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FLOW PROBLEMS AND DUALITY

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1. Introduction

In some recent papers [4]–[8] the author stated a new duality principle for the usual problems of optimal control. In their standard form as parametric variational problems the said duality has the following condition.

Let the *primary problem* be of the type

$$J_0(x) = \int_0^T r(x, \dot{x})dt \to \inf \quad \text{subject to } x \in W_1^{1,n}(0, T) \tag{1}$$

with state constraints $x(t) \in G$ $\forall t \in [0, T]$, decision constraints $\dot{x}(t) \in V(x(t))$ a.e. on $[0, T]$, and boundary conditions $x(0) \in \mathfrak{M}_0$, $x(T) \in \mathfrak{M}_T$.

We denote the set of its feasible elements x by $\mathfrak{X}$.

The terms occurring in it are stated and restricted by the following *basic assumption*:

G is a strongly Lipschitz domain of E^n;

$V(\cdot)$ is a normal set-valued mapping on G (in the sense of [3]) and $V(\xi)$ are cones of E^n with the vertex at the origin 0 $\forall \xi \in G$;

$r(\cdot, \cdot)$ is continuous on $G \times E^n$, positive homogeneous of degree one with respect to the second argument, and $r(\xi, v) > 0$ $\forall \xi \in G$, $v \neq 0$;

$\mathfrak{M}_0$ and $\mathfrak{M}_T$ are subsets of $\operatorname{int} G$;

$\mathfrak{X} \neq \emptyset$.

Then with reference to [5] the corresponding dual problem of (1) consists in the goal

$$L(S) := S_T - S_0 \to \sup \quad \text{subject to } S \in W_\infty^1(G), \tag{2a}$$

which satisfies additionally a.e. on G the constraints

(2b) $$\nabla S(\xi) \in \mathfrak{F}(\xi) \ := \{z \in E^n | \ z'v \leqslant r(\xi, v) \ \forall v \in V(\xi)\}$$

as well as

(2c) $$S = \text{const} \quad \text{on} \quad \mathfrak{M}_0 \text{ and } \mathfrak{M}_T.$$

S_0 and S_T are abbreviations of the expressions $S|_{\mathfrak{M}_0}$ and $S|_{\mathfrak{M}_T}$, respectively. $\mathfrak{S}$ denotes the set of all feasible elements S of (2). $\mathfrak{F}(\xi)$ is said to be the *figuratrix* set of problem (1) at the point $\xi \in G$. From our basic assumption it immediately follows that $0 \in \operatorname{int} \mathfrak{F}(\xi)$. The duality between (1) and (2) is expressed by the characteristic condition

(3) $$\inf_{\mathfrak{X}} J_0 \geqslant \sup_{\mathfrak{S}} L.$$

An important question is the validity of strong duality between (1) and (2) i.e., equality in (3). For one-point sets $\mathfrak{M}_0$ and $\mathfrak{M}_T$ this question was answered in [8] under some additional weak assumptions. Under stronger requirements, namely the convexity of r, $V(\xi)$ and G, this strong duality can be guaranteed also by Rockafellar's duality theory [9].

Naturally we cannot always expect strong duality in (3).

This is due to the circumstance that often some "relaxed problems" of (1) have a smaller infimum but the same dual problem (2). On this account for a further elucidation of strong duality properties between (1) and (2), it is useful to begin with duality discussions of relaxed problems. This practice was already explained in paper [10] by R. V. Vinter, in which it is shown that very strong relaxed problems of (1) have the same dual problem (2). Further, he proved for constant $V(\cdot)$ that one obtains even strong duality if (1) is replaced by its sufficient general relaxed problem. This result is very important from the theoretical point of view. But the above-mentioned strong relaxed problems, based on L. C. Young's theory of generalized flows [11], can hardly be considered from the geometrical point of view. Therefore we prefer another approach to generalized primary problems by means of weaker relaxed problems, which we call flow problems of the first kind and of the second kind.

2. Flow problems of the first kind

As a preliminary for the statement of a "reasonable" relaxed problem of (1) let us consider once more our primary problem (1). It implies the question of finding an optimal trajectory within G from $\mathfrak{M}_0$ to $\mathfrak{M}_T$ as an infinitesimal thin line. For our eyes it is sufficient to construct in $G_0 = G \setminus (\mathfrak{M}_0 \cup \mathfrak{M}_T)$ approximately the cheapest thin flow $\mathfrak{v}$ drawing a chalk line on the blackboard from $\mathfrak{M}_0$ to $\mathfrak{M}_T$ with a constant pressure and velocity of the writing hand. We understand "cheapest" here in respect of the given cost rate r. The direction of the vector $\mathfrak{v}(\xi)$ characterizes the direction of the writing motion at the point ξ, and $|\mathfrak{v}(\xi)|$ is the density of chalk at that point. In consequence of the constant pressure and velocity the divergence $\nabla\cdot\mathfrak{v}$ of this vector field $\mathfrak{v}$ is equal to zero in G_0. If we assume henceforth $\mathfrak{M}_0$ and $\mathfrak{M}_T$ to be strongly Lipschitz domains, then furthermore we have the condition

$$\int\limits_{\partial\mathfrak{M}_0} \mathfrak{v}\, do = -\int\limits_{\partial\mathfrak{M}_T} \mathfrak{v}\, do.$$

By a suitable choice of the pressure and velocity we can obtain

$$\int\limits_{\partial\mathfrak{M}_0} \mathfrak{v}\, do = 1.$$

Finally, if we omit the inconvenient attribute "thin", then our real problem has precisely the following structure:

$$J_1(\mathfrak{v}) = \int\limits_{G_0} r\big(\xi, \mathfrak{v}(\xi)\big)\, d\xi \to \inf \tag{4a}$$

subject to the following conditions:

$$\mathfrak{v}(\xi) \in V(\xi) \quad \text{on} \quad G_0, \tag{4b}$$

$$\nabla\cdot\mathfrak{v} = 0 \quad \text{on} \quad G_0, \tag{4c}$$

$$\int\limits_{\Gamma_0} \mathfrak{v}\, do = 1, \qquad \int\limits_{\Gamma_T} \mathfrak{v}\, do = -1^{(1)} \qquad (\Gamma_0 = \partial\mathfrak{M}_0,\ \Gamma_T = \partial\mathfrak{M}_T), \tag{4d}$$

and

$$\mathfrak{v}\, do = 0 \quad \text{on} \quad \partial G. \tag{4e}$$

Problem (4) is called a *flow problem of the first kind.* Its admissible elements $\mathfrak{v}$ are said to be *flows* and the set of all flows is denoted by $\mathfrak{V}$.

THEOREM 1. *The optimization problem* (2) *is a dual problem of the flow problem* (4).

Proof. For any $\mathfrak{v} \in \mathfrak{V}$ and $S \in \mathfrak{S}$ we have

$$J_1(\mathfrak{v}) = \int\limits_{G_0} \big[r\big(\xi, \mathfrak{v}(\xi)\big) + S(\xi)\, \nabla\cdot\mathfrak{v}(\xi)\big]\, d\xi \tag{5}$$

in consequence of property (4c). Since

$$\nabla\cdot\big(S(\xi)\mathfrak{v}(\xi)\big) = \big(\nabla S(\xi)\big)\cdot\mathfrak{v}(\xi) + S(\xi)\nabla\cdot\mathfrak{v}(\xi),$$

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(1) Here the surface element "do" is oriented in the direction of the outer normal of Γ_0 and Γ_T.

equation (5) can be formulated as

$$(6)\qquad J_1(\mathfrak{v}) = \int_{G_0} \big[r\big(\xi, \mathfrak{v}(\xi)\big) - \nabla S(\xi)\cdot\mathfrak{v}(\xi)\big]d\xi + \int_{\partial G_0} S(\xi)\mathfrak{v}(\xi)do$$

by means of Gauss' Theorem. From (6), (4d) and (4e) we conclude, using the abbreviations $S_0 = S|_{\mathfrak{M}_0}$, $S_T = S|_{\mathfrak{M}_T}$, that

$$J_1(\mathfrak{v}) = \int_{G_0} \big[r\big(\xi, \mathfrak{v}(\xi)\big) - \nabla S(\xi)\cdot\mathfrak{v}(\xi)\big]\,d\xi + S_T - S_0.$$

Therefore, in virtue of (2b), we obtain

$$(7)\qquad J_1(\mathfrak{v}) \geqslant L(S) = S_T - S_0,$$

and this property characterizes the duality between problems (2) and (4).

3. Flow problems of the second kind

Now we introduce an extension of problem (4). Namely, we define

$$(8)\qquad \mathfrak{W} = \Big\{\mathfrak{v} \in L_1^n(G_0) \,|\, \int_{G_0} \nabla S(\xi)\cdot\mathfrak{v}(\xi)d\xi = S_T - S_0\, \forall S \in \mathfrak{S},$$
$$\mathfrak{v}(\xi) \in V(\xi)\, \forall \xi \in G_0\Big\},$$

and study

$$(9)\qquad J_2(\mathfrak{v}) = \int_{G_0} r\big(\xi, \mathfrak{v}(\xi)\big)d\xi \to \inf \quad \text{on} \quad \mathfrak{W}.$$

This problem is called a *flow problem of the second kind.*

Remark. As we already observed in connection with (2b), the origin is an interior point of the figuratrix set $F(\xi)$ $\forall \xi \in G_0$. Therefore the definition of $\mathfrak{W}$ according to (8) remains unchanged if we replace in it the set $\mathfrak{S}$ of trial functions S by the set (Banach space)

$$(10)\qquad \mathfrak{S}_0 = \{S \in W_\infty^1(G_0) \,|\, S = \text{const on } \Gamma_0 \text{ and } \Gamma_T\}.$$

LEMMA 1. $\mathfrak{B} \subset \mathfrak{W}$.

Proof. If $\mathfrak{v} \in \mathfrak{B}$, then by Gauss' Theorem

$$(11)\qquad \int_{G_0} \nabla S(\xi)\cdot\mathfrak{v}(\xi)d\xi = -\int_{G_0} S(\xi)\nabla\cdot\mathfrak{v}(\xi)d\xi + \int_{\partial G_0} S(\xi)\mathfrak{v}(\xi)do$$

for every $S \in \mathfrak{S}$. Since $\mathfrak{v}$ fulfils (4c)–(4e) and S fulfils condition (2c), we conclude from equation (11) that

$$\int_{G_0} \nabla S(\xi)\cdot\mathfrak{v}(\xi)d\xi = S_T - S_0.$$

Finally, this together with (4b) gives $\mathfrak{v} \in \mathfrak{W}$ and $\mathfrak{B} \subset \mathfrak{W}$.

LEMMA 2. *Every* $\mathfrak{v} \in \mathfrak{W} \cap C_1^n(\bar{G}_0)$ *is an element of* $\mathfrak{V}$.

Proof. If $\mathfrak{v} \in \mathfrak{W} \cap C_1^n(\bar{G}_0)$, then formula (11) holds again, even for every $S \in \mathfrak{S}_0$ in view of the Remark from above.

In the first step we consider (11) for arbitrary $S \in \dot{W}^1_\infty(G_0)$. Since $\mathfrak{v} \in \mathfrak{W}$ and according to (8), we have

$$\int_{G_0} S(\xi)\nabla\cdot\mathfrak{v}(\xi)d\xi = 0 \qquad \forall S \in \dot{W}^1_\infty(G_0). \tag{12}$$

Since $\dot{W}^1_\infty(G_0)$ is dense in $L_1(G_0)$, this variational equation (12) leads to the result $\nabla\cdot\mathfrak{v} = 0$ in G_0, which means that $\mathfrak{v}$ satisfies (4c).

In the second step we consider (11) by using result (12). Because of $\mathfrak{v} \in \mathfrak{W}$ equation (11) implies

$$S_T - S_0 = \int_{\partial G} S(\xi)\mathfrak{v}(\xi)do - \int_{\Gamma_0} S(\xi)\mathfrak{v}(\xi)do - \int_{\Gamma_T} S(\xi)\mathfrak{v}(\xi)do \qquad \forall S \in \mathfrak{S}_0. \tag{13}$$

Under the additional restriction $S_T = S_0$ the variational equality (13) brings about $\mathfrak{v}do = 0$ in ∂G, i.e., condition (4e). On the other hand, under the additional requirement $S|_{\partial G} = 0$, we obtain from (13) by the arbitrariness of S_T and S_0 the conditions $\int_{\Gamma_0} \mathfrak{v}(\xi)do = 1$, $\int_{\Gamma_T} \mathfrak{v}(\xi)do = -1$, i.e., property (4d). All these results together confirm the statement $\mathfrak{v} \in \mathfrak{V}$.

THEOREM 2. *The optimization problem* (2) *is a dual problem of the flow problem* (9).

Proof. For any $\mathfrak{v} \in \mathfrak{W}$ and $S \in \mathfrak{S}$ according to (8) we find

$$J_2(\mathfrak{v}) = \int_{G_0} \left[r\big(\xi, \mathfrak{v}(\xi)\big) - \nabla S(\xi)\cdot\mathfrak{v}(\xi)\right] d\xi + S_T - S_0$$

and by (2b) immediately $J_2(\mathfrak{v}) \geqslant L(S) = S_T - S_0$. This proves the duality between (2) and (9).

4. Flow problems and Rockafellar's duality

We shall prove that, also in Rockafellar's special sense of duality, problem (2) is a dual problem of the flow problem (9). This acknowledgement will enable us to use Rockafellar's stability theory [9] for getting strong duality criteria. We base this conception on the ideas of Ekeland and Temam [1] in Ch. III.2.

LEMMA 3. *Fenchel–Rockafellar's duality conception generates in respect of problem* (9) *the same dual problem* (2) *as that which we constructed above.*

Proof. We begin by describing W. Fenchel's and R. T. Rockafellar's method of constructions dual optimization problems. Let X and Y be two topological vector spaces with the duals X^* and Y^*, respectively. Assume that Φ is a real functional on $X \times Y$ and

$$\Phi(\mathfrak{v}, 0) \to \inf \quad \text{on} \quad X \tag{14}$$

is the primary problem in question.

We denote by Φ^* the conjugate functional of Φ in the sense of Fenchel and Rockafellar; it is defined by the declaration

$$\Phi^*(q^*, p^*) := \sup_{\mathfrak{v} \in X, p \in Y} [\langle q^*, \mathfrak{v} \rangle + \langle p^*, p \rangle - \Phi(\mathfrak{v}, p)] \tag{15}$$

for every $(q^*, p^*) \in X^* \times Y^*$. Then it appears that

$$-\Phi^*(0, p^*) \to \sup \quad \text{on} \quad Y^* \tag{16}$$

is a *dual problem* of the introduced primary problem (14). In order to apply this conception, we first transform problem (9) into a problem of the shape (14). For this purpose we realize $X = L_1^n(G_0)$, $Y = \mathfrak{S}_0^*$ and equip Y with the weak topology of $\mathfrak{S}_0^*$. Hence $Y^* = \mathfrak{S}_0^{**} = \mathfrak{S}_0$. Further we define a linear continuous mapping Λ from X into Y by

$$\langle S, \Lambda \mathfrak{v} \rangle = -\int_{G_0} \nabla S(\xi) \cdot \mathfrak{v}(\xi) d\xi \quad \forall S \in \mathfrak{S}_0. \tag{17}$$

Since the right-hand side of (17) is equal to $-\langle \nabla S, \mathfrak{v} \rangle$, the operator Λ is equal to $-\nabla^*$. Finally, for $(\mathfrak{v}, p) \in X \times Y$, we put

$$\Phi(\mathfrak{v}, p) := \begin{cases} J_2(\mathfrak{v}) & \text{if} \quad \mathfrak{v}(\xi) \in V(\xi) \text{ a.e. on } G_0 \text{ and} \\ & \qquad \langle S, \Lambda \mathfrak{v} - p \rangle + S_T - S_0 = 0 \text{ holds } \forall S \in Y^*, \\ \infty & \text{otherwise.} \end{cases} \tag{18}$$

Because of the remark concerning problem (9) the first case in definition (18) occurs for $p = 0$ iff $\mathfrak{v} \in \mathfrak{W}$ in the sense of (8). Hence the primary problem (14) in the described realization with (18) is equivalent to the optimization problem (9).

Now we shall compute $\Phi^*(0, p^*)$ by means of definitions (15) and (18). Obviously for each $\mathfrak{v} \in X$ the condition

$$\langle S, \Lambda \mathfrak{v} - p \rangle + S_T - S_0 = 0 \quad \forall S \in Y^*, \tag{19}$$

which is used in (18), defines uniquely a solution $p \in Y$ of this variational equation. This solution depends on $\mathfrak{v}$, which authorizes us to write $p = A\mathfrak{v}$.

This together with (18) immediately implies

$$\Phi^*(0, p^*) = \sup_{\mathfrak{v}\in X,\, p\in Y} [\langle p^*, p\rangle - \Phi(\mathfrak{v}, p)] \tag{20}$$
$$= \sup_{\mathfrak{v}\in X,\, p=A\mathfrak{v}} [\langle p^*, p\rangle - \Phi(\mathfrak{v}, p)].$$

Since $p = A\mathfrak{v}$ satisfies equation (19), we obtain, for $S = -p^*$, the result

$$\langle p^*, p\rangle = \langle p^*, A\mathfrak{v}\rangle + p_T^* - p_0^*,$$

which we substitute in expression (20). Thus we obtain

$$\Phi^*(0, p^*) = \sup_{\mathfrak{v}\in X,\, p=A\mathfrak{v}} [p_T^* - p_0^* + \langle p^*, A\mathfrak{v}\rangle - \Phi(\mathfrak{v}, p)],$$

and by (17) and (18)

$$\Phi^*(0, p^*) = \sup_{\substack{\mathfrak{v}\in X \\ \mathfrak{v}(\xi)\in V(\xi)\,\text{a.e.}}} \Big[p_T^* - p_0^* - \int_{G_0} \big(\nabla p^* \mathfrak{v}(\xi) + r(\xi, \mathfrak{v}(\xi))\big) d\xi\Big] \tag{21}$$
$$= \begin{cases} p_T^* - p_0^* & \text{if } \quad -\nabla p^*(\xi) \in \mathfrak{F}(\xi) \text{ a.e. on } G_0, \\ \infty & \text{otherwise.} \end{cases}$$

This equation proves the equivalence of the optimization problem (16) to the problem

$$L(-p^*) = p_0^* - p_T^* \to \sup \text{ subject to all } p^* \in \mathfrak{S}_0 \tag{22}$$
$$\text{which satisfy } -\nabla p^*(\xi) \in \mathfrak{F}(\xi) \text{ a.e. on } G_0.$$

With reference to (10) and replacing $-p^*$ by S, we find that problem (22) is evidently identical with (2). This is what Lemma 3 asserts.

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