LEMMA 2.2. *Let* $\bar{S}(s)$ *satisfy Assumptions* 1.1 *and* 1.2. *If* $\bar{S}(s)$ *is not holomorphic in* $\operatorname{Re} s \geqslant 0$ *there exist solutions of* (1.7) *whose norms are bounded away from zero. Indeed, if* $\bar{S}(s)$ *has a pole in* $\operatorname{Re} s > 0$, *then* (1.7) *have solutions which diverge exponentially.*

*Proof.* Assumption 1.1 and Remark 2.1 guarantee that $\bar{S}(s)$ has only a finite number of poles in $\operatorname{Re} s > -\beta_1$. Suppose that such a pole occurs at $\mu$, $\operatorname{Re}\mu \geqslant 0$. Then making use of Remark 2.1 we may assume there exists an integer $m \geqslant 1$ such that the generalized null space

$$N_m(\mu) = \{x\colon [I - (\mu I - A_0)^{-1}Q(\mu)]^m x = 0\}$$

is not empty, where $m$ is the smallest positive integer such that

$$N_m(\mu) = N_{m+1}(\mu).$$

Since $(\mu I - A_0)^{-1}$ is compact, $N_m(\mu)$ is finite dimensional and thus for $s$ in a neighborhood of $\mu$, $\bar{S}(s)$ must have the Laurent expansion

$$\bar{S}(s) = \sum_{k=-m}^{\infty} (s-\mu)^k R_k, \tag{2.8}$$

where $R_{-m} \neq 0$ (see e.g. [6]). In the case of (1.7a) there exists $x_0$, $|x_0| = 1$, such that $R_{-m}x_0 \neq 0$. We let $\hat{\varphi} = \langle x_0, 0 \rangle$, and choose $c(\mu)$ a positively oriented circle in $Z$ with center $\mu$ which contains no poles of $\bar{S}(s)$, other than $\mu$, on its boundary or interior. The solution of (1.7a) may be given in the form

$$x(t, \hat{\varphi}) = \frac{1}{2\pi i} \int_{c(\mu)} e^{st}\bar{S}(s)x_0 ds + \frac{1}{2\pi i} \int_{c_1} e^{st}\bar{S}(s)x_0 ds, \tag{2.9}$$

where $c_1$ is chosen to consist of at most finitely many contours surrounding the poles of $\bar{S}(s)$ in $\mathrm{Re}\, s \geqslant 0$ plus some vertical line, $\mathrm{Re}\, s = -\beta_4$, $0 < \beta_4 < \beta_1$. Since $R_{-m}x_0 \neq 0$, the first integral in (2.9) has the form

$$\frac{1}{2\pi i} \int_{c(\mu)} e^{st}\bar{S}(s)x_0 ds = e^{\mu t} \sum_{j=0}^{m-1} p_j(t) q_j, \tag{2.10}$$

where the $p_j$ are polynomials in $t$ such that degree $p_j \leqslant j$, $q_j \in H$ and

$$p_{m-1}(t) q_{m-1} = \frac{t^{m-1}}{(m-1)!} R_{-m} x_0.$$

Thus (2.10) is not identically zero for $t \geqslant 0$ and in fact either diverges (if $\mathrm{Re}\,\mu > 0$) or at least does not converge to zero (if $\mathrm{Re}\,\mu = 0$) as $t$ tends to infinity. The second integral in (2.9) has the form

$$x_1(t, \hat{\varphi}) = \sum_{j=1}^{r} e^{\mu_j t} \sum_{k=1}^{m(j)} p_{kj}(t) q_{jk} + \frac{1}{2\pi i} \int_{-\beta_4 - i\infty}^{-\beta_4 + i\infty} e^{st}\bar{S}(s)x_0 ds, \tag{2.11}$$

where $\mathrm{Re}\,\mu_j \geqslant 0$, $\mu_j \neq \mu$, $j = 1, \ldots, r$ (i.e., if $\bar{S}(s)$ has poles other than $\mu$ in $\mathrm{Re}\, s \geqslant 0$, otherwise the first term on the right side of (2.11) is zero). This proves the lemma for (1.7a).

In the case of (1.7b) we find an approximation $\{\varphi_n\}$ such that $\{\varphi_n(0) = x_0\}$, $\{|\varphi_n| = |x_0|\}$ and $\varphi_n(\sigma) \to 0$ pointwise in $[-h, 0)$. Then for $n$ sufficiently large the solution $x(t, \varphi_n)$ of (1.7b) can be shown to have the same qualitative property that (2.9) has (see e.g. [1] for a complete description of this process for a special type of problem).

Lemmas 2.1 and 2.2 may be summed up in the following theorem.

THEOREM 2.1. *Let Assumptions* 1.1 *and* 1.2 *hold. Then the systems* (1.7) *are uniformly exponentially stable if and only if there exists* $\beta > 0$ *such that* $\bar{S}(s)$ *is holomorphic in* $\mathrm{Re}\, s > -\beta$.

## 3. Stability and parametric dependence

In this section we consider the manner in which stability varies when a family of linear processes depends continuously on a parameter. Before describing the processes we need some definitions. Let $\Omega \subset R^m$ be simply connected.

(i) Let $h: \Omega \to R^+$ be continuous and define

$$\mathscr{H}_a = \{\langle\varphi(0), \varphi\rangle : \varphi(0) \in H \text{ and } \varphi: [-h(a), 0] \to H \quad \text{is Bochner square integrable}\}.$$

(ii) Let $q(s, a, \hat{\varphi})$, $\hat{\varphi} \in \mathscr{H}_a$, be the finite Laplace transform of $l(t, a, \hat{\varphi})$, where

$$l: \Omega \times R^+ \to [H]$$

is continuous. We shall assume the support of $l$ for $a$ fixed is $-\infty < T_1(a) \leqslant T_2(a) < \infty$, where $T_1$ and $T_2$ are continuous.

(iii) Let $Q: Z \times \Omega \to [H]$ be continuous in and, for $a \in \Omega$ fixed, holomorphic in $\operatorname{Re} s > -\beta_1$ and such that $|Q(s, a)| \leqslant M$, $M < \infty$, if $\operatorname{Re} s > -\beta_1$. We consider the family of linear processes defined on $\Omega$ whose Laplace transforms can be written

$$\bar{x}(s, \hat{\varphi}, a) = \bar{S}(s, a)\,[\varphi(0) + q(s, a, \hat{\varphi})], \tag{3.1}$$

where

$$\bar{S}(s, a) = [sI - A_0 - Q(s, a)]^{-1}. \tag{3.2}$$

We denote the inverse transforms of (3.2) by $S(t, a)$. Thus (3.1) is the Laplace transform of

$$x(t, \hat{\varphi}, a) = S(t, a)\Big[\varphi(0) + \int_0^t S(t-\sigma)\, l(\sigma, a, \hat{\varphi})\, d\sigma\Big]. \tag{3.3}$$

Clearly, by Theorem 2.1, (3.3) is u.e.s. if and only if $\bar{S}(s, a)$ is holomorphic in $\operatorname{Re} s > -\beta$, $\beta > 0$. Thus we can state the following lemma.

LEMMA 3.1. *Let* (3.3) *be* u.e.s. *for* $a_0 \in \Omega$. *Then there exists a maximal relatively open set* $U(a_0)$ *in* $\Omega$, *containing* $a_0$, *such that* (3.3) *is* u.e.s. *for all* $a$ *in* $U(a_0)$.

*Proof.* We only need to prove that (3.3) is u.e.s. in some relatively open neighborhood of $a_0$. The proof proceeds by contradiction. Thus suppose there exist $\{a_n\} \subset \Omega$ which converge to $a_0$ and $\{s_n\} \subset \operatorname{Re} s \geqslant 0$ such that

$$(s_n I - A_0)^{-1} Q(s_n, a_n) \tag{3.4}$$

has, for each $n$, one as an eigenvalue. Since $|Q(s, a)| \leqslant M$ for $\operatorname{Re} s \geqslant 0$, it follows that $\{s_n\}$ lies in a compact set in $\operatorname{Re} s \geqslant 0$. Hence we can find a subsequence $\{(s_n, a_n)\}$ which converges to $(s_0, a_0)$. But by continuity and the compactness of (3.4) it follows that

$$(s_0 I - A_0)^{-1} Q(s_0, a_0)$$

has one as an eigenvalue which implies that $\bar{S}(s, a_0)$ is not analytic in $\operatorname{Re} s > -\beta$, $\beta > 0$. This contradiction proves the lemma.

LEMMA 3.2. *Let* (3.1) *be* u.e.s. *for* $a_0$ *and* $U(a_0)$ *be the maximal relatively open set described in Lemma* 3.1. *If* $a_1$ *is on the boundary of* $U(a_0)$ *but not in* $U(a_0)$, *then* $\bar{S}(s, a_1)$ *has one or more poles on the imaginary axis and the remainder in* $\operatorname{Re} s < 0$.

*Proof.* Clearly $\bar{S}(s, a_1)$ cannot have all its poles in $\operatorname{Re} s < 0$ for then by Assumption 1.1 and Theorem 2.1 $a_1$ would be in $U(a_0)$. Suppose $\bar{S}(s, a_1)$ has a pole at $\lambda_1$, $\operatorname{Re}\lambda_1 > 0$. Let $C(\lambda_1)$ be a positively oriented circle in $\operatorname{Re} s > 0$ which contains $\lambda_1$ as its center and has no other poles of $\bar{S}(s, a_1)$ on its boundary or interior. Then near $s = \lambda_1$, $\bar{S}(s, a_1)$ has the Laurent expansion

$$\bar{S}(s, a_1) = \sum_{j=-m}^{\infty} R_j (s - \lambda_1)^j, \tag{3.5}$$

where $m \geqslant 1$, $R_{-m} \neq 0$ and

$$R_{-m} = \frac{1}{2\pi i} \int_{C(\lambda_1)} (s - \lambda_1)^{m-1} \bar{S}(s, a_1)\, ds \tag{3.6}$$

(see e.g. [6]). $\{a_n\} \in U(a_0)$ converge to $a_1$. By continuity and compactness $\{\bar{S}(s, a_n)\}$ converges to $\bar{S}(s, a_1)$ on $C(\lambda_1)$. Thus for $n$ sufficiently large

$$R_{-m}(a_n) = \frac{1}{2\pi i} \int_{C(\lambda_1)} (s - \lambda_1)^{m-1} \bar{S}(s, a_n)\, ds \neq 0, \tag{3.7}$$

which implies that $\bar{S}(s, a_n)$ has, for large $n$, at least one pole inside $C(\lambda_1)$. This is impossible since $a_n \in U(a_0)$. Hence $\bar{S}(s, a_1)$ can have no pole in $\operatorname{Re} s > 0$, but must have, by Theorem 2.1, at least one pole on the imaginary axis.

Combining Lemmas 3.1 and 3.2 we obtain the following theorem.

THEOREM 3.1. *If* $a_0 \in \Omega$ *is such that* (3.3) *is* u.e.s. *for* $a = a_0$, *then there exists a maximal relatively open set* $U(a_0)$, *containing* $a_0$, *such that* (3.3) *is* u.e.s. *for all* $a$ *in* $U(a_0)$. *If* $a_1$ *is on the boundary of* $U(a_0)$ *but not in* $U(a_0)$, *the* $\bar{S}(s, a)$, *defined by* (3.2), *has at least one pole on the imaginary axis and none in* $\operatorname{Re} s > 0$.

## 4. Examples

EXAMPLE 4.1. Let $0 \leqslant h_1 \leqslant h_2 \leqslant \ldots \leqslant h_m = h$. Let $\{A_j\} \subset H$, $A$: $[-h, 0] \to [H]$ be continuous and $A_0$ satisfy Assumption 1.1. Consider the $C_0$ semi-group generated by solutions of

$$\text{(4.1a)} \quad \frac{d}{dt}\big((x(t)\big) = A_0 x(t) + \sum_{j=1}^{m} A_j x(t-h_j) + \int_{-h}^{0} A(\sigma) x(t+\sigma) d\sigma$$

if $t > 0$,

$$\text{(4.1b)} \quad x(t) = \varphi(t), \quad t \in [-h, 0)$$

and

$$\text{(4.1c)} \quad x(0) = \varphi(0).$$

This type of system has its Laplace transform described by

$$\text{(4.2)} \ \bar{x}(s, h_1, \ldots, h_m, \hat{\varphi}) = \Big[sI - A_0 - \sum_{j=1}^{m} A_j e^{-sh_j} - \int_{-h}^{0} A(\sigma) e^{s\sigma} d\sigma\Big]^{-1} \Big[\varphi(0) + $$

$$+ \sum_{j=1}^{m} A_j \int_{-h_j}^{0} e^{-s(\sigma+h_j)} \varphi(\sigma) d\sigma + \int_{-h}^{0} \int_{\sigma}^{0} A(\sigma) e^{-s(\beta-\sigma)} \varphi(\beta) d\beta d\sigma\Big].$$

In this instance the structure of $\bar{S}(s)$ is apparent and $\Omega = \{(h_1, \ldots, h_m) \in R^m$: $0 \leqslant h_1 \leqslant h_2 \leqslant \ldots \leqslant h_m\}$.

A specific example of Example 4.1 whose stability properties were discussed in [3] is:

EXAMPLE 4.2. Let $q \geqslant 0$ and $h \geqslant 0$. Consider

$$\text{(4.3a)} \qquad \mu_t(x, t) = \mu_{xx}(x, t) + q \int_{-h}^{0} \mu(x, t+\sigma) d\sigma,$$

where $t \geqslant 0$, $x \in [0, 1]$. Let

$$\text{(4.3b)} \qquad \mu(x, t) = \varphi(x, t)$$

for $x \in [0, 1]$ and $t \in [-h, 0]$, and

$$\text{(4.3c)} \qquad \mu(0, t) = \mu(1, t) = 0$$

for $t \in [-h, \infty)$.

Let

$$\text{(4.4)} \qquad U(x, s) = \int_{0}^{\infty} e^{-st} \mu(x, t) dt.$$

The equivalent of $\bar{S}(s)$ in our theory satisfies the two point boundary value problem

$$\frac{d^2}{dx^2}\big(U(x,s)\big)+\left[q\,\frac{1-e^{-sh}}{s}-s\right]U(x,s)=0, \tag{4.5a}$$

$$U(0,s)=U(1,s)=0. \tag{4.5b}$$

(See e.g. [2], the sections on distributed systems.) For (4.3) the poles are obtained by solving the equations

$$s-\frac{q(1-e^{-sh})}{s}=-n^2\pi^2, \quad n=1,\ldots, \tag{4.6}$$

which are the points where (4.5) has only trivial solutions. Since $\Omega=\{(q,h)\in R^2\colon q\geqslant 0, h\geqslant 0\}$, we see that $(0,0)$ is a point of u.e.s. for the system.

The points in $\Omega$ where (4.6) has solutions of the form $s=i\omega$, $\omega$ real, are only those points which satisfy

$$qh=n^2\pi^2, \quad n=1,2,\ldots \tag{4.7}$$

(Use l'Hopital's rule on the right side of (4.6) for $s=0$, and observe that there can be no solutions of (4.6) of the form $i\omega$, $\omega>0$.) Thus the boundary of the region of stability which contains $(0,0)$ is the hyperbola

$$qh=\pi^2.$$

The techniques given in Sections 2 and 3 may also be applied to "neutral problems". That is problems where $\bar{S}(s)$ has the structure

$$\bar{S}(s)=\big[s\big(I-B(s)\big)-A_0-Q(s)\big]^{-1} \tag{4.8}$$

and $A_0$ and $Q(s)$ satisfy Assumptions 1.1 and 1.2 and $B(s)$ satisfies the following assumption.

ASSUMPTION 4.1. In $\operatorname{Re}s>-\beta_1$:

(i) $B(s)$ and $A_0B(s)$ are holomorphic and

(ii) $|B(s)|\leqslant\varrho<1$ and $|A_0B(s)|\leqslant M$, $M<\infty$.

*Remark* 4.1. The condition $|B(s)|\leqslant\varrho<1$ is indispensable. For even in finite dimensions it can be shown that linear autonomous neutral functional differential equations may be unstable if it is violated (see e.g. [5]). The holomorphicity and boundedness of $A_0B(s)$ in $\operatorname{Re}s>-\beta_1$ are technical conditions and it is not clear to what extent they are essential. In finite dimensions they follow from the conditions on $B(s)$. However, since in infinite dimensions the operator $A_0$ is unbounded, these assumptions are not automatically satisfied from the conditions placed on $B(s)$.

Observe that we need to prove u.e.s. of systems of the form (1.7) or (3.3) by showing that the inverse Laplace transform $S(t)$ of $\bar{S}(s)$ decays exponentially. To accomplish this for (4.8) we make the following transformation:

$$\bar{Y}(s) = [I-B(s)]\bar{S}(s). \tag{4.9}$$

Let $Y(t)$ denote the inverse Laplace transform of $\bar{Y}(s)$, i.e.,

$$Y(t) = \mathscr{L}^{-1}\big(\bar{Y}(s)\big)(t). \tag{4.10}$$

Because of Assumption 4.1 $Y(t)$ will decay exponentially if and only if $S(t)$ does also. However, $\bar{Y}(s)$ fits into the structure of the $\bar{S}(s)$ considered in Section 2. To see this observe that by (4.8) and (4.9)

$$\begin{aligned} \bar{Y}(s) &= [I-B(s)]\big[(sI-A_0)\big(I-B(s)\big)-\big(A_0B(s)+Q(s)\big)\big]^{-1} \\ &= \big[(sI-A_0)-\big(A_0B(s)+Q(s)\big)\big(I-B(s)\big)^{-1}\big]^{-1}_{,} \end{aligned} \tag{4.11}$$

if $\operatorname{Re} s > -\beta_1$. Moreover for $\operatorname{Re} s > -\beta_1$ Assumption 4.1 guarantees that

$$Q_1(s) = \big(A_0B(s)+Q(s)\big)\big(I-B(s)\big)^{-1} \tag{4.12}$$

satisfies Assumption 1.2.

EXAMPLE 4.3. A simple example of a so-called "neutral" problem is

$$\frac{d}{dt}\Big[x(t)-\int_0^t K(t-\sigma)x(\sigma)\,d\sigma\Big] = A_0x(t), \tag{4.13}$$

where $H = R^n$, $A_0$ is an $n\times n$ matrix and $K$: $[0, \infty)\to[R^n]$ decays exponentially and is such that

$$|\bar{K}(s)| = \Big|\int_0^\infty e^{-st}K(t)\,dt\Big| \leqslant \varrho < 1$$

if $\operatorname{Re} s > -\beta_1$, $\beta_1 > 0$. The main thing to notice in this example is that u.e.s. will hold if and only if the equation

$$\det\big(s\big(I-\bar{K}(s)\big)-A_0\big) = 0$$

has all solutions with $\operatorname{Re} s < 0$. This spectral condition holds even though (4.13) does not necessarily generate a semi-group.

EXAMPLE 4.4. In this example $A_0$ does not satisfy Assumption 1.1. Consider

$$\mu_{tt}(x,t) = \mu_{xx}(x,t)-\mu_t(x,t-h), \tag{4.14a}$$

where $t \geqslant 0$ and $x \in [0,1]$. Let

$$\mu(x,t) = \varphi(x,t) \quad \text{and} \quad \mu_t(x,t) = \psi(x,t) \tag{4.14b}$$

for $x \in [0,1]$ and $t \in [-h, 0]$ and

$$\mu(0,t) = \mu(1,t) = 0 \tag{4.14c}$$

for $t \in [-h, \infty)$. Let $U(x,s)$ be given by (4.4).

The equivalent of $\bar{S}(s)$ for (4.14) satisfies the two point boundary value problem (again see [2])

$$\frac{d^2 U}{dx^2}(x,s) = (s^2 + se^{-sh})\, U(x,t), \tag{4.15a}$$

$$U(0,s) = U(1,s) = 0. \tag{4.15b}$$

The poles of (4.14) are the points where (4.15) has only trivial solutions, i.e., where

$$s^2 + se^{-sh} = -n^2\pi^2, \quad n = 1, 2, \ldots \tag{4.16}$$

For $h = 0$ (4.16) has all solutions with $\operatorname{Re} s < 0$. However, when $h > 0$, if we seek solutions of the form $s = i\omega$, $\omega > 0$, we obtain the two equations

$$-\omega^2 - \omega \sin \omega h = -n^2\pi^2, \tag{4.17a}$$

$$\omega \cos \omega h = 0. \tag{4.17b}$$

Equations (4.17) have solutions in $\omega$ and $h$ of the form

$$\omega = \frac{-1 + \sqrt{1 + 4n^2\pi^2}}{2} \tag{4.18a}$$

and

$$h = \frac{\pi}{2\omega}. \tag{4.18b}$$

But for large $n$, $h \cong 1/2n$. Thus we can find a sequence $\{h_n\} \to 0$ such that the $\{\bar{S}(s, h_n)\}$ associated with (4.14) have poles on the imaginary axis.

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