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NOTE ON NON-LINEAR SEMIGROUPS AND FREE BOUNDARY PROBLEM FOR CONTROLLED MARKOV PROCESSES

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§ 1. Introduction

In this note we treat the optimal stopping problem for controlled Markov processes. Let a control region Γ be a compact subset of R^k. $\mathfrak{X}^u = \{W, X, F, F_t, P_x^u, x \in R^n\}$ denotes an n-dimensional Markov process, that is, W = path space = $\{w: [0, \infty) \to R^n$, right continuous with left limits$\}$, X = coordinate function on W, $X(t, w) = w(t)$, $F_t = \bigcap_{s>t} \sigma_s$ where σ_s is the σ-field generated by $w(\theta)$, $\theta \leqslant s$. $F = \sigma(F_s, s < \infty)$, P_x^u is a Markovian measure on F starting at x. Let $P^u(t)$, $t \geqslant 0$, be the transition semigroup of $\mathfrak{X}^u$, and $H^u(t)$, $t \geqslant 0$ the transition semigroup of $\mathfrak{X}^u$ with killing rate $c^u \geqslant 0$, namely

$$(1.1) \qquad H^u(t)\varphi(x) = \tilde{E}_x^u e^{-\int_0^t c^u(X(\theta))d\theta} \varphi(X(t))$$

where $\tilde{E}_x^u$ means the expectation w.r. to P_x^u. Sometimes we denote $c^u(x)$ by $c(x, u)$, and so on.

By $d: [0, \infty) \times W \to \Gamma$ we denote a σ_t-adapted function. Let $\mathfrak{A}_N = \{d: d(t) = d(k2^{-N})$ for $t \in [k2^{-N}, (k+1)2^{-N})$, $k = 0, 1, 2 \ldots\}$ and $\mathfrak{A} = \bigcup_{N=1}^{\infty} \mathfrak{A}_N$. We call $d \in \mathfrak{A}$ an *admissible control*. For $d \in \mathfrak{A}_N$ we can construct a unique probability measure Q_x^d on F such that

$$(1.2) \qquad Q_x^d(X(t) \in A/F_s) = H^{d(s)}(t-s)\chi_A(X(s)) \quad \text{for} \quad k2^{-N} \leqslant s < t \leqslant (k+1)2^{-N}$$

where χ_A is the indicator function of set A, namely, Q_x^d is a piecewise Markovian measure with killing. Define

$$m(t) = \{\tau \wedge t: \tau \text{ is an } F_t\text{-stopping time}\}.^{(1)}$$

(1) $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$.

Let f be a bounded continuous function on $R^n \times \Gamma$ such that

$$\sup_{u\in\Gamma}\sup_{x\neq y}\frac{|f(x,u)-f(y,u)|}{|x-y|} < \infty.$$

We consider the following optimization problems:

$$S(t,x,\varphi) = \sup_{d\in\mathfrak{A}} E_x^d \int_0^t f\big(X(s),d(s)\big)ds + \varphi\big(X(t)\big) \tag{1.3}$$

and

$$V(t,x,\varphi) = \sup_{d\in\mathfrak{A},\tau\in\mathfrak{M}(t)} E_x^d \int_0^\tau f\big(X(s),d(s)\big)ds + \varphi\big(X(\tau)\big) \tag{1.4}$$

where E_x^d means the expectation w.r. to Q_x^d.

Let C be a Banach lattice of all bounded and uniformly continuous functions on R^n, with the supremum norm and the usual order. We assume that $H^u(t)$ acts on C. Let A^u be the generator of $H^u(t)$. Hereafter we assume the following conditions (A1)–(A3):

(A1) $D = \bigcap_{u\in\Gamma} D(A^u)$ is dense in C and

$$\sup_{u\in\Gamma}\|A^u\varphi\| < \infty \quad \text{for} \quad \varphi\in D. \tag{1.5}$$

(A2) $H(t,u)\varphi(x)$ is continuous in $(t,x,u)\in[0,\infty)\times R^n\times\Gamma$ for $\varphi\in C$.

(A3) For any $T>0$ there exists a positive constant $\lambda=\lambda(T)$ such that, for $t<T$,

$$\sup_{u\in\Gamma}|H(t,u)\varphi(x)-H(t,u)\varphi(y)| \leqslant |x-y|e^{\lambda t} \tag{1.6}$$

whenever φ is Lipschitz continuous with Lipschitz constant 1 and $\|\varphi\|\leqslant 1$.

Now we recall the following propositions:

PROPOSITION 1 [8]. *$S(t,\cdot,\varphi)\in C$ whenever $\varphi\in C$. The operator $S(t)$ defined by*

$$S(t)\varphi(x) = S(t,x,\varphi)$$

has the following properties:

(i) *$S(t)$, $t\geqslant 0$, is a semigroup*(2) *on C.*

(ii) $\varphi\leqslant\psi \Rightarrow S(t)\varphi\leqslant S(t)\psi$.

(iii) *contraction*: $\|S(t)\varphi-S(t)\psi\|\leqslant\|\varphi-\psi\|$.

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(2) $S(t+\theta)\varphi = S(t)\,(S(\theta)\varphi)$, $S(0)\varphi=\varphi$ and $S(t)\varphi$ is strongly continuous in t.

(iv) *$S(t)\varphi \geqslant T(t,u)\varphi$ for any t, u, φ where*

$$T(t,u)\varphi = \int_0^t H(s,u)f(\cdot,u)\,ds + H(t,u)\varphi$$

$$= \tilde{E}^u \int_0^t e^{-\int_0^s c(X(\theta),u)\,d\theta} f\big(X(s),u\big)\,ds + e^{-\int_0^t c(X(\theta),u)\,d\theta}\varphi\big(X(t)\big).$$

(v) *If $\Lambda(t)$, $t \geqslant 0$ is a semigroup with* (iv), *then*

$$S(t)\varphi \leqslant \Lambda(t)\varphi.$$

(vi) *The generator G of $S(t)$ is expressed by*

$$G\varphi = \sup_{u\in\Gamma}(A^u\varphi + f^u) \quad \text{for} \quad \varphi \in D \cap D(G). \tag{1.7}$$

Moreover, assuming

(A4) $\sup_{u\in\Gamma} \|(1/t)\big(H^u(t)\varphi - \varphi\big) - A^u\varphi\| \to 0$ *as* $t\downarrow 0$, $\varphi \in D$,

(A5) $\sup_{u\in\Gamma} \|H^u(t)f^u - f^u\| \to 0$ *as* $t\downarrow 0$,

we have $D(G) \supset D$.

COROLLARY 1. *Suppose* (A1)–(A5). *Then the operator $\tilde{G}$ defined by*

$$\tilde{G}\varphi = \sup_{u\in\Gamma}(A^u\varphi + f^u)$$

is a dissipative([3]) *operator from D into C and $S(t)\varphi$ is an integral solution of the Cauchy problem* (1.8):

$$\frac{dW}{dt}(t) = \tilde{G}W(t), \qquad W(0) = \varphi \in C. \tag{1.8}$$

The mapping W: $[0,\infty) \to C$ is called an *integral solution* of (1.8) if W is continuous, $W(0) = \varphi$ and

$$\|W(t)-\psi\|^2 - \|W(s)-\psi\|^2 \leqslant 2\int_s^t \|W(\theta)-\psi\|\,\tau\big(\tilde{G}\psi,\, W(\theta)-\psi\big)\,d\theta \tag{1.9}$$

for any $\psi \in D$

where τ is the tangent functional, that is,

$$\tau(g,h) = \lim_{\lambda\downarrow 0}\frac{1}{\lambda}(\|h+\lambda g\| - \|h\|) = \inf_{\lambda>0}\frac{1}{\lambda}(\|h+\lambda g\| - \|h\|). \tag{1.10}$$

([3]) *Dissipative* means strictly dissipative in this note.

Remark. (1.9) is equivalent to (1.11):

$$\|W(t)-\psi\|-\|W(s)-\psi\| \leqslant \int_s^t \tau(\tilde{G}\psi, W(\theta)-\psi)d\theta. \tag{1.11}$$

We have similar results for $V(t, x, \varphi)$.

PROPOSITION 2 [9]. *$V(t, \cdot, \varphi) \in C$ whenever $\varphi \in C$. The operator $V(t)$ defined by*

$$V(t)\varphi(x) = V(t, x, \varphi)$$

has the following properties:

(i) *$V(t)$, $t \geqslant 0$ is a semigroup on C.*

(ii) *$\varphi \leqslant \psi \Rightarrow V(t)\varphi \leqslant V(t)\psi$.*

(iii) *contraction: $\|V(t)\varphi - V(t)\psi\| \leqslant \|\varphi-\psi\|$.*

(iv) *$V(t)\varphi \geqslant \varphi$ and $V(t)\varphi \geqslant T(t, u)\varphi$ for any t, u, φ.*

(v) *If $\Lambda(t)$, $t \geqslant 0$ is a semigroup on C with* (iv), *then*

$$V(t)\varphi \leqslant \Lambda(t)\varphi.$$

(vi) *The generator $\mathfrak{G}$ of $V(t)$ is expressed by*

$$\mathfrak{G}\varphi = 0 \vee \sup_{u\in\Gamma}(A^u\varphi + f^u) \quad for \quad \varphi \in D \cap D(\mathfrak{G}). \tag{1.12}$$

Moreover, if (A4) *and* (A5) *hold, then $D(\mathfrak{G}) \supset D$.*

COROLLARY 2. *Suppose* (A1)–(A5). *Then the operator $\tilde{\mathfrak{G}}$ defined by*

$$\tilde{\mathfrak{G}}\varphi = 0 \vee \sup_{u\in\Gamma}(A^u\varphi + f^u) = 0 \vee \tilde{G}\varphi \tag{1.13}$$

is a dissipative operator from D into C and $V(t)\varphi$ is an integral solution of the Cauchy problem (1.14):

$$\frac{dW}{dt}(t) = \tilde{\mathfrak{G}}W(t), \quad W(0) = \varphi \in C. \tag{1.14}$$

The optimal stopping is related to the free boundary problem. When the value function $V(t)\varphi$ is smooth, we have Theorem 1.

THEOREM 1. *Suppose* (A1)–(A5) *and, for any $\varepsilon > 0$,*

$$\sup_{d\in\mathfrak{A}} Q_x^d\left(\sup_{s<t}|X(s)-x| > \varepsilon\right) \to 0 \quad as \quad t\downarrow 0. \tag{1.15}$$

If $V(t)\varphi \in D$ and the right derivative $d^+V(t)\varphi/dt$ and $\tilde{G}V(t)\varphi$ are continuous *in t, then $V(t)\varphi$ is a solution of* (1.16):

$$
(1.16)\qquad
\begin{aligned}
&\text{(i)} && W(t) \geqslant \varphi,\\
&\text{(ii)} && \frac{dW}{dt}(t) \geqslant \tilde{G}W(t),\\
&\text{(iii)} && \left(\frac{dW}{dt}(t) - \tilde{G}W(t)\right)\left(W(t)-\varphi\right) = 0,\\
&\text{(iv)} && W(0) = \varphi.
\end{aligned}
$$

THEOREM 2. *Let W: $[0, \infty)\to D$ be increasing and continuously differentiable. If W is a solution of* (1.16), *then $W(t) = V(t)\varphi$.*

We prove theorems in § 2, and in § 3 a simple example will be treated.

§ 2. Proof of corollaries and theorems

Since we can apply the same method to the proof of corollaries, we prove only Corollary 1. Put

$$
(2.1)\qquad G_h\varphi = \frac{1}{h}\left(S(h)\varphi - \varphi\right) \quad \text{for} \quad \varphi \in C.
$$

Then G_h is a dissipative operator on C and

$$
(2.2)\qquad G_h\varphi \xrightarrow[h\downarrow 0]{} G\varphi \quad \text{for} \quad \varphi \in D(G).
$$

Moreover, there exists a unique (strong) solution W: $[0, \infty)\to C$ of Cauchy problem (2.3)

$$
(2.3)\qquad \frac{dW}{dt}(t) = G_hW(t), \qquad W(0) = \varphi \in C,
$$

and $S_h(t)\varphi$, defined by $S_h(t)\varphi = W(t)$, provides a unique semigroup $S_h(t)$, $t \geqslant 0$, on C, whose generator is G_h. According to [4], we see that

$$
(2.4)\qquad S_h(t)\varphi \xrightarrow[h\to 0]{} S(t)\varphi \text{ uniformly on any finite interval.}
$$

Since a solution is an integral solution, we have, by virtue of (1.10) and (1.11),

$$
(2.5)\qquad \|S_h(t)\varphi - \psi\| - \|S_h(s)\varphi - \psi\|
$$

$$
\leqslant \int_s^t \frac{1}{\lambda}\left(\|S_h(\theta)\varphi - \psi + \lambda G_h\psi\| - \|S_h(\theta)\varphi - \psi\|\right)d\theta \qquad \text{for } \psi \in C \text{ and } \lambda > 0.
$$

For $\psi \in D$ $(\subset D(G))$, we get from (2.2) and (2.4):

$$\|S(t)\varphi-\psi\|-\|S(s)\varphi-\psi\| \leqslant \int_s^t \frac{1}{\lambda}\left(\|S(\theta)\varphi-\psi+\lambda\tilde{G}\psi\|-\|S(\theta)\varphi-\psi\|\right)d\theta.$$

Since the integrand is not greater than $\|\tilde{G}\psi\|$, letting $\lambda \to 0$ we get Corollary 1 by the convergence theorem.

Proof of Theorem 1. From the definition of $V(t)\varphi$, (i) and (iv) are clear. Since $\dfrac{d^+V(t)\varphi}{dt}$ is continuous in t, $V(t)\varphi$ is differentiable and $\dfrac{dV(t)\varphi}{dt} = \mathfrak{G}V(t)\varphi$. From the assumption that $V(t)\varphi \in D$ we have

$$\frac{dV(t)\varphi}{dt} = \tilde{\mathfrak{G}}V(t)\varphi = 0 \vee \tilde{G}V(t)\varphi. \tag{2.6}$$

This implies (ii).

For the proof of (iii), we apply the random stopping method due to Krylov [5]. Let $r(t)$, $t \geqslant 0$, be a bounded non-negative valued F_t-progressible measurable and right continuous path. Roughly speaking, $r(\cdot)$ gives the following random stopping:

$$P(\text{stop in } (t, t+dt)/\text{non-stop until } t) = r(t)e^{-\int_0^t r(s)ds}\,dt.$$

So we have the following gain:

$$\begin{aligned} (2.7)\quad & I(t, \varphi, d, r) \\ &= \int_0^t r(s)e^{-\int_0^s r(\theta)\,d\theta}\left(\int_0^s f\big(X(\theta), d(\theta)\big)d\theta+\varphi\big(X(s)\big)\right)ds + \\ &\qquad + e^{-\int_0^t r(\theta)\,d\theta}\left(\int_0^t f\big(X(\theta), d(\theta)\big)d\theta+\varphi\big(X(t)\big)\right) \\ &= \int_0^t e^{-\int_0^s r\,d\theta}\left(f\big(X(s), d(s)\big)+r(s)\varphi\big(X(s)\big)\right)ds + e^{-\int_0^t r\,d\theta}\varphi\big(X(t)\big). \end{aligned}$$

Let $\mathfrak{R}$ be the totality of $r(\cdot)$.

LEMMA. $V(t)\varphi(x) = \sup\limits_{r\in\mathfrak{R},\, d\in\mathfrak{A}} E_x^d I(t, \varphi, d, r)$.

Proof. For $\tau \in m(t)$ we put

$$\gamma_{N,t}(t) = 2^N \chi_{[\tau, \tau+t2^{-N})}(t).$$

Then we have

$$\lim_{t\to\infty}\lim_{N\to\infty} I(t, \varphi, d, r_{N,t}) = \int_0^\tau f\big(X(s), d(s)\big)ds + \varphi\big(X(\tau)\big).$$

Therefore,

$$V(t)\varphi(x) \leqslant \sup_{\mathfrak{R},\mathfrak{A}} E_x^d I(t, \varphi, d, r). \tag{2.8}$$

For the converse, putting

$$\mathfrak{R}_N = \{r \in \mathfrak{R}:\ r(t) = r(k2^{-N}),\ k2^{-N} \leqslant t < (k+1)2^{-N},\ k = 0, 1, 2, \ldots\}$$

and

$$\tilde{\mathfrak{R}} = \bigcup_{N=1}^{\infty} \mathfrak{R}_N$$

we remark that $\tilde{\mathfrak{R}}$ is dense, that is

$$\sup_{\mathfrak{R},\mathfrak{A}} E_x^d I(t, \varphi, d, r) = \sup_{\tilde{\mathfrak{R}},\mathfrak{A}} E_x^d I(t, \varphi, d, r). \tag{2.9}$$

Fix $d = \mathfrak{A}_N$, $r \in \mathfrak{R}_N$ and $T > 0$ arbitrarily and put

$$Z(t) = \int_0^t \Big(f\big(X(s), d(s)\big) + r(s)\,V(T-s)\varphi\big(X(s)\big)e^{-\int_0^s r\,d\theta}\,ds + \\ + e^{-\int_0^t r\,ds}\,V(T-t)\varphi\big(X(t)\big).$$

For simplicity, we denote the integrand of right side by $J(s)$. We will show that

$$E_x^d\big(Z(t)/F_s\big) \leqslant Z(s) \quad \text{for} \quad k2^{-N} \leqslant s < t \leqslant (k+1)2^{-N}, \tag{2.10}$$

$$E_x^d\big(Z(t)/F_s\big) = \int_0^s J(\theta)d\theta + E_x^d\Big(\int_s^t J(\theta)d\theta + \\ + e^{-\int_0^t r\,d\theta}\,V(T-t)\varphi\big(X(t)\big)/F_s\Big). \tag{2.11}$$

For the computation of the conditional expectation of right side of (2.11), Q_x^d can be regarded as the Markovian measure $Q_{X(s)}^{d(s)}$ and $r(\theta) = r(s)$ and $d(\theta) = d(s)$. Therefore

$$I_1 = E_x^d\Big(\int_s^t e^{-r(s)(\theta-s)}\big(f\big(X(\theta), d(s)\big) - r(s)\int_s^\theta f\big(X(h), d(s)\big)dh\big)d\theta/F_s\Big)$$

$$= e^{-r(s)(t-s)}E_x^d\Big(\int_s^t f\big(X(\theta), d(s)\big)d\theta/F_s\Big),$$

$$
\begin{aligned}
I_2 &= E_x^d\Big(\int_s^t e^{-r(s)(\theta-s)}r(s)\Big(\int_s^\theta f\big(X(h),\,d(s)\big)dh+V(T-\theta)\varphi\big(X(\theta)\big)d\theta/F_s\Big)\\
&= \int_s^t e^{-r(s)(\theta-s)}r(s)\,T^{d(s)}(\theta-s)\,V(T-\theta)\varphi\big(X(s)\big)d\theta\\
&\leqslant \int_s^t e^{-r(s)(\theta-s)}r(s)\,V(T-s)\varphi\big(X(s)\big)d\theta\\
&= V(T-s)\varphi\big(X(s)\big)\big(1-e^{-r(s)(t-s)}\big).
\end{aligned}
$$

Combining I_1 and I_2 with (2.11), we see that

$$
\begin{aligned}
E_x^d\big(Z(t)/F_s\big) &\leqslant \int_0^s J(\theta)d\theta+\\
&\quad + e^{-\int_0^s r\,d\theta}\Big[e^{-r(s)(t-s)}E_x^d\Big(\int_s^t f\big(X(\theta),\,d(s)\big)d\theta+\\
&\qquad\qquad + V(T-t)\varphi\big(X(t)\big)/F_s\Big)-\\
&\quad - e^{-r(t-s)}V(T-s)\varphi\big(X(s)\big)+V(T-s)\varphi\big(X(s)\big)\Big]\\
&= \int_0^s J(\theta)d\theta+e^{-\int_0^s r\,d\theta}\Big[e^{-r(s)(t-s)}T^{d(s)}(t-s)\,V(T-t)\varphi\big(X(s)\big)-\\
&\quad - e^{-r(s)(t-s)}V(T-s)\varphi\big(X(s)\big)+V(T-s)\varphi\big(X(s)\big)\Big]\\
&\leqslant \int_0^s J(\theta)d\theta+e^{-\int_0^s r\,d\theta}\,V(T-s)\varphi\big(X(s)\big) = Z(s).
\end{aligned}
$$

This implies

$$
E_x^d\big(Z(t)/F_s\big)\leqslant Z(s)\qquad \text{for any } s<t\leqslant T, \tag{2.12}
$$

that is, $Z(t)$ is F_t-supermartingale w.r. to Q_x^d. So, using $V(t)\varphi\geqslant\varphi$, (2.12) turns out

$$
V(T)\varphi(x) = Z(0)\geqslant E_x^d Z(T)\geqslant E_x^d I(T,\varphi,d,r).
$$

Hence we obtain the converse of (2.8), by virtue of (2.9). This completes the proof of the Lemma.

Setting $M = \{(t,x)\colon\ V(t)\varphi(x)>\varphi(x)\}$, we show that

$$
\frac{dV(t)\varphi}{dt}(x) = \tilde{G}V(t)\varphi(x)\qquad \text{on}\qquad M. \tag{2.13}
$$

By the Lemma we see that

$$(2.14)\quad 0=\sup_{\mathfrak{R},\mathfrak{A}} E_x^d\Big[\int_0^t e^{-\int_0^\theta r\,dh}\big(f(X(\theta),d(\theta))+r(\theta)\varphi(X(\theta))\big)d\theta+ +e^{-\int_0^t r\,d\theta}\varphi(X(t))-V(t)\varphi(x)\Big].$$

Fix $(t, x)\in M$ arbitrarily and put $W(\theta, x)=V(t-\theta)\varphi(x)$. So $W(0, x)=V(t)\varphi(x)$ and $W(t, x)=\varphi(x)$. Let $r\in\mathfrak{R}_N$ and $d\in\mathfrak{A}_N$. For simplicity we put

$$(2.15)\quad I(s, t, \Phi, \Psi)=\int_s^t e^{-\int_s^\theta r\,dh}\Phi(\theta, X(\theta), d(\theta))+r(\theta)\Psi(\theta, X(\theta))d\theta$$

and

$$(2.16)\qquad J(t)=E_x^d I(0, t, f, \varphi)+e^{-\int_0^t r\,d\theta}W(t, X(t))-W(0, x).$$

Then, putting $\Delta=2^{-N}$, we see that

$$(2.17)\quad J(t)=J(k\Delta)+E_x^d e^{-\int_0^{k\Delta} r\,d\theta}\big[I(k\Delta, t, f, W(t,\cdot))+ +e^{-\int_{k\Delta}^t r\,d\theta}W(t, X(t))-W(k\Delta, X(k\Delta))\big]$$

for k such that $k\Delta<t\leqslant(k+1)\Delta$. Since $W(\theta,\cdot)$ belongs to D and is differentiable, we have

$$(2.18)\quad E_x^d(\text{inside of [ ] of (2.17)}/F_{k\Delta})=I\Big(k\Delta, t, f+\frac{\partial W}{\partial\theta}+ +A^{d(k\Delta)}W, \varphi-W\Big).$$

Therefore the second term on the right side of (2.17) becomes

$$E_x^d e^{-\int_0^{k\Delta} r\,d\theta} I\Big(k\Delta, t, f+\frac{\partial W}{\partial\theta}+A^{d(k\Delta)}W, \varphi-W\Big).$$

Repeating the same calculation, we get

$$J(k\Delta)=J((k-1)\Delta)+E_x^d e^{-\int_0^{(k-1)\Delta} r\,d\theta} I\Big((k-1)\Delta, k\Delta, f+\frac{\partial W}{\partial\theta}+ +A^{d((k-1)\Delta)}W, \varphi-W\Big)$$

and so on. Hence (2.17) turns out

$$(2.19)\qquad \begin{aligned} J(t) &= E_x^d I\left(0, t, f + \frac{\partial W}{\partial \theta} + A^{d(\theta)} W, \varphi - W\right) \\ &\leqslant E_x^d I\left(0, t, \frac{\partial W}{\partial \theta} + \tilde{G} W, \varphi - W\right). \end{aligned}$$

Since $\dfrac{\partial W}{\partial \theta} = -\dfrac{\partial V(T-\theta)\varphi}{\partial t}$, properties (i) and (iii) imply that the right side of (2.19) is not greater than 0. Hence

$$(2.20)\qquad \begin{aligned} 0 = \sup_{\mathfrak{R},\mathfrak{A}} E_x^d \int_0^t e^{-\int_0^\theta r\,dh} \Big[&\frac{\partial W}{\partial \theta}\big(\theta, X(\theta)\big) + \tilde{G} W\big(\theta, X(\theta)\big) + \\ &+ r(\theta)\Big(\varphi\big(X(\theta)\big) - W\big(0, X(\theta)\big)\Big)\Big] d\theta. \end{aligned}$$

Put $\delta = \frac{1}{2}\big(V(t)\varphi(x) - \varphi(x)\big)$. Then $\delta > 0$ and there exists $\Delta > 0$ such that $V(s)\varphi(y) > \varphi(y) + \delta$ whenever $|s-t| < \Delta$ and $|x-y| < \Delta$. We denote by σ $\Delta \wedge$(the hitting time to the set $\{y\colon |x-y| < \Delta\}$). Let (d_k, r_k) be an approximate optimal for (2.20). Then we have, by virtue of (i) and (ii),

$$(2.21)\qquad 0 = \lim_{k\to\infty} E_x^{d_k} \int_0^{t\wedge\sigma} e^{-\int_0^\theta r_k dh} \left(\frac{\partial W}{\partial t}\big(\theta, X(\theta)\big) + \tilde{G} W\big(\theta, X(\theta)\big)\right) d\theta$$

and

$$(2.22)\qquad 0 = \lim_{k\to\infty} E_x^{d_k} \int_0^{t\wedge\sigma} e^{-\int_0^\theta r_k\,dh} r_k(\theta)\Big(\varphi\big(X(\theta)\big) - W\big(\theta, X(\theta)\big)\Big) d\theta.$$

Since $\varphi\big(X(\theta)\big) - W\big(\theta, X(\theta)\big) < -\delta$ in the integrand of (2.22), we get

$$(2.23)\qquad 0 = \lim_{k\to\infty} E_x^{d_k} \int_0^{t\wedge\sigma} e^{-\int_0^\theta r_k\,dh} r_k(\theta)\,d\theta = \lim_{k\to\infty} E_x^{d_k}\Big(1 - e^{-\int_0^{t\wedge\sigma} r_k\,dh}\Big).$$

For simplicity we put $U = \dfrac{\partial W}{\partial t} + \tilde{G} W$ and we have

$$\begin{aligned} 0 &\leqslant E_x^{d_k} \int_0^{t\wedge\sigma} \Big(e^{-\int_0^\theta r_k\,dh} - 1\Big)\, U\big(\theta, X(\theta)\big)\,d\theta \\ &\leqslant \sup_{\theta<t, |x-y|<\Delta} |U(\theta, y)|\, E_x^{d_k}\Big(1 - e^{-\int_0^{t\wedge\sigma} r_k\,dh}\Big) t. \end{aligned}$$

Since U is bounded on $[0, t] \times R^n$, the right side converges to 0 as $k \to \infty$. Hence, (2.21) turns out

$$0 = \lim_{k\to\infty} E_x^{d_k} \int_0^{t\wedge\sigma} U(\theta, X(\theta))\,d\theta. \tag{2.24}$$

Recalling $U \leqslant 0$, we see

$$0 = \lim_{k\to\infty} E_x^{d_k} \int_0^{s\wedge\sigma} U(\theta, X(\theta))\,d\theta \quad \text{for any } s \leqslant t.$$

That is,

$$0 = \lim_{s\downarrow 0} \lim_{h\to\infty} \frac{1}{s} E_x^{d_k} \int_0^{s\wedge\lambda} U(\theta, X(\theta))\,ds. \tag{2.25}$$

Since $U(\theta, \cdot)$ belongs to C and is continuous in θ, for any $\varepsilon > 0$ there exists a positive $\mu = \mu(\varepsilon) < \Delta$ such that

$$|U(\theta, y) - U(0, x)| < \varepsilon \quad \text{whenever} \quad \theta < \mu \text{ and } |x-y| < \mu. \tag{2.26}$$

Hence, for $s < \mu$,

$$\frac{1}{s}\left|E_x^{d_k} \int_0^{s\wedge\sigma} U(\theta, X(\theta))\,d\theta - U(0, x) E_x^{d_k}(s\wedge\sigma)\right| < \varepsilon. \tag{2.27}$$

On the other hand,

$$1 - \frac{1}{s} E_x^d(\sigma\wedge s) \leqslant Q_x^d(\sigma < s) = Q_x^d\big(\sup_{\theta\leqslant s}|X(\theta) - x| < \Delta\big). \tag{2.28}$$

Therefore, by virtue of (1.15), $\frac{1}{s} E_x^d(\sigma\wedge s)$ tends to 1 as $s\downarrow 0$, uniformly in $d \in \mathfrak{A}$. So, from (2.25) and (2.27), we have

$$0 = U(0, x) = \frac{\partial W}{\partial t}(0, x) + \tilde{G}W(0, x). \tag{2.29}$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Let W be a solution of (1.16) of Theorem 2. Putting $M = \{(t, x)\colon W(t, x) > \varphi(x)\}$, we have

$$\frac{\partial W}{\partial t}(t, x) = \begin{cases} \tilde{G}W(t, x) & \text{on } M, \\ 0 & \text{on } \bar{M}^c. \end{cases}$$

Therefore by the smoothness of W, we have

$$\frac{\partial W}{\partial t}(t, x)\cdot\left(\frac{\partial W}{\partial t}(t, x) - \tilde{G}W(t, x)\right) = 0. \tag{2.30}$$

Since both terms are non-negative, (2.30) turns out

$$\frac{\partial W}{\partial t}(t, x) = 0 \vee \tilde{G}W(t, x) = \tilde{\mathfrak{G}}W(t, x). \tag{2.31}$$

So, W is a solution of (1.14).

Let $\mathfrak{G}_h$ be an approximate for $\mathfrak{G}$, i.e.,

$$\mathfrak{G}_h = \frac{1}{h}\big(V(h) - I\big)$$

and consider the Cauchy problem

$$\frac{dW_h(t)}{dt} = \mathfrak{G}_h W_h(t), \quad W_h(0) = \varphi. \tag{2.32}$$

Then there exists a unique solution $W_h(t)$ and, according to the method of proof of Corollary 1, $W_h(t)$ converges to $V(t)\varphi$. Put

$$f_h(t) = \mathfrak{G}W(t) - \mathfrak{G}_h W(t).$$

Then we have

$$\frac{dW(t)}{dt} = \mathfrak{G}_h W(t) + f_h(t).$$

Therefore, setting $\langle g, h\rangle = \|h\|\tau(g, h)$, we get $\frac{d}{dt}\|W(t) - W_h(t)\|^2 = 2\langle \mathfrak{G}_h W(t) - \mathfrak{G}_h W_h(t), W(t) - W_h(t)\rangle + 2\langle f_h(t), W(t) - W_h(t)\rangle$. Since $\mathfrak{G}_h$ is dissipative, the first term of the right side is non-positive. So

$$\begin{aligned} \|W(t) - W_h(t)\|^2 &\leqslant 2\int_0^t \langle f_h(s), W(s) - W_h(s)\rangle ds \\ &\leqslant 2\int_0^t \|f_h(s)\|\, \|W(s) - W_h(s)\| ds. \end{aligned} \tag{2.33}$$

Recalling that $W(t) \in D$, we have $\mathfrak{G}_h W(t) \to W(t)$, namely $\|f_h(t)\| \to 0$ as $h \downarrow 0$. Moreover, $\|f_h(s)\|$ and $\|W(s) - W_h(s)\|$ are bounded on any finite interval of (h, s). Therefore, from (2.33) we obtain

$$\|W(t) - W_h(t)\| \to 0 \quad \text{as} \quad h \downarrow 0.$$

Hence $W(t) = V(t)\varphi$.

Remark 1. Under the conditions of Theorem 1, $V(t)\varphi$ is a unique solution of the free boundary problem (1.16) and, at the same time, a unique solution of the Bellman equation of (1.14).

Remark 2. Using a solution W of Theorem 2, we can find an approximate optimal policy. Besides the assumptions of Theorem 1, we assume:

(a_1) The convergence of (1.15) is uniform in x.

(a_2) $A^u W(t, x)$ is uniformly continuous on (any finite time interval) $\times$ $\times R^n \times \Gamma$.

For $T > 0$ we put $V(t, x) = W(T-t, x)$. Let $d \in \mathfrak{A}$ and $\tau \in m(T)$. The process $Z(t)$, $0 \leqslant t \leqslant T$, defined by

$$Z(t) = V(t, X(t)) - \int_0^t \frac{\partial V}{\partial s}(s, X(s)) + A(d(s)) V(s, X(s)) ds$$

is F_s-martingale w.r. to Q_x^d-measure. Hence

$$E_x^d V(\tau, X(\tau)) - V(0, x) = E_x^d \int_0^\tau \frac{\partial V}{\partial s}(s, X(s)) + A(d(s)) V(s, X(s)) ds. \tag{2.34}$$

Therefore we have

$$E_x^d \varphi(X(\tau)) - W(T, x) \leqslant -E_x^d \int_0^\tau f(X(s), d(s)) ds.$$

This implies

$$W(T, x) \geqslant V(T)\varphi(x). \tag{2.35}$$

We choose an approximate optimal $\tilde{d}$ and $\tilde{\tau}$ in the following way. Put

$$M(T) = \{(t, x)\colon\ V(t, x) = \varphi(x)\} \subset [0, T] \times R^n$$

and $\tilde{\tau}$ = the hitting time to $M(T)$, that is,

$$\tilde{\tau} = \inf\{t\colon\ (t, X(t)) \in M(T)\}.$$

Since $\{T\} \times R^n \subset M(T)$, we have $\tilde{\tau} \leqslant T$. From the assumptions, for $\varepsilon > 0$ there exist δ and $\tilde{\delta}$ such that

$$\begin{aligned} &|A(u)V(t, x) - A(u)V(s, y)| < \varepsilon \quad &&\text{for} \quad u \in \Gamma, \\ &|f(x, u) - f(y, u)| < \varepsilon &&\text{for} \quad u \in \Gamma \end{aligned} \tag{2.36}$$

whenever $|x-y| < \tilde{\delta}$ and $|t-s| < \delta$, and

$$Q_x^d\big(\sup_{\theta < \delta} |X(\theta) - x| > \tilde{\delta}\big) < \varepsilon \quad \text{for} \quad d \in \mathfrak{A},\ x \in R^n. \tag{2.37}$$

Therefore we have

$$|\tilde{G}V(t, x) - \tilde{G}V(s, y)| \leqslant \sup_u |A(u)V(t, x) - A(u)V(s, y)| < \varepsilon \tag{2.38}$$

and

$$\left|\frac{\partial V}{\partial t}(t, x) - \frac{\partial V}{\partial t}(s, y)\right| \leqslant |\tilde{G}V(t, x) - \tilde{G}V(s, y)| < \varepsilon. \tag{2.39}$$

Fix N so that $\Delta = 2^{-N} < \delta$. Take a Borel function d_k: $R^n \to \Gamma$,

$$\tilde{G}V(k\Delta, x) = A\big(d_k(x)\big)V(k\Delta, x) + f\big(x, d_k(x)\big) \tag{2.40}$$

and define $\tilde{d} \in \mathfrak{A}_N$ by

$$\tilde{d}(t, w) = \begin{cases} d_k\big(w(k\Delta)\big) & \text{for} \quad t \in [k\Delta, (k+1)\Delta), \\ d_j\big(w(j\Delta)\big) & \text{for} \quad t \geqslant j\Delta \end{cases} \tag{2.41}$$

where $j\Delta \leqslant T < (j+1)\Delta$. Then we have, in the same way as (2.34),

$$E_x^{\tilde{d}} V\big(\tilde{\tau}, X(\tilde{\tau})\big) - V(0, x) = E_x^{\tilde{d}} \int_0^{\tilde{\tau}} \frac{\partial V}{\partial t}\big(s, X(s)\big) + A\big(\tilde{d}(s)\big) V\big(s, X(s)\big) ds. \tag{2.42}$$

From (2.36)–(2.39) we have, putting $\tilde{a} = \Delta[2^N a]$,

$$\left| \frac{\partial V}{\partial t}(s, y) - \frac{\partial V}{\partial t}(\tilde{s}, y) \right| < \varepsilon \tag{2.43}$$

and

$$\left| A\big(\tilde{d}(s)\big) V(s, y) - A\big(\tilde{d}(\tilde{s})\big) V(\tilde{s}, y) \right| < \varepsilon. \tag{2.44}$$

Since $|f(x, u)|$, $\left|\frac{\partial V}{\partial t}(s, x)\right|$ and $|A(u)V(s, x)|$ are bounded (say, by k) on $[0, T] \times R^n \times \Gamma$, we have by (2.37)

$$E_{X(\tilde{s})}^{\tilde{d}(\tilde{s})} \left| \frac{\partial V}{\partial t}\big(\tilde{s}, X(s)\big) - \frac{\partial V}{\partial t}\big(\tilde{s}, X(\tilde{s})\big) \right| \leqslant \varepsilon + k\varepsilon \tag{2.45}$$

and similarly for $A\big(\tilde{d}(\tilde{s})\big)V(\tilde{s}, \cdot)$ and $f\big(\cdot, \tilde{d}(\tilde{s})\big)$. Therefore, recalling the definition of $\tilde{d}$, we have

$$\text{(the right side of (2.42))} \geqslant -3(k+1)\varepsilon T - E_x^{\tilde{d}} \int_0^{\tilde{\tau}} f\big(X(s), \tilde{d}(s)\big) ds. \tag{2.46}$$

Hence,

$$E_x^{\tilde{d}} \varphi\big(X(\tau)\big) + \int_0^{\tilde{\tau}} f\big(X(s), \tilde{d}(s)\big) ds \geqslant W(T, x) - 3(k+1)\varepsilon T.$$

Appealing to (2.35), we can see that $(\tilde{d}, \tilde{\tau})$ is an approximate optimal.

Remark 2 gives a probabilistic proof of Theorem 2.

§ 3. Example

Let $P^u(t)$ be a transition semigroup of a 1-dimensional Lévy process of pure jumping type with finite Lévy measure $n^u(ds, dz) = ds\, n^u(dz)$, that is, the process $X^u(t)$ is expressed by

$$X^u(t) = x + \int_{-\infty}^{\infty}\int_0^t zN^u(ds\,dz)$$

with a Poisson random measure N^u of $EN^u(ds\,dz) = n^u(ds\,dz)$. Let $c(x, u) = 0$ and let $f(x, u)$ be bounded and smooth on $R^1 \times \Gamma$. Assume that

$$K \equiv \sup_{u\in\Gamma} n^u(R^1) < \infty \quad \text{and} \quad K' \equiv \sup_{u\in\Gamma} \int_{-\infty}^{\infty} |z| n^u(dz) < \infty, \tag{3.1}$$

and $P^u(t)\varphi(x)$ is continuous in (t, x, u). Then

$$A^u\varphi(x) = \int_{-\infty}^{\infty} \big(\varphi(x+y) - \varphi(x)\big) n^u(dy) \tag{3.2}$$

and $D(A^u) = C$. Moreover, from (3.1) conditions (A1)–(A5) are satisfied. The operator $\tilde{G}$: $\tilde{G}\varphi = \sup_{u\in\Gamma}(A^u\varphi + f^u)$ and $\tilde{\mathfrak{G}}$: $\tilde{\mathfrak{G}}\varphi = 0 \vee \sup_{u\in\Gamma}(A^u\varphi + f^u)$ satisfy the Lipschitz conditions, that is, $\|\tilde{G}\varphi - \tilde{G}\psi\| \leqslant 2k\|\varphi - \psi\|$ and similarly for $\tilde{\mathfrak{G}}$. According to Propositions 1 and 2, the optimization problems (1.3) and (1.4) provide semigroups $S(t)$ or $V(t)$, respectively. Moreover, $D(G)$ and $D(\mathfrak{G})$ are equal to C. Therefore we have

$$\frac{d^+ S(t)\varphi}{dt} = \tilde{G}S(t)\varphi \quad \text{and} \quad \frac{d^+ V(t)\varphi}{dt} = \tilde{\mathfrak{G}}V(t)\varphi.$$

Since $\tilde{G}$ and $\tilde{\mathfrak{G}}$ are Lipschitz continuous, we can derive that $S(t)\varphi$ and $V(t)\varphi$ are continuously differentiable.

We show (1.15). Let $d \in \mathfrak{A}_N$ and $0 = s_1^{(k)} < s_2^{(k)} < \ldots < s_k^{(k)} = t$, such that $\max_i |s_i^{(k)} - s_{i-1}^{(k)}| \to 0$ as $k \uparrow \infty$ and for large k $\{s_1^{(k)}, \ldots, s_k^{(k)}\} \supset \{2^{-N}j,\ j = 0, 1, \ldots, [t2^N]\} \cup \{s_1^{(k-1)}, \ldots, s_k^{(k-1)}\}$ where $[a]$ is the largest integer not greater than a. Then

$$E_x^d\big(\sup_{s\leqslant t}|X(s) - x|\big) \leqslant \lim_{k\uparrow\infty} E_x^d\Big(\sum_i |X(s_i^k) - X(s_{i-1}^k)|\Big). \tag{3.2}$$

On the other hand, if $2^{-N}j \leqslant s < \theta \leqslant 2^{-N}(j+1)$, then

$$\begin{aligned} E_x^d(|X(\theta) - X(s)|/F_s) &= E_{X(s)}^{d(s)}|X(\theta) - X(s)| \\ &\leqslant (t-s)\int_{-\infty}^{\infty} |z| n^{d(s)}(dz) \leqslant (t-s)k'. \end{aligned}$$

Therefore

$$E_x^d\big(\sup_{s\leqslant t}|X(s) - x|\big) \leqslant tk'. \tag{3.3}$$

This implies (1.15). Hence $V(t)\varphi$ is a unique solution of (1.16).

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