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THE SELBERG ZETA-FUNCTION FOR COCOMPACT DISCRETE SUBGROUPS OF PSL(2, C)

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Introduction

This paper contains a simple direct approach to the Selberg zeta-function for cocompact discrete subgroups $\Gamma < \mathrm{PSL}(2, C)$. Our approach is based on the computation of the trace of the iterated resolvent kernel for Γ.

We briefly survey the relevant spectral theory of automorphic functions in Section 1. There we consider arbitrary discrete subgroups $\Gamma < \mathrm{PSL}(2, C)$ and the action of Γ on the upper half-space $\boldsymbol{H} \subset \boldsymbol{R}^3$ equipped with its hyperbolic metric. The Laplace–Beltrami operator Δ is an essentially self-adjoint linear operator on a suitable dense domain $\mathscr{D} \subset L^2(\Gamma \backslash \boldsymbol{H})$. The resolvent

$$R_\lambda = (-\tilde{\Delta} - \lambda)^{-1} \qquad (\lambda \in \varrho(-\tilde{\Delta}))$$

of the unique self-adjoint extension $-\tilde{\Delta}$ of $-\Delta$ can be described in terms of an integral operator of Carleman type, the so-called resolvent kernel; $\varrho(-\tilde{\Delta})$ denotes the resolvent set of $-\tilde{\Delta}$. It turns out that R_λ $(\lambda \in \varrho(-\tilde{\Delta}))$ is compact if and only if $\Gamma \backslash \boldsymbol{H}$ is compact. In this case the operator $-\Delta$: $\mathscr{D} \to L^2(\Gamma \backslash \boldsymbol{H})$ has a complete orthonormal system of eigenfunctions with associated eigenvalues

$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$$

(counted with multiplicities) such that

$$\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty.$$

There are also some results for non-cocompact groups Γ of finite covolume in Section 1.

From the beginning of Section 2 we restrict ourselves to the case of a cocompact discrete subgroup $\Gamma < \mathrm{PSL}(2, \boldsymbol{C})$. We allow Γ to contain elliptic elements. Section 2 deals with the computation of the trace of the kernel for the operator $R_\lambda R_\mu$. Following the ideas of Selberg [20] this leads to a trace formula which is stated in full in Theorem 2.2. Since we restrict ourselves to the case of discontinuous groups on three-dimensional space, the group theoretic data required in the course of the computation of the trace can be described in full detail. The trace formula yields a corresponding version of Huber's theorem which says that for cocompact discrete groups, the eigenvalue and length spectra uniquely determine each other.

One side of our trace-formula is determined by the logarithmic derivative of a certain infinite product analogous to an Euler product, the so-called Selberg zeta-function Z. The trace-formula implies that the Selberg zeta-function actually is an entire function with zeros determined by the eigenvalues of the Laplace–Beltrami operator. The zeros satisfy an analogue of the Riemann hypothesis, and Z has a simple functional equation (see Theorem 4.4).

Applying to Z the methods of classical analytic number theory, we obtain Weyl's asymptotic law for the asymptotic distribution of the eigenvalues. This implies that Z is an entire function of order 3. We can even write down the canonical factorization of Z rather explicitly.

The Selberg zeta-function has many properties in common with the usual zeta- and L-functions, and the analogue of the Riemann hypothesis is true for Z. An up to now unproved conjecture for the Riemann zeta-function is the Lindelöf hypothesis. It is known that the truth of the Riemann hypothesis implies the truth of the Lindelöf hypothesis. Since Z satisfies an analogue of the Riemann hypothesis it is natural to expect that Z also satisfies an analogue of the Lindelöf hypothesis. We prove this in Section 6.

This work is an elaboration of parts of our previous progress report [7]. We refer to this report for some applications of the general theory given here. A detailed exposition of our work on analytical theory and arithmetic applications of discontinuous groups on three-dimensional hyperbolic space is in preparation. – Two of the authors took part in the Semester on Elementary and Analytic Theory of Numbers held at the Stefan Banach International Mathematical Center in the autumn of 1982. We want to express our sincere thanks to our Polish hosts for the stimulating atmosphere at the conference and for their great hospitality under difficult exterior conditions.

1. A brief survey of the spectral theory of automorphic functions

We take the upper half-space

$$H := C \times]0, \infty[= \{(z, r): z \in C, r > 0\} \tag{1.1}$$

in R^3 as a model of three-dimensional hyperbolic space. Usually we write points $P \in H$ in the form

$$P = (z, r) = (x, y, r) = z + rj, \tag{1.2}$$

where $z = x+iy$ $(x, y \in R)$ and $j = (0, 0, 1)$. The hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2} \tag{1.3}$$

on H has a corresponding hyperbolic distance $d(P, P')$ $(P = z+rj$, $P' = z'+r'j \in H)$ which is given by

$$\cosh d(P, P') = \delta(P, P') \tag{1.4}$$

with

$$\delta(P, P') = \frac{|z-z'|^2 + r^2 + r'^2}{2rr'}. \tag{1.5}$$

Moreover, the hyperbolic metric gives rise to the hyperbolic volume measure v,

$$dv = \frac{dx\,dy\,dr}{r^3} \tag{1.6}$$

and to the Laplace–Beltrami operator

$$\Delta = r^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial r^2}\right)-r\frac{\partial}{\partial r}. \tag{1.7}$$

The group $\mathrm{SL}(2, \mathbf{C})$ acts on $\mathbf{H}$ in the following way: Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{C})$ and $P = (z, r) \in \mathbf{H}$, and define

$$MP = z^* + r^* j \tag{1.8}$$

by

$$z^* = \frac{(az+b)(\bar{c}\bar{z}+\bar{d})+a\bar{c}r^2}{|cz+d|^2+|c|^2 r^2}, \tag{1.9}$$

$$r^* = \frac{r}{|cz+d|^2+|c|^2 r^2}. \tag{1.10}$$

Then $\mathrm{PSL}(2, \mathbf{C}) := \mathrm{SL}(2, \mathbf{C})/\{\pm I\}$ is the group of all orientation preserving isometries for the hyperbolic metric. Since the hyperbolic metric is $\mathrm{PSL}(2, \mathbf{C})$-invariant, the hyperbolic distance d, the hyperbolic volume measure v and the Laplace–Beltrami operator Δ are invariant as well. In particular, δ (cf. (1.5)) is a point-pair invariant, i.e.

$$\delta(P, Q) = \delta(MP, MQ) \quad \text{for all } P, Q \in \mathbf{H},\ M \in \mathrm{PSL}(2, \mathbf{C}). \tag{1.11}$$

The invariance of Δ means that the action of Δ on C^2-functions commutes with the action of $\mathrm{PSL}(2, \mathbf{C})$:

$$\Delta(f \circ M) = (\Delta f)\circ M \quad \text{for all } f \in C^2(\mathbf{H}),\ M \in \mathrm{PSL}(2, \mathbf{C}). \tag{1.12}$$

The invariance of Δ enables us to introduce a self-adjoint linear operator in the following way: Let Γ be an arbitrary discrete subgroup of $\mathrm{PSL}(2, \mathbf{C})$ and denote by $L^2(\Gamma\backslash\mathbf{H})$ the Hilbert space of all (equivalence classes of almost everywhere equal) measurable functions $f\colon \mathbf{H} \to \mathbf{C}$ such that

$$f \circ M = M \quad \text{for all } M \in \Gamma \tag{1.13}$$

and

$$\int_{\mathcal{F}} |f|^2\, dv < \infty, \tag{1.14}$$

where $\mathcal{F}$ is a fundamental domain for Γ on $\mathbf{H}$. We equip $L^2(\Gamma\backslash\mathbf{H})$ with the scalar product

$$\langle f, g\rangle = \int_{\mathcal{F}} f\bar{g}\, dv \quad (f, g \in L^2(\Gamma\backslash\mathbf{H})).$$

For every C^2-function $f \in L^2(\Gamma\backslash\mathbf{H})$ the function Δf is Γ-invariant by (1.12) but possibly Δf is not square integrable over $\mathcal{F}$. Hence it is natural to introduce the dense subspace

$$\mathcal{D} := \{f \in L^2(\Gamma\backslash H) \cap C^2(H) \colon \Delta f \in L^2(\Gamma\backslash H)\} \tag{1.15}$$

of $L^2(\Gamma\backslash H)$ as a domain for Δ. Then it turns out that $-\Delta\colon \mathcal{D} \to L^2(\Gamma\backslash H)$ is a nonnegative essentially self-adjoint linear operator (see e.g. [17]). This means that the closure of the graph of Δ in $L^2(\Gamma\backslash H) \times L^2(\Gamma\backslash H)$ is the graph of a self-adjoint linear operator $\tilde{\Delta}\colon \tilde{\mathcal{D}} \to L^2(\Gamma\backslash H)$.

We want to describe the resolvent of the operator $-\tilde{\Delta}$. Let $\varrho(-\tilde{\Delta})$ denote the resolvent set of $-\tilde{\Delta}\colon \tilde{\mathcal{D}} \to L^2(\Gamma\backslash H)$. By definition, $\lambda \in \varrho(-\tilde{\Delta})$ if and only if

$$R_\lambda := (-\tilde{\Delta} - \lambda)^{-1} \tag{1.16}$$

is a bounded linear operator defined on $L^2(\Gamma\backslash H)$. For $\lambda \in \varrho(-\tilde{\Delta})$ the resolvent operator R_λ is a bounded linear operator mapping $L^2(\Gamma\backslash H)$ bijectively onto $\tilde{\mathcal{D}}$.

For suitable values of λ, the operator R_λ can be described by a kernel of Carleman type as follows: Define

$$\varphi_s(\delta) := \frac{1}{4\pi} \frac{(\delta + \sqrt{\delta^2 - 1})^{-s}}{\sqrt{\delta^2 - 1}} \qquad (s \in C,\ \delta > 1) \tag{1.17}$$

and let

$$F(P, Q, s) := \sum_{M \in \Gamma} \varphi_s(\delta(P, MQ)) \tag{1.18}$$

($\operatorname{Re} s > 1$, $P, Q \in H$, $P \notin \Gamma Q$) where $\delta(\cdot,\cdot)$ is defined by (1.5). The series (1.18) converges uniformly on compact sets provided that $\operatorname{Re} s > 1$, $P \notin \Gamma Q$. If Γ has a fundamental domain of finite hyperbolic volume, the abscissa of convergence of the generalized Dirichlet series (1.18) is equal to 1, and the series diverges at $s = 1$. Note that $F(\cdot, Q, s)$ has singularities at all points of the orbit ΓQ (cf. (1.5)).

THEOREM 1.1. *For* $\operatorname{Re} s > 1$, *the series* $F(P, Q, s)$ *is a kernel of Carleman type, i.e.*

$$\int_{\mathcal{F}} |F(P, Q, s)|^2 \, dv(Q) < \infty \tag{1.19}$$

for all $P \in H$, *and* $F(P, \cdot, s) \in L^2(\Gamma\backslash H)$ *depends continuously on* $(P, s) \in H \times \{s \in C\colon \operatorname{Re} s > 1\}$. *Suppose that* $\lambda = 1 - s^2$, $\operatorname{Re} s > 1$. *Then* $\lambda \in \varrho(-\tilde{\Delta})$, *and*

$$R_\lambda f(P) = \int_{\mathcal{F}} F(P, Q, s) f(Q) \, dv(Q) \tag{1.20}$$

for all $f \in L^2(\Gamma\backslash H)$.

The proof of Theorem 1.1 uses the same ideas as the corresponding proof for the hyperbolic plane (see [18], Teil II, [5], Teil I). Full details will be given in a future work by the authors; for a brief outline see [7].

If $\Gamma\backslash H$ is compact, i.e., if Γ has a compact fundamental domain $\mathscr{F}$, the integral of the continuous function (of P) (1.19) over $\mathscr{F}$ is finite. Hence R_λ is of Hilbert–Schmidt type. The converse is also true but more difficult to prove. The main part of the proof is based on a careful analysis of the growth behaviour of (1.19) when P approaches a cusp of Γ; for an elaboration on the same idea in the case of the hyperbolic plane see [5], Teil II. Since we do not use this part of Theorem 1.2, we omit the tedious details here.

THEOREM 1.2. *The resolvent operator* R_λ $(\lambda \in \varrho(-\tilde{\Delta}))$ *is of Hilbert–Schmidt type if and only if* $\Gamma\backslash H$ *is compact.*

COROLLARY 1.3. *Suppose that* Γ *is a cocompact discrete subgroup of* $\mathrm{PSL}(2, \mathbf{C})$. *Then* $-\Delta\colon \mathscr{D} \to L^2(\Gamma\backslash H)$ *has a complete orthonormal system* $(e_n)_{n\geqslant 0}$ *of eigenfunctions with corresponding eigenvalues*

$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots$$

(counted with multiplicites) such that

$$\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty. \tag{1.21}$$

The eigenfunction e_0 is constant and we may choose

$$e_0 = \big(v(\mathscr{F})\big)^{-1/2}.$$

The proof of Corollary 1.3 follows from Theorem 1.2 and from the fact that the eigenfunctions of $-\tilde{\Delta}\colon \tilde{\mathscr{D}} \to L^2(\Gamma\backslash H)$ are twice continuously differentiable (and even real analytic) functions on H (compare [18]).

For the rest of Section 1 we assume that $\Gamma\backslash H$ is non-compact and of finite covolume. Then there exists a continuous spectrum of $-\tilde{\Delta}$. It is equal to $[1, \infty[$ and its multiplicity (in the sense of [18]) coincides with the number of Γ-inequivalent cusps of Γ. A complete system of orthogonal eigenpackets of $-\tilde{\Delta}$ is obtained from the analytically continued Eisenstein series for a maximal system of Γ-inequivalent cusps of Γ (cf. [18] and [14]). This means that the continuous part of the spectral decomposition of $-\tilde{\Delta}$: $\tilde{\mathscr{D}} \to L^2(\Gamma\backslash H)$ is quite well understood. However, the discrete part of the spectrum turns out to be much more difficult to handle. None of the eigenvalues or eigenfunctions is known save for the eigenvalue zero and the associated constant eigenfunction. Do there exist infinitely many eigenvalues? This seems to be a difficult open question. In the case of the hyperbolic plane, the analogous problem has been discussed by A. B. Venkov (see [23], [24] and the references cited there). In the three-dimensional case we know e.g. that for the Picard group $\Gamma = \mathrm{PSL}(2, \boldsymbol{Z}[i])$ (and hence for all its subgroups of finite index) infinitely many eigenvalues exist.

Can there exist "too many" eigenvalues so that e.g. the series (1.21) diverges? This cannot happen. We briefly indicate the idea of proof. The main point is to show that there are not too many cusp forms, i.e., eigenfunctions of $-\Delta$ vanishing exponentially at all the cusps of Γ. To show this, one modifies the resolvent kernel $F(P, Q, s)$ in such a way that it becomes a Hilbert–Schmidt kernel and still has all the cusp eigenfunctions of $-\Delta$ as eigenfunctions. The modification is done in the following way: Choose $\mathscr{F}$ as a Poincaré normal polyhedron for Γ such that $\zeta_1 = A_1^{-1}\infty, \ldots, \zeta_p = A_p^{-1}\infty$ $(A_1, \ldots, A_p \in \mathrm{PSL}(2, \boldsymbol{C}))$ are all the cusps of $\mathscr{F}$ and such that these are all Γ-inequivalent. For $\nu = 1, \ldots, p$ let Γ_{ζ_ν} be the stabilizer of ζ_ν in Γ, and let $R > 1$ be sufficiently large. Then the cusp sector of $\mathscr{F}$ at ζ_ν has the form

$$\mathscr{S}_\nu = A_\nu^{-1}(\mathscr{R}_\nu \times [R, \infty[),$$

where $\mathscr{R}_\nu \subset \boldsymbol{C}$ is a suitable fundamental domain for the action of $A_\nu \Gamma_{\zeta_\nu} A_\nu^{-1}$ on $\boldsymbol{C}$. We put for $P, Q \in \boldsymbol{H}$

$$A_\nu P = (z_\nu, r_\nu), \qquad A_\nu Q = (w_\nu, t_\nu),$$

and define for $s > 1$, $s \in \boldsymbol{R}$:

$$F^*(P, Q, s)$$

$$:= \begin{cases} F(P, Q, s) - \dfrac{1}{2s|\mathscr{R}_\nu|}\, r_\nu^{1-s} t_\nu^{1+s} & \text{for } P, Q \in \mathscr{S}_\nu,\ r_\nu \geqslant t_\nu, \\[2ex] F(P, Q, s) - \dfrac{1}{2s|\mathscr{R}_\nu|}\, r_\nu^{1+s} t_\nu^{1-s} & \text{for } P, Q \in \mathscr{S}_\nu,\ r_\nu \leqslant t_\nu, \\[2ex] F(P, Q, s) & \text{for } P, Q \in \mathscr{F} \times \mathscr{F} \setminus \bigcup_{\nu=1}^{p} \mathscr{S}_\nu \times \mathscr{S}_\nu \end{cases}$$

and extend $F^*(P, Q, s)$ by Γ-invariance in both variables to all of $\boldsymbol{H} \times \boldsymbol{H}$. Then a very tedious chain of estimates yields the following result (compare [18], Teil II, § 8).

THEOREM 1.4. *The function $F^*(P, Q, s)$ $(P, Q \in \boldsymbol{H}, s > 1)$ is real-valued, Γ-invariant with respect to P and Q, symmetric in P and Q and satisfies*

$$\int_{\mathscr{F}} \int_{\mathscr{F}} |F^*(P, Q, s)|^2 \, dv(P) \, dv(Q) < \infty. \tag{1.22}$$

For every cusp eigenfunction f of $-\Delta$, the action of $F^*(\cdot, \cdot, s)$ on f is the same as the action of $F(\cdot, \cdot, s)$, i.e.,

$$\int_{\mathscr{F}} F^*(P, Q, s) f(Q) \, dv(Q) = \int_{\mathscr{F}} F(P, Q, s) f(Q) \, dv(Q). \tag{1.23}$$

Moreover, equality (1.23) *even holds for all* $f \in L^2(\Gamma \backslash H)$ *such that the zeroth Fourier coefficients of* f *vanish almost everywhere in the neighbourhood of all the cusps of* Γ. *The function*

$$(1.24) \qquad P \mapsto \int_{\mathscr{F}} |F^*(P, Q, s)|^2 \, dv(Q)$$

is bounded on compact subsets of H.

COROLLARY 1.5. *Suppose that* $-\tilde{\Delta}$: $\tilde{\mathscr{D}} \to L^2(\Gamma \backslash H)$ *has an orthonormal system* $(\varphi_n)_{n \geq 1}$ *of cusp eigenfunctions of* $-\Delta$ *with corresponding eigenvalues* λ_n:

$$-\Delta \varphi_n = \lambda_n \varphi_n.$$

Then

$$(1.25) \qquad \sum_{n=1}^{\infty} \lambda_n^{-2} < \infty,$$

and for every $g \in \tilde{\mathscr{D}}$ *the contribution*

$$(1.26) \qquad \sum_{n=1}^{\infty} \langle g, \varphi_n \rangle \varphi_n$$

of $(\varphi_n)_{n \geq 1}$ *to the expansion of* g *in eigenfunctions and eigenpackets* (*of* $-\tilde{\Delta}$) *converges also pointwise absolutely and uniformly on compact sets.*

Proof. The system $(\varphi_n)_{n \geq 1}$ can be included in a complete orthonormal system of eigenfunctions of the symmetric Hilbert–Schmidt kernel $F^*(\cdot, \cdot, s)$ $(s > 1)$. This yields (1.25). To prove the assertion on the pointwise convergence properties of (1.26) we put $\lambda = 1 - s^2$ with $s > 1$. Then we have for $\mu, \nu \in N$, $\mu < \nu$ and $P \in H$

$$\begin{aligned}
\sum_{n=\mu}^{\nu} |\langle g, \varphi_n \rangle \varphi_n(P)| &= \sum_{n=\mu}^{\nu} \Big|\langle g, (-\tilde{\Delta} - \lambda) \varphi_n \rangle \int_{\mathscr{F}} F(P, Q, s) \varphi_n(Q) \, dv(Q)\Big| \\
&= \sum_{n=\mu}^{\nu} \Big|\langle (-\tilde{\Delta} - \lambda) g, \varphi_n \rangle \int_{\mathscr{F}} F^*(P, Q, s) \varphi_n(Q) \, dv(Q)\Big| \\
&\leq \Big(\sum_{n=\mu}^{\nu} |\langle (-\tilde{\Delta} - \lambda) g, \varphi_n \rangle|^2\Big)^{1/2} \Big(\int_{\mathscr{F}} |F^*(P, Q, s)|^2 \, dv(Q)\Big)^{1/2}.
\end{aligned}$$

Since (1.24) is bounded on compact sets, the assertion follows. ∎

2. Computation of the trace

For the rest of this work we keep the following assumptions and notations fixed: Suppose that $\Gamma < \mathrm{PSL}(2, C)$ is a cocompact discrete group, and let $\mathscr{F}$ be a compact fundamental domain of Γ on H. Let $(e_n)_{n \geq 0}$ be a complete orthonormal system of eigenfunctions of $-\Delta\colon \mathscr{D} \to L^2(\Gamma\backslash H)$ with associated eigenvalues

(2.1) $$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots$$

counted with appropriate multiplicities. Then we know from Corollary 1.3 that

(2.2) $$\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty.$$

We choose

(2.3) $$e_0 = \big(v(\mathscr{F})\big)^{-1/2}.$$

Let

(2.4) $$N := \max\{n \geqslant 0\colon\ \lambda_n < 1\}$$

and write

(2.5) $$\lambda_n = 1 - s_n^2$$

with

(2.6) $$s_0 = 1,$$

(2.7) $$s_n \in\,]0, 1] \quad \text{for } n = 0, \dots, N,$$

(2.8) $$s_n = it_n, \quad t_n \geqslant 0 \text{ for } n \geqslant N+1.$$

The resolvent set of $-\tilde{\Delta}\colon \tilde{\mathscr{D}} \to L^2(\Gamma\backslash H)$ is

$$\varrho(-\tilde{\Delta}) = \{\lambda \in \boldsymbol{C}\colon\ \lambda \neq \lambda_n \text{ for all } n \geqslant 0\}.$$

For $\lambda \in \varrho(-\tilde{\Delta})$ we have

$$R_\lambda e_n = \frac{1}{\lambda_n - \lambda}\, e_n$$

(cf. (1.16)). Unfortunately, it is impossible to compute the trace of R_λ since the series $\sum_{n=1}^{\infty} \lambda_n^{-1}$ diverges (see Corollary 5.6). But if we take another $\mu \in \varrho(-\tilde{\Delta})$, the operator $R_\lambda R_\mu$ is a product of two Hilbert–Schmidt operators and hence a trace-class operator (cf. Weidmann [28], p. 167). Since

$$R_\lambda R_\mu e_n = (\lambda_n - \lambda)^{-1} (\lambda_n - \mu)^{-1} e_n,$$

the trace of $(\lambda - \mu) R_\lambda R_\mu$ is given by

$$(\lambda - \mu)\operatorname{tr}(R_\lambda R_\mu) = \sum_{n=0}^{\infty} \left(\frac{1}{\lambda_n - \lambda} - \frac{1}{\lambda_n - \mu}\right).$$

We introduce new variables s, t by

$$\lambda = 1-s^2, \quad \mu = 1-t^2 \tag{2.9}$$

and have

$$(\lambda-\mu)\operatorname{tr} R_\lambda R_\mu = \sum_{n=0}^{\infty}\left(\frac{1}{s^2-s_n^2}-\frac{1}{t^2-s_n^2}\right). \tag{2.10}$$

Note that for fixed t the sum on the right-hand side is a meromorphic function of s with poles $\pm s_n$ $(n \geqslant 0)$ (cf. (2.2)).

Computing the trace (2.10) in another way we shall arrive at the logarithmic derivative of the Selberg zeta-function. The present section is devoted to the computation of the trace. The final result will be given in Theorem 2.2.

We start from Hilbert's resolvent equation

$$(\lambda-\mu) R_\lambda R_\mu = R_\lambda - R_\mu. \tag{2.11}$$

Suppose now for the rest of our computation that

$$\operatorname{Re} s > 1, \quad \operatorname{Re} t > 1. \tag{2.12}$$

Then it follows from Theorem 1.1 that the operator (2.11) is represented by the kernel

$$H(P, Q) := \lim_{Z \to Q} \big(F(P, Z, s) - F(P, Z, t)\big). \tag{2.13}$$

This is a continuous function of $(P, Q) \in H \times H$ since the singularities cancel out. In fact,

$$h(\delta) := \varphi_s(\delta) - \varphi_t(\delta) \tag{2.14}$$

(cf. (1.17)) is continuous for $\delta \geqslant 1$,

$$h(1) = -\frac{1}{4\pi}(s-t), \tag{2.15}$$

$$h(\delta) = O(\delta^{-2-\varepsilon}) \quad \text{as } \delta \to \infty \text{ for some } \varepsilon > 0, \tag{2.16}$$

and

$$H(P, Q) = \sum_{M \in \Gamma} h(\delta(P, MQ)).$$

We now obtain

$$(\lambda-\mu)\operatorname{tr}(R_\lambda R_\mu) = \int_{\mathcal{F}} \sum_{M \in \Gamma} h(\delta(P, MP))\, dv(P). \tag{2.17}$$

To justify the termwise integration of this series we note that for P, $Q \in H$ (cf. (1.4))

$$\delta(P, Q) \geqslant \tfrac{1}{2} e^{d(P,Q)} \geqslant \tfrac{1}{2} e^{d(j,Q) - d(P,j)}.$$

Let $\eta := \max\{d(P, j)\colon P \in \mathcal{F}\}$. Then we have for all $P \in \mathcal{F}$, $Q \in H$

$$\delta(P, Q) \geqslant \tfrac{1}{2} e^{-\eta} e^{d(j,Q)} \geqslant \tfrac{1}{2} e^{-\eta} \delta(j, Q). \tag{2.18}$$

Looking at (2.15), (2.16), we see that the integrand in (2.17) is majorized termwise by a constant multiple of the series

$$\sum_{M \in \Gamma} \delta(j, MP)^{-2-\varepsilon}. \tag{2.19}$$

Since we have

$$\int_{\mathcal{F}} \sum_{M \in \Gamma} \delta(j, MP)^{-2-\varepsilon} dv(P)$$

$$= \int_{H} \delta(j, P)^{-2-\varepsilon} dv(P) = \int_{H} \left(\frac{2r}{x^2+y^2+r^2+1}\right)^{2+\varepsilon} \frac{dx\, dy\, dr}{r^3}$$

$$= 2\pi \int_0^\infty \int_0^\infty \left(\frac{2r}{\varrho^2+r^2+1}\right)^{2+\varepsilon} \varrho\, d\varrho\, \frac{dr}{r^3} = \frac{2^{2+\varepsilon}\pi}{1+\varepsilon} \int_0^\infty \frac{dr}{(r^2+1)^{1+\varepsilon} r^{1-\varepsilon}} < \infty,$$

we are permitted to integrate (2.17) termwise. This yields by (2.15)

$$(\lambda - \mu) \operatorname{tr}(R_\lambda R_\mu) = -\frac{v(\mathcal{F})}{4\pi}(s-t) + \sum_{\substack{M \in \Gamma \\ M \neq I}} \int_{\mathcal{F}} h(\delta(P, MP))\, dv(P). \tag{2.20}$$

Following Selberg [20] we rearrange the terms of the series on the right-hand side collecting the terms belonging to the same Γ-conjugacy class. Let $\{T\}$ run through the set of Γ-conjugacy classes of the elements $M \in \Gamma$, $M \neq I$. Then $M = S^{-1} TS$ runs through $\{T\}$ precisely once whenever S runs through a representative system $\mathcal{C}(T)\backslash\Gamma$ of the right cosets of the centralizer $\mathcal{C}(T)$ of T in Γ. Hence we find

$$\begin{aligned} \sum_{\substack{M \in \Gamma \\ M \neq I}} \int_{\mathcal{F}} h(\delta(P, MP))\, dv(P) &= \sum_{\{T\}} \sum_{S \in \mathcal{C}(T)\backslash\Gamma} \int_{\mathcal{F}} h(\delta(P, S^{-1} TSP))\, dv(P) \\ &= \sum_{\{T\}} \sum_{S \in \mathcal{C}(T)\backslash\Gamma} \int_{S\mathcal{F}} h(\delta(P, TP))\, dv(P) \\ &= \sum_{\{T\}} \int_{\mathcal{F}(\mathcal{C}(T))} h(\delta(P, TP))\, dv(P), \end{aligned} \tag{2.21}$$

where $\mathcal{F}(\mathcal{C}(T))$ denotes a fundamental domain of $\mathcal{C}(T)$.

We have to compute the terms of the sum on the right-hand side of (2.21). First we consider the case that $T \in \Gamma$ is a hyperbolic or loxodromic element. Then T is conjugate in $\mathrm{PSL}(2, \mathbf{C})$ to a unique element

$$D(T) = \begin{pmatrix} a(T) & 0 \\ 0 & a(T)^{-1} \end{pmatrix} \quad \text{with } |a(T)| > 1. \tag{2.22}$$

Following Selberg we call

$$N(T) := |a(T)|^2 \tag{2.23}$$

the *norm* of T. Note that $N(T) = N(T^{-1})$. Since T is hyperbolic or loxodromic, an element of $\mathrm{PSL}(2, \mathbf{C})$ commutes with T if and only if it has the same fixed points in $\mathbf{C} \cup \{\infty\}$ as T. We determine the structure of $\mathscr{C}(T)$: Let $\mathscr{E}(T)$ be the set of elements of finite order in $\mathscr{C}(T)$. Then either $\mathscr{E}(T)$ contains only the identity or $\mathscr{E}(T)$ is the finite cyclic group generated by the hyperbolic rotation in Γ with minimal rotation angle around the axis of T, i.e., around the hyperbolic line through the fixed points of T in $\mathbf{C} \cup \{\infty\}$. Let $T_0 \in \mathscr{C}(T)$ be an element such that $N(T_0) > 1$ is minimal among the set of norms of all hyperbolic or loxodromic elements of $\mathscr{C}(T)$. Note that the elements $T_0 E$, $T_0^{-1} E$ $(E \in \mathscr{E}(T))$ are precisely the elements of norm $N(T_0)$ contained in $\mathscr{C}(T)$. We call T_0 a *primitive hyperbolic or loxodromic element for* T in Γ. T_0 itself is not uniquely determined by T, but $N(T_0)$ is. $\mathscr{C}(T)$ is the direct product of $\mathscr{E}(T)$ with $\langle T_0 \rangle$, the cyclic subgroup generated by T_0. In particular, $\mathscr{C}(T)$ is abelian.

Now choose $V \in \mathrm{PSL}(2, \mathbf{C})$ such that

$$T = V^{-1} D(T) V. \tag{2.24}$$

Then $V \mathscr{C}(T) V^{-1}$ has the fundamental domain

$$\mathscr{F}(T) := \{\varrho e^{i\varphi} + rj \colon \varrho > 0,\ 0 \leqslant \varphi < 2\pi/\operatorname{ord} \mathscr{E}(T),\ 1 \leqslant r < N(T_0)\} \tag{2.25}$$

and hence we have

$$\begin{aligned} &\int_{\mathscr{F}(\mathscr{C}(T))} h(\delta(P, TP))\, dv(P) \\ &= \int_{\mathscr{F}(T)} h(\delta(P, D(T)P))\, dv(P) \\ &= \int_{\mathscr{F}(T)} h\left(\frac{|a(T)^2 - 1|^2 |z|^2 + (N(T)^2 + 1) r^2}{2N(T) r^2}\right) \frac{dx\, dy\, dr}{r^3} \\ &= \frac{\log N(T_0)}{\operatorname{ord} \mathscr{E}(T)} \int_{\mathbf{C}} h\left(\frac{|a(T)^2 - 1|^2 |z|^2 + N(T)^2 + 1}{2N(T)}\right) dx\, dy \end{aligned} \tag{2.26}$$

$$= \frac{2\pi \log N(T_0)}{\operatorname{ord} \mathscr{E}(T)} \int_0^\infty h\left(\frac{|a(T)^2-1|^2 \varrho^2 + N(T)^2+1}{2N(T)}\right) \varrho \, d\varrho$$

$$= \frac{2\pi \log N(T_0)}{\operatorname{ord} \mathscr{E}(T) |a(T)-a(T)^{-1}|^2} \int_{\alpha(T)}^\infty h(u)\, du$$

where

$$\alpha(T) := \tfrac{1}{2}(N(T)+N(T)^{-1}).$$

Substituting $u = \cosh x$ we find from (1.17)

$$(2.27) \qquad \int_{\alpha(T)}^\infty \varphi_s(u)\, du = \frac{1}{4\pi} \int_{\operatorname{arcosh} \alpha(T)}^\infty e^{-sx}\, dx = \frac{1}{4\pi s} N(T)^{-s}.$$

Since $h = \varphi_s - \varphi_t$, *the contribution of a hyperbolic or loxodromic conjugacy class of* Γ *to the sum on the right-hand side of* (2.21) *is finally equal to*

$$(2.28) \qquad \int_{\mathscr{F}(\mathscr{C}(T))} h(\delta(P, TP))\, dv(P)$$

$$= \frac{1}{2s} \frac{\log N(T_0)}{\operatorname{ord} \mathscr{E}(T) |a(T)-a(T)^{-1}|^2} N(T)^{-s} - \frac{1}{2t} \frac{\log N(T_0)}{\operatorname{ord} \mathscr{E}(T) |a(T)-a(T)^{-1}|^2} N(T)^{-t}.$$

Here, T_0 denotes a primitive hyperbolic or loxodromic element associated with T. (The same notation will be tacitly used in the sequel.)

The contribution of the elliptic conjugacy classes of Γ (if any) is determined in essentially the same way, but the details are more cumbersome. The final result will be equal to (2.28) once the corresponding concepts are defined properly. Suppose now that $R \in \Gamma$ is elliptic. Then R is a hyperbolic rotation around a hyperbolic line which remains fixed under R pointwise. This hyperbolic line meets $C \cup \{\infty\}$ in the fixed points of R in $C \cup \{\infty\}$. The subgroup of Γ containing all the elements of Γ with the same fixed points in $C \cup \{\infty\}$ as R contains a rotation R_0 with minimal rotation angle. R_0 is uniquely determined up to inversion, and R is a power of R_0. We call R_0 a *primitive elliptic element of* Γ *associated with* R.

We claim that $\mathscr{C}(R)$ is infinite. Assume to the contrary that $\mathscr{C}(R)$ is finite. We know from our discussion above that the integral

$$(2.29) \qquad \int_{\mathscr{F}(\mathscr{C}(R))} h(\delta(P, RP))\, dv(P)$$

converges (absolutely). Since we assume that $\mathscr{C}(R)$ is finite we conclude that

$$\int_H h(\delta(P, RP))\, dv(P)$$

converges as well. But this is absurd: Choose $V \in \mathrm{PSL}(2, \mathbb{C})$ such that

$$R = V^{-1} R(\zeta) V \tag{2.30}$$

with

$$R(\zeta) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad |\zeta| = 1, \zeta \neq \pm 1, \tag{2.31}$$

and note that

$$R(\zeta)(z + rj) = \zeta^2 z + rj. \tag{2.32}$$

Then

$$\begin{aligned} \int_H h(\delta(P, RP))\,dv(P) &= \int_H h(\delta(P, R(\zeta)P))\,dv(P) \\ &= \int_H h\left(\frac{|\zeta^2 - 1|^2 |z|^2 + 2r^2}{2r^2}\right) \frac{dx\,dy\,dr}{r^3} \\ &= \int_H h\left(1 + \frac{|\zeta^2 - 1|^2 |z|^2}{2}\right) \frac{dx\,dy\,dr}{r}. \end{aligned} \tag{2.33}$$

The latter integral obviously diverges (look at the dependence on r!). Hence we arrive at a contradiction and conclude that $\mathscr{C}(R)$ must be infinite, i.e., *$\mathscr{C}(R)$ must contain a hyperbolic or loxodromic element.* The precise structure of $\mathscr{C}(R)$ is described in the next theorem.

THEOREM 2.1. *Suppose that $R \in \Gamma$ is elliptic, and let R_0 be a primitive elliptic element associated with R. Then the centralizer $\mathscr{C}(R)$ of R in Γ contains hyperbolic or loxodromic elements. Let $T_0 \in \mathscr{C}(R)$ be hyperbolic or loxodromic such that $N(T_0)$ is minimal in the set of norms of hyperbolic or loxodromic elements contained in $\mathscr{C}(R)$. Then there are two possibilities:*

(a) *Either $\langle R_0 \rangle$ contains all the elliptic elements of $\mathscr{C}(R)$. Then $\mathscr{C}(R)$ is abelian,*

$$\mathscr{C}(R) = \langle R_0 \rangle \times \langle T_0 \rangle,$$

and $\mathscr{E}(R) := \langle R_0 \rangle$ is a maximal finite subgroup of $\mathscr{C}(R)$, in fact, the unique maximal finite subgroup of $\mathscr{C}(R)$.

(b) *Or R is elliptic of order 2, and there exists an elliptic element $S \in \mathscr{C}(R)$ also of order 2 whose fixed line meets the fixed line of R orthogonally in a common point. Then for every such S*

$$S^{-1} R_0 S = R_0^{-1},$$

and

$$\mathscr{E}(R) := \langle R_0 \rangle \cup \langle R_0 \rangle S$$

is a maximal finite subgroup of $\mathscr{C}(R)$. $\mathscr{E}(R)$ *is of dihedral type. All the maximal finite subgroups of* $\mathscr{C}(R)$ *are conjugate in* $\mathrm{PSL}(2, \boldsymbol{C})$.

$$\mathscr{C}(R) = \{T_0^n E \colon E \in \mathscr{E}(R),\ n \in \boldsymbol{Z}\},$$

and $\langle R_0 \rangle \times \langle T_0 \rangle$ *is an abelian subgroup of index* 2 *in* $\mathscr{C}(R)$.

Proof. Assume first that $\langle R_0 \rangle$ contains all the elliptic elements of $\mathscr{C}(R)$. Every hyperbolic or loxodromic element of $\mathscr{C}(R)$ commutes with R and hence has the same fixed points in $\boldsymbol{C} \cup \{\infty\}$ as R. Thus $\mathscr{C}(R)$ is abelian, and assertion (a) is obvious.

Suppose now that there exists another elliptic element $S \in \mathscr{C}(R) \setminus \langle R_0 \rangle$. Transform R to normal form (2.31). Then an elementary computation yields that $R(\zeta)$ commutes with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \boldsymbol{C}) \tag{2.34}$$

if and only if either $b = c = 0$ (i.e., if $R(\zeta)$ and (2.34) have the fixed points 0 and ∞ in common) or $\zeta = \pm i$ and $a = d = 0$. Since we assume that R and S do not have the same fixed points in $\boldsymbol{C} \cup \{\infty\}$, we conclude from the second case that R is a hyperbolic rotation with rotation angle π, and $S \neq R$ is an elliptic element of order 2 commuting with R. The fixed lines of R and S meet orthogonally in a common point. Transforming R to normal form, we obtain

$$VRV^{-1} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{2.35}$$

$$VSV^{-1} = \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}, \tag{2.36}$$

where $V \in \mathrm{PSL}(2, \boldsymbol{C})$ is a suitable element. This yields

$$S^{-1} R_0 S = R_0^{-1}.$$

Hence

$$\mathscr{E}(R) := \langle R_0 \rangle \cup \langle R_0 \rangle S \tag{2.37}$$

is a finite subgroup of $\mathscr{C}(R)$ of dihedral type. (The subgroup $\{I, R, S, RS\}$ of $\mathscr{C}(R)$ is isomorphic to the Klein four group.)

We proceed to show that $\mathscr{E}(R)$ is a maximal subgroup of $\mathscr{C}(R)$ containing only elements of finite order. To prove this, suppose that $\mathscr{H} \subset \mathscr{C}(R)$ is a subgroup containing only elements of finite order such that $\mathscr{E}(R) \subset \mathscr{H}$.

Let $A \in \mathscr{H}$, $A \notin \langle R_0 \rangle$. Then our deduction of (2.36) yields that

$$VAV^{-1} = \begin{pmatrix} 0 & \beta \\ -\beta^{-1} & 0 \end{pmatrix} \tag{2.38}$$

for some $\beta \neq 0$. Hence $VASV^{-1}$ leaves 0 and ∞ fixed, and since $AS \in \mathscr{H}$ is an element of finite order, we conclude that $AS \in \langle R_0 \rangle$, i.e., $A \in \langle R_0 \rangle S$. This proves that $\mathscr{H} \subset \mathscr{E}(R)$, and hence $\mathscr{E}(R)$ is a maximal subgroup of $\mathscr{C}(R)$ containing only elements of finite order. In particular, $\mathscr{E}(R)$ is a maximal finite subgroup of $\mathscr{C}(R)$.

We proved already that $\mathscr{C}(R)$ contains a hyperbolic or loxodromic element. The fixed line of R is the axis of every hyperbolic or loxodromic element of $\mathscr{C}(R)$. We choose a hyperbolic or loxodromic element $T_0 \in \mathscr{C}(R)$ with $N(T_0)$ minimal and claim that

$$\mathscr{C}(R) = \{T_0^n E\colon\ E \in \mathscr{E}(R),\ n \in \mathbf{Z}\}. \tag{2.39}$$

To prove this assertion, assume first that $T \in \mathscr{C}(R) \setminus \langle R_0 \rangle$ is elliptic. Then VTV^{-1} has the form (2.38), hence $VTSV^{-1}$ leaves 0 and ∞ fixed, i.e. TS has the same fixed points in $C \cup \{\infty\}$ as R. This implies that $TS = T_0^n R_0^\nu$ for some integers ν, n, $0 \leqslant \nu < \operatorname{ord} R_0$, and hence T belongs to the set on the right-hand side of (2.39). Second, assume that $T \in \mathscr{C}(R)$ is hyperbolic or loxodromic. Then T has the same fixed points in $C \cup \{\infty\}$ as T_0. Hence $T_0^{-n} T$ is elliptic for some integer n, and we conclude that $T = T_0^n R_0^\nu$ for some integers n, ν. This proves (2.39). Obviously $\langle T_0 \rangle \times \langle R_0 \rangle$ is an abelian subgroup of index 2 in $\mathscr{C}(R)$, and $\{I, S\}$ is a representative system of the cosets.

It remains to prove that all the maximal finite subgroups of $\mathscr{C}(R)$ are conjugate in PSL(2, C). Let $\mathscr{G}$ be a maximal finite subgroup of $\mathscr{C}(R)$. We draw from our discussion above that there exists an elliptic element $A \in \mathscr{C}(R)$ of order two such that the fixed lines of A and R meet orthogonally in an common point and such that

$$\mathscr{G} = \langle R_0 \rangle \times \langle R_0 \rangle A.$$

By (2.39),

$$A = T_0^n R_0^\nu S$$

for some integers n, ν. Choose a hyperbolic or loxodromic element $T_1 \in \mathrm{PSL}(2, C)$ such that $T_0 = T_1^2$. Then T_1 commutes with R_0, and

$$T_1^{-n} R_0^m A T_1^n = R_0^{m+\nu} S \quad \text{for all } m \in \mathbf{Z}.$$

This yields

$$T_1^{-n} \mathscr{G} T_1^n = \mathscr{E}(R). \ \blacksquare$$

Now we compute the contribution of the elliptic conjugacy class $\{R\}$ to the right-hand side of (2.21): If case (a) of Theorem 2.1 occurs, then $V\mathscr{C}(R)V^{-1}$ has the fundamental domain

$$\mathscr{F}(R) := \{\varrho e^{i\varphi}+rj: \ \varrho > 0,\ 0 \leqslant \varphi < 2\pi/\operatorname{ord} R_0,\ 1 \leqslant r < N(T_0)\} \tag{2.40}$$

where R_0, T_0 are as in Theorem 2.1. (The same notation will be tacitly used in the sequel.) Hence we find (cf. (2.33) and (2.27))

$$\begin{aligned}\int\limits_{\mathscr{F}(\mathscr{C}(R))} h(\delta(P, RP))\,dv(P) &= \frac{2\pi \log N(T_0)}{\operatorname{ord}\mathscr{E}(R)} \int\limits_0^\infty h\left(1+\frac{|\zeta-\zeta^{-1}|^2}{2}\varrho^2\right)\varrho\,d\varrho \\ &= \frac{2\pi\log N(T_0)}{\operatorname{ord}\mathscr{E}(R)|\zeta-\zeta^{-1}|^2}\int\limits_1^\infty h(u)\,du \\ &= \frac{2\pi\log N(T_0)}{\operatorname{ord}\mathscr{E}(R)\,|(\operatorname{tr} R)^2-4|}\cdot\left(\frac{1}{4\pi s}-\frac{1}{4\pi t}\right) \\ &= \left(\frac{1}{2s}-\frac{1}{2t}\right)\frac{\log N(T_0)}{\operatorname{ord}\mathscr{E}(R)\,|(\operatorname{tr} R)^2-4|},\end{aligned} \tag{2.41}$$

where $(\operatorname{tr} R)^2$ is the square of the trace of R (which makes sense for elements of $\mathrm{PSL}(2, \mathbb{C})$).

Assume finally that R is such that case (b) of Theorem 2.1 occurs. Then $\langle R_0\rangle \times \langle T_0\rangle$ is an abelian subgroup of index 2 in $\mathscr{C}(R)$, and (2.40) is a fundamental domain for this subgroup. This yields by (2.41)

$$\begin{aligned}\int\limits_{\mathscr{F}(\mathscr{C}(R))} h(\delta(P, RP))\,dv(P) &= \tfrac{1}{2}\int\limits_{\mathscr{F}(R)} h\big(\delta(P, R(\zeta)P)\big)\,dv(P) \\ &= \left(\frac{1}{2s}-\frac{1}{2t}\right)\frac{\log N(T_0)}{2\operatorname{ord} R_0\,|(\operatorname{tr} R)^2-4|} \\ &= \left(\frac{1}{2s}-\frac{1}{2t}\right)\frac{\log N(T_0)}{\operatorname{ord}\mathscr{E}(R)\,|(\operatorname{tr} R)^2-4|}.\end{aligned}$$

Summing up, we have now proved: *If R is elliptic, the contribution of $\mathscr{C}(R)$ to the right-hand side of* (2.21) *is equal to*

$$\int\limits_{\mathscr{F}(\mathscr{C}(R))} h(\delta(P, RP))\,dv(P) = \left(\frac{1}{2s}-\frac{1}{2t}\right)\frac{\log N(T_0)}{\operatorname{ord}\mathscr{E}(R)\,|(\operatorname{tr} R)^2-4|}. \tag{2.42}$$

Note that this result agrees formally with (2.28) if we put $N(R) := 1$ since $|a(T)-a(T)^{-1}|^2 = |(\operatorname{tr} T)^2-4|$ in (2.28). From (2.21), (2.28), (2.42) we obtain the following *trace formula.*

THEOREM 2.2. *Suppose that $\Gamma < \mathrm{PSL}(2, \mathbb{C})$ is a cocompact discrete group with fundamental domain $\mathcal{F}$ and eigenvalue spectrum $(\lambda_n)_{n\geq 0}$, $\lambda_n = 1-s_n^2$, and let $\lambda = 1-s^2$, $\mu = 1-t^2$, $\operatorname{Re} s > 1$, $\operatorname{Re} t > 1$. Then*

$$(2.43)\qquad (\lambda-\mu)\operatorname{tr} R_\lambda R_\mu = \sum_{n=0}^{\infty}\left(\frac{1}{s^2-s_n^2}-\frac{1}{t^2-s_n^2}\right)$$

$$= -\frac{v(\mathcal{F})}{4\pi}(s-t)+$$

$$+\left(\frac{1}{2s}-\frac{1}{2t}\right)\sum_{\{R\}\text{ellipt.}}\frac{\log N(T_0)}{\operatorname{ord}\mathscr{E}(R)\,|(\operatorname{tr} R)^2-4|}+$$

$$+\frac{1}{2s}\sum_{\{T\}\text{lox.}}\frac{\log N(T_0)}{\operatorname{ord}\mathscr{E}(T)\,|a(T)-a(T)^{-1}|^2}N(T)^{-s}-$$

$$-\frac{1}{2t}\sum_{\{T\}\text{lox.}}\frac{\log N(T_0)}{\operatorname{ord}\mathscr{E}(T)\,|a(T)-a(T)^{-1}|^2}N(T)^{-t},$$

where the summation with respect to $\{R\}$ extends over the finitely many Γ-conjugacy classes of elliptic elements of Γ and the summation with respect to $\{T\}$ extends over the Γ-conjugacy classes of hyperbolic or loxodromic elements of Γ. Moreover, the notation of Theorem 2.1 applies, and $a(T)$, $N(T)$ are as in (2.22), (2.23).

It is obvious from the above proof that the same method yields a trace formula for arbitrary integral operators associated with a Poincaré series defined by a point-pair invariant provided that suitable growth conditions are satisfied. However, the iterated resolvent kernel seems to be one of the most interesting examples since its trace immediately yields the Selberg zeta-function (see Section 4). Another interesting example is the kernel

$$\Theta(P, Q, t) := \sum_{M\in\Gamma} e^{-t\delta(P,MQ)}\qquad (t>0).$$

The eigenvalues of the associated integral operator are computed by means of the Selberg transform (cf. [7]). This yields the trace formula

$$(2.44)\qquad \int_{\mathcal{F}}\Theta(P, P, t)\,dv(P) = \frac{4\pi}{t}\sum_{n=0}^{\infty}K_{s_n}(t)$$

$$= v(\mathcal{F})e^{-t}+\frac{2\pi}{t}e^{-t}\sum_{\{R\}\text{ellipt.}}\frac{\log N(T_0)}{\operatorname{ord}\mathscr{E}(R)\,|(\operatorname{tr} R)^2-4|}+$$

$$+\frac{2\pi}{t}\sum_{\{T\}\text{lox.}}\frac{\log N(T_0)}{\operatorname{ord}\mathscr{E}(T)\,|a(T)-a(T)^{-1}|^2}e^{-\frac{t}{2}(N(T)+N(T)^{-1})},$$

where K_ν denotes the modified Bessel function.

The same method applies to the kernel

$$H(P, Q, s) := \sum_{M \in \Gamma} \delta(P, MQ)^{-1-s} \quad (\operatorname{Re} s > 1).$$

Here the computation of the trace yields

$$(2.45) \qquad \int_{\mathscr{F}} H(P, P, s)\, dv(P)$$

$$= \frac{2^s \pi}{\Gamma(s+1)} \sum_{n=0}^{\infty} \Gamma\left(\frac{s+s_n}{2}\right) \Gamma\left(\frac{s-s_n}{2}\right)$$

$$= v(\mathscr{F}) + \frac{2\pi}{s} \sum_{\{R\}\text{ellipt.}} \frac{\log N(T_0)}{\operatorname{ord} \mathscr{E}(R)\, |(\operatorname{tr} R)^2 - 4|} +$$

$$+ \frac{2\pi}{s} \sum_{\{T\}\text{lox.}} \frac{\log N(T_0)}{\operatorname{ord} \mathscr{E}(T)\, |a(T) - a(T)^{-1}|^2} \left(\tfrac{1}{2}(N(T) + N(T)^{-1})\right)^{-s}.$$

Obviously, the left-hand sides of (2.43), (2.45) are meromorphic in the whole s-plane. Hence the right-hand sides of (2.43), (2.45) are meromorphic functions of $s \in C$.

3. Huber's theorem

We maintain the assumptions and notations of Section 2 and digress briefly into uniqueness questions associated with the eigenvalue and length spectra of $\Gamma \backslash H$. Problems of this kind were discussed first by H. Huber [11] in the case of the hyperbolic plane. L. Bérard-Bergery [2], [3] and H. Riggenbach [16] extended Huber's result to the case of hyperbolic spaces of arbitrary dimension with cocompact discrete groups without fixed points. We choose a slightly different approach based on Theorem 2.2, and we admit groups with elliptic elements.

The trace formula (2.43) has a geometric meaning if Γ contains no elliptic elements. Assume for a moment that Γ is a cocompact discrete subgroup of $\mathrm{PSL}(2, C)$ without elliptic elements. Then the sum

$$(3.1) \qquad \sum_{n=0}^{\infty} \left(\frac{1}{s^2 - s_n^2} - \frac{1}{t^2 - s_n^2}\right)$$

in (2.43) is determined by the sequence $(s_n)_{n \geq 0}$, i.e., by the *eigenvalue spectrum* $(\lambda_n)_{n \geq 0}$ of the Laplacian on the Riemannian manifold $M := \Gamma \backslash H$. The first term on the right-hand side of (2.43) has an obvious geometric meaning. It is simply given by the *volume* of M. The norms $N(T)$ in (2.43) can be interpreted in terms of the lengths of the *closed geodesics* on M.

Recall that $\boldsymbol{H}$ is the universal covering of M and that Γ is isomorphic to the fundamental group of M. The conjugacy classes of Γ are in a natural bijective correspondence with the free homotopy classes of closed continuous paths on M as follows. Consider a free homotopy class W of closed continuous paths on M. This class contains a closed oriented geodesic γ of M which is uniquely determined up to the choice of its initial point. A lift of γ to $\boldsymbol{H}$ is a hyperbolic line segment L whose endpoints A, B satisfy $A = TB$ for some unique $T \in \Gamma$. (Remember that Γ has no fixed points on $\boldsymbol{H}$.) All images SL $(S \in \Gamma)$ are also lifts of γ to $\boldsymbol{H}$, and the element of Γ matching the endpoints of SL is equal to STS^{-1}. We associate the Γ-conjugacy class $\{T\}$ with γ, i.e., with the free homotopy class W. This correspondence is bijective. The length of γ can be recovered from $\{T\}$ as follows. Suppose that γ is not a point. Then $T \neq I$, and L is part of the axis of T, and the length of L is equal to $\log N(T)$. Hence the numbers $\log N(T)$, where $\{T\}$ runs through the Γ-conjugacy classes of elements $T \in \Gamma$, $T \neq I$, are the lengths of the closed geodesics on M. The trace formula establishes a quantitative relationship between the eigenvalue spectrum of the Laplacian on M and between the volume and the lengths of the closed geodesics on M, i.e., between analytical and geometrical invariants of M.

We now drop our hypothesis that Γ acts fixed point freely on $\boldsymbol{H}$ and admit that Γ may contain elliptic elements. Suppose that

$$\mu_j := \log N(T_j) \qquad (j \geqslant 1)$$

are the logarithms of the norms of the hyperbolic or loxodromic elements of Γ, arranged in strictly increasing order. Then we call the family of ordered pairs

$$\left(\left(\mu_j, \sum_{\substack{\{T\}\text{lox.}\\ \log N(T)=\mu_j}} \frac{\log N(T_0)}{\operatorname{ord} \mathscr{E}(T)\,|a(T)-a(T)^{-1}|^2}\right)\right)_{j \geqslant 1} \tag{3.2}$$

the *length spectrum of* Γ. This notion imitates the corresponding definition in the fixed-point free case. Our notion of length spectrum really is a group theoretic concept although we maintain the geometric language from the fixed-point free case.

The number

$$E := \sum_{\{R\}\text{ellipt.}} \frac{\log N(T_0)}{\operatorname{ord} \mathscr{E}(R)\,|(\operatorname{tr} R)^2 - 4|} \tag{3.3}$$

is called the *elliptic number* of Γ. (We do not know if this number has an interpretation in terms of geometric data of $\Gamma \backslash \boldsymbol{H}$.) The following result is our version of *Huber's theorem.*

THEOREM 3.1. *Let Γ_1, Γ_2 be cocompact discrete groups.*

(a) *Suppose that the eigenvalue spectra for Γ_1 and Γ_2 agree up to at most finitely many terms. Then the eigenvalue spectra, the length spectra, the volumes and the elliptic numbers for Γ_1 and Γ_2 are the same.*

(b) *Suppose that the length spectra for Γ_1 and Γ_2 agree up to at most finitely many terms. Then the length spectra, the eigenvalue spectra, the volumes and the elliptic numbers for Γ_1 and Γ_2 coincide.*

Proof. (a) By assumption, the sums (3.1) for Γ_1 and Γ_2 agree up to at most finitely many terms. Letting s tend to infinity in the corresponding equation resulting from the trace formula (2.43), we see that the volumes for Γ_1 and Γ_2 coincide. Omit the contribution from the volumes and let $t \to \infty$ in the new equation. This yields an equation of the form

$$(3.4)\qquad \sum_{\text{fin.}}(\pm)\frac{2s}{s^2-s_n^2} = E_1 - E_2 + \sum_{\{T\}\text{lox.}}{}_1\, c(T)N(T)^{-s} - \sum_{\{T\}\text{lox.}}{}_2\, c(T)N(T)^{-s}$$

where "fin." indicates a certain finite sum involving the numbers s_n for Γ_1, Γ_2, and where E_1, E_2 are the elliptic numbers for Γ_1, Γ_2, and where $\sum_1$, $\sum_2$ mean summation over the conjugacy classes $\{T\}$ of hyperbolic or loxodromic elements of Γ_1, Γ_2, respectively. In addition we put

$$c(T) = \frac{\log N(T_0)}{\operatorname{ord}\mathscr{E}(T)\,|a(T)-a(T)^{-1}|^2}.$$

Letting $s \to +\infty$, we obtain $E_1 = E_2$. Hence (3.4) can be rewritten in the form

$$(3.5)\qquad \sum_{\text{fin.}}(\pm)\frac{2s}{s^2-s_n^2} = \sum_{N(T)} \gamma(T)N(T)^{-s},$$

where the summation on the right-hand side extends over all the *different norms* of hyperbolic or loxodromic elements from $\Gamma_1 \cup \Gamma_2$ and where $\gamma(T)$ indicates the difference of the associated "weights" for Γ_1 and Γ_2 occurring in (3.2). We have to show that $\gamma(T) = 0$ for all T. Assume that this is false and let $T^* \in \Gamma_1 \cup \Gamma_2$ be such that $\gamma(T^*) \neq 0$ and $N(T^*)$ is minimal with this property. Then

$$(3.6)\qquad N(T^*)^s\sum_{\text{fin.}}(\pm)\frac{2s}{s^2-s_n^2} = \gamma(T^*) + \sum_{N(T)>N(T^*)} \gamma(T)\left(\frac{N(T)}{N(T^*)}\right)^{-s}.$$

Letting $s = \sigma + it$ with sufficiently large fixed σ, we see that for $t \to \infty$ the left-hand side of (3.6) tends to zero whereas the right-hand side does not. This contradiction yields $\gamma(T) = 0$ for all T, i.e., the length spectra for Γ_1 and Γ_2 are the same. Hence the left-hand side of (3.5) vanishes as well, i.e., all the terms cancel. This means that the eigenvalue spectra for Γ_1 and Γ_2 coincide completely.

(b) Assume that the length spectra for Γ_1 and Γ_2 agree up to at most finitely many terms. Then (2.43) yields that the volumes for Γ_1 and Γ_2 are the same. Omit the contribution from the volumes, multiply the corresponding equation by $2s$ and compare the poles in the s-plane. Then obviously the eigenvalue spectra for Γ_1 and Γ_2 coincide (including multiplicities). Omitting the contribution from the eigenvalues, we finally find that the elliptic numbers and the length spectra for Γ_1, Γ_2 are the same as well. ∎

Two cocompact discrete groups Γ_1, Γ_2 are called *isospectral* if their eigenvalue or length spectra coincide. Examples are known of nonconjugate isospectral groups (cf. M.-F. Vignéras [25], [26]). Therefore the following corollary to Theorem 3.1 is worth mentioning.

COROLLARY 3.2. *If Γ_1 and Γ_2 are isospectral cocompact discrete subgroups of* PSL(2, $\mathbb{C}$), *then either both Γ_1 and Γ_2 contain elliptic elements or none of them contains elliptic elements.*

The following problem (orally communicated to the authors by M.-F. Vignéras) seems to be open: Suppose that the eigenvalue or length spectra for Γ_1 and Γ_2 agree up to a sequence which is of lower density in some appropriate sense. Are Γ_1 and Γ_2 isospectral?

4. The Selberg zeta-function

The right-hand side of our trace-formula (2.43) is for $\operatorname{Re} s > 1$ the logarithmic derivative of an infinite product, the so-called *Selberg zeta-function*, and the trace formula is the key to the investigation of the amazing analytical properties of this function (analytic continuation, zeros, functional equation, growth behaviour, canonical factorization).

We maintain the hypotheses and notations of Section 2, and we denote the elliptic number of Γ by E (see (3.3)). Suppose that $T_0 \in \Gamma$ is a primitive hyperbolic or loxodromic element, and let $\mathscr{E}(T_0) = \langle R_0 \rangle$, where R_0 equals either the identity or the hyperbolic rotation in Γ with minimal rotation angle around the axis of T_0. R_0 is uniquely determined up to inversion. We claim that all the elements

$$T = T_0^{n+1} R_0^{\nu} \quad (n \geqslant 0,\ 0 \leqslant \nu < \operatorname{ord} R_0) \tag{4.1}$$

are non-conjugate in Γ. To prove this assertion we may assume from the outset that T_0 and R_0 have diagonal form:

$$T_0 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad |\alpha| > 1,$$

$$R_0 = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad |\zeta| = 1.$$

For an element $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ not commuting with T the element XTX^{-1} has diagonal form if and only if

$$X = \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix},$$

i.e., if and only if X is a hyperbolic rotation with rotation angle π around a hyperbolic line meeting the axis of T_0 orthogonally. For such an X we have

$$XTX^{-1} = T^{-1},$$

and T^{-1} does not admit a representation of the form $T_0^{m+1} R_0^{\mu}$ with $m \geqslant 0$ and $0 \leqslant \mu < \operatorname{ord} R_0$. Hence the elements (4.1) are non-conjugate in Γ.

We now choose a maximal system $\mathscr{R}$ of primitive hyperbolic or loxodromic elements of Γ such that no two of the elements

(4.2) $\quad T = T_0^{n+1} R_0^{\nu} \quad (T_0 \in \mathscr{R},\ \mathscr{E}(T_0) = \langle R_0 \rangle,\ n \geqslant 0,\ 0 \leqslant \nu < \operatorname{ord} R_0)$

are conjugate in Γ. We claim that (4.2) automatically is a representative system of all the Γ-conjugacy classes of hyperbolic or loxodromic elements of Γ. To prove this, suppose that $S \in \Gamma$ is hyperbolic or loxodromic and is not conjugate to any of the elements (4.2). Choose a primitive hyperbolic or loxodromic element S_0 for S in Γ in such a way that $S_0^{-m-1} S$ is of finite order for some $m \geqslant 0$, and put $\mathscr{E}(S_0) = \langle Q_0 \rangle$. By assumption, $\mathscr{R}$ is maximal and we proved that no two elements of the form

(4.3) $\quad S_0^{k+1} Q_0^{\mu} \quad (k \geqslant 0,\ 0 \leqslant \mu < \operatorname{ord} Q_0)$

are conjugate in Γ. Hence some element of the form (4.3) is conjugate to some element occurring in (4.2). Suppose that

$$M^{-1} S_0^{k+1} Q_0^{\mu} M = T_0^{n+1} R_0^{\nu}$$

for some $M \in \Gamma$, k, $n \geqslant 0$, $0 \leqslant \mu < \operatorname{ord} Q_0$, $0 \leqslant \nu < \operatorname{ord} R_0$. Since we can replace S, S_0, Q_0 by their conjugates with respect to M, we may assume from the outset that $M = I$. Then S_0, T_0, R_0, Q_0 have the same fixed points, and since $k \geqslant 0$, $n \geqslant 0$, we conclude that $Q_0 = R_0^{\pm 1}$, $k = n$, $S_0 = T_0 R_0^{\varrho}$ for some ϱ, $0 \leqslant \varrho < \operatorname{ord} R_0$. This implies that we may assume from the outset that $S_0 = T_0$, $Q_0 = R_0$. But S itself has the form (4.3) and we see that S itself occurs among the elements (4.2). This contradicts our hypothesis on S, and our claim is proved.

If T_0 is a primitive hyperbolic or loxodromic element of Γ, the element T_0^{-1} is also primitive and $\mathscr{E}(T_0) = \mathscr{E}(T_0^{-1})$. It is easy to draw from our above remarks that if T_0 is Γ-conjugate to $(T_0^{-1})^{n+1} R_0^{\nu}$ $(n \geqslant 0,\ 0 \leqslant \nu < \operatorname{ord} R_0)$, then $n = \nu = 0$, i.e., T_0 is Γ-conjugate to T_0^{-1}. In this case, there exists an elliptic element $X \in \Gamma$ of order two whose rotation axis meets the axis of T_0

orthogonally such that $XT_0X^{-1} = T_0^{-1}$, and T_0, T_0^{-1} cannot simultaneously be included in $\mathscr{R}$. The system $\mathscr{R}$ can be chosen in such way that for all $T_0 \in \mathscr{R}$ which are not Γ-conjugate their inverse the element T_0^{-1} also belongs to $\mathscr{R}$.

DEFINITION 4.1. For $\operatorname{Re} s > 1$, the *Selberg zeta-function* for Γ is defined by

$$Z(s) := \prod_{\substack{T_0 \in \mathscr{R} \\ k, l \geqslant 0 \\ k \equiv l \bmod \operatorname{ord} \mathscr{E}(T_0)}} \left(1 - a(T_0)^{-2k} \overline{a(T_0)}^{-2l} N(T_0)^{-s-1}\right), \tag{4.4}$$

where the product with respect to T_0 extends over a maximal system $\mathscr{R}$ of primitive hyperbolic or loxodromic elements of Γ such that no two of the elements (4.2) are conjugate in Γ. (For $a(T_0)$, $N(T_0)$ see (2.22), (2.23).) The associated *Selberg xi-function* is defined by

$$\Xi(s) := \exp\left(-\frac{v(\mathscr{F})}{6\pi} s^3 + Es\right) Z(s) \tag{4.5}$$

(cf. (3.3)).

The Selberg xi-function is defined in such a way that the right-hand side of (2.43) is equal to

$$\frac{1}{2s} \frac{\Xi'}{\Xi}(s) - \frac{1}{2t} \frac{\Xi'}{\Xi}(t)$$

(see Lemma 4.3). First we have to check that (4.4) converges.

LEMMA 4.2. *The number $\pi(x)$ of Γ-conjugacy classes $\{T\}$ of hyperbolic or loxodromic elements of Γ with $N(T) \leqslant x$ $(x > 0)$ satisfies*

$$\pi(x) = O(x^2) \quad \textit{for} \quad x \to \infty. \tag{4.6}$$

Proof. If $T \in \mathrm{PSL}(2, \mathbb{C})$ is hyperbolic or loxodromic, then

$$\log N(T) = \inf\{d(P, TP) \colon P \in H\},$$

and the equation $\log N(T) = d(P, TP)$ holds if and only if P lies on the axis of T (cf. (1.4)).

Let $T \in \Gamma$ be hyperbolic or loxodromic and assume that $P \in H$ lies on the axis of T. Then there exists an $S \in \Gamma$ such that $Q := SP \in \mathscr{F}$ and hence

$$\log N(T) = d(P, TP) = d(Q, STS^{-1}Q)$$

where $V := STS^{-1} \in \{T\}$. We denote the hyperbolic minimal distance of two non-empty subsets A, $B \subset H$ by $d(A, B)$ and the hyperbolic diameter of $\mathscr{F}$ by d_0. Choose a point $P_0 \in \mathscr{F}$, and let $B(P_0, \varrho)$ be the open hyperbolic ball with centre P_0 and hyperbolic radius ϱ. Then we have for $x > 1$

$$\begin{aligned}\pi(x) &= \#\{\{T\}\colon\ T\in\Gamma \text{ hyperbolic or loxodromic, } N(T)\leqslant x\}\\ &\leqslant \#\{V\in\Gamma\colon\ d(\mathscr{F}, V\mathscr{F})\leqslant \log x\}\\ &\leqslant \#\{V\in\Gamma\colon\ d(P_0, VP_0)\leqslant \log x+2d_0\}\\ &\leqslant \#\{V\in\Gamma\colon\ V\mathscr{F}\subset B(P_0, \log x+3d_0)\}\\ &\leqslant \frac{v(B(P_0, \log x+3d_0))}{v(\mathscr{F})}\end{aligned}$$

and hence

$$\pi(x)=O(x^2) \quad \text{for} \quad x\to\infty$$

since

$$v(B(P_0, \varrho)) = 2\pi(\sinh\varrho\cosh\varrho-\varrho) \sim \frac{\pi}{2}e^{2\varrho} \quad \text{for} \quad \varrho\to\infty. \ \blacksquare$$

LEMMA 4.3. *The product* (4.4) *for the Selberg zeta-function converges absolutely for* $\operatorname{Re}s>1$ *and satisfies*

$$\frac{Z'}{Z}(s) = \sum_{\{T\}\text{lox.}} \frac{\log N(T_0)}{\operatorname{ord}\mathscr{E}(T)\,|a(T)-a(T)^{-1}|^2}\,N(T)^{-s}. \tag{4.7}$$

Proof. We have for $\sigma=\operatorname{Re}s>1$

$$\sum_{\substack{T_0\in\mathscr{R}\\ k,l\geqslant 0}} |a(T_0)^{-2k}\,\overline{a(T_0)}^{\,-2l}N(T_0)^{-s-1}| = \sum_{T_0\in\mathscr{R}} (1-|a(T_0)|^{-2})^{-2}N(T_0)^{-\sigma-1}. \tag{4.8}$$

Applying (4.6) and partial summation we see that

$$\sum_{\{T\}\text{lox.}} N(T)^{-s-1} \tag{4.9}$$

converges absolutely for $\operatorname{Re}s>1$. Hence (4.8) converges (absolutely and) uniformly on compact sets in $\operatorname{Re}s>1$. This implies that $Z(s)$ is holomorphic in the half-plane $\operatorname{Re}s>1$ and that the logarithmic derivative of Z may be computed termwise. Remember that the elements (4.2) run through a representative system of the Γ-conjugacy classes $\{T\}$ of hyperbolic or loxodromic elements of Γ precisely once. For $T_0\in\mathscr{R}$ let $\mathscr{E}(T_0)=\langle R_0\rangle$, let $\zeta(T_0)$, $\zeta(T_0)^{-1}$ be the eigenvalues of R_0, and put $m(T_0):=\operatorname{ord}R_0$. Then $\zeta(T_0)$ is a primitive $(2m(T_0))$-th root of unity, and $\zeta(T_0)$ is uniquely determined up to inversion and change of sign. Hence we obtain for the sum on the right-hand side of (4.7)

$$\sum_{\{T\}\text{lox.}} \frac{\log N(T_0)}{\operatorname{ord} \mathscr{E}(T)|a(T)-a(T)^{-1}|^2} N(T)^{-s}$$

$$= \sum_{\substack{T_0\in\mathscr{R}\\ n\geqslant 0\\ 0\leqslant \nu<m(T_0)}} \frac{\log N(T_0)}{m(T_0)|\zeta(T_0)^{\nu}a(T_0)^{n+1}-\zeta(T_0)^{-\nu}a(T_0)^{-n-1}|^2} N(T_0)^{-s(n+1)}$$

$$= \sum_{\substack{T_0\in\mathscr{R}\\ n\geqslant 0\\ 0\leqslant \nu<m(T_0)}} \frac{\log N(T_0)}{m(T_0)\left(1-\zeta(T_0)^{-2\nu}a(T_0)^{-2(n+1)}\right)\left(1-\overline{\zeta(T_0)}^{-2\nu}\overline{a(T_0)}^{-2(n+1)}\right)} \times$$

$$\times\ N(T_0)^{-(s+1)(n+1)}$$

$$= \sum_{\substack{T_0\in\mathscr{R}\\ k,l,n\geqslant 0\\ 0\leqslant \nu<m(T_0)}} \frac{1}{m(T_0)} \log N(T_0)\zeta(T_0)^{-2\nu(k-l)}a(T_0)^{-2k(n+1)}\overline{a(T_0)}^{-2l(n+1)} \times$$

$$\times N(T_0)^{-(s+1)(n+1)}$$

$$= \sum_{\substack{T_0\in\mathscr{R}\\ k,l\geqslant 0\\ k\equiv l \bmod m(T_0)}} \frac{\log N(T_0)a(T_0)^{-2k}\overline{a(T_0)}^{-2l}N(T_0)^{-(s+1)}}{1-a(T_0)^{-2k}\overline{a(T_0)}^{-2l}N(T_0)^{-(s+1)}} = \frac{Z'}{Z}(s). \quad \blacksquare$$

THEOREM 4.4. *The Selberg zeta-function and the Selberg xi-function defined for* $\operatorname{Re} s > 1$ *by* (4.4), (4.5), *are entire functions of* s *and satisfy*

$$\frac{1}{2s}\frac{\Xi'}{\Xi}(s)-\frac{1}{2t}\frac{\Xi'}{\Xi}(t) = \sum_{n=0}^{\infty}\left(\frac{1}{s^2-s_n^2}-\frac{1}{t^2-s_n^2}\right) \tag{4.10}$$

for all $s,\ t\in \mathbf{C}\setminus\{\pm s_n\colon n\geqslant 0\}$. *The zeros of* Z *and* Ξ *are the numbers* $\pm s_n$, $n\geqslant 0$. *For* $\lambda_n\neq 1$, *the numbers* s_n *and* $-s_n$ *both are zeros of multiplicity equal to the multiplicity of the eigenvalue* λ_n *of* $-\Delta\colon \mathscr{D}\to L^2(\Gamma\setminus \mathbf{H})$. *If* $\lambda_m = 1$ *is an eigenvalue of* $-\Delta\colon \mathscr{D}\to L^2(\Gamma\setminus \mathbf{H})$ *of multiplicity* k, *then* $s_m = 0$ *is a zero of* Z *and* Ξ *of multiplicity* $2k$. Ξ *and* Z *satisfy the functional equations*

$$\Xi(-s) = \Xi(s), \tag{4.11}$$

$$Z(-s) = \exp\left(-\frac{v(\mathscr{F})}{3\pi}s^3+2Es\right)Z(s). \tag{4.12}$$

Proof. It follows from (2.43), (4.5) and (4.7) that (4.10) is valid for $\operatorname{Re} s > 1$, $\operatorname{Re} t > 1$. Keep t fixed, $\operatorname{Re} t > 1$. Then the right-hand side of (4.10) is a meromorphic function of $s\in \mathbf{C}$ with simple poles at the points $\pm s_n$. Hence Ξ'/Ξ is a meromorphic function with simple poles at $\pm s_n$ $(n\geqslant 0)$. The multiplicity of the poles s_n, $-s_n$ equals the multiplicity of the eigenvalue λ_n if

$\lambda_n \neq 1$; if $\lambda_m = 1$ is an eigenvalue of multiplicity k, then $s_k = 0$ is a pole of Ξ'/Ξ of multiplicity $2k$. This implies that Ξ and Z are entire functions whose zeros are as described in Theorem 4.4 and that (4.10) holds as stated.

In view of (4.5) we are left to prove (4.11). We see immediately from (4.10) that Ξ'/Ξ is an odd function whence Ξ itself is either even or odd. Observe now that for $\Xi(0) \neq 0$ the order of Ξ at 0 is even, and our above discussion of the zeros of Ξ shows that for $\Xi(0) = 0$ the order of Ξ at 0 is even as well. Hence Ξ itself is even which proves (4.11). ∎

The Selberg zeta-function in its properties closely resembles the usual zeta- or L-functions of number theory. The primitive elements of Γ can be thought of as some kind of substitutes of prime numbers, and (4.4) is an analogue of the *Euler product expansion.* Theorem 4.4 says that Z has an *analytic continuation* to the whole s-plane and satisfies a simple *functional equation* which takes its most convenient form in terms of the function Ξ. The notations Z and Ξ are analogous to the usual notations ζ, ξ for the Riemann zeta-function and its associated function

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s).$$

The line $\mathrm{Re}\, s = 0$ is the critical line for Z, and with the only exception of $s_0 = 1, \ldots, s_N, -s_0 = -1, \ldots, -s_N \in [-1, 1]$ (cf. (2.4)–(2.8)) all the zeros of Z are on the critical line. This means that the analogue of the Riemann hypothesis is valid for Z save for the zeros in $[-1, 1]$ just mentioned. Note that the proof of this fact is based on the trace formula (2.43). We do not have a particular example of a cocompact discrete group $\Gamma < \mathrm{PSL}(2, \mathbf{C})$ where a zero of Z actually occurs in $]0, 1[$ nor do we have an example where $Z(0) = 0$. There exists no series of trivial zeros of Z and Ξ in contrast to the zeta- or L-functions of analytic number theory and in contrast to the properties of the Selberg zeta-function for cocompact discrete subgroups of $\mathrm{PSL}(2, \mathbf{R})$ (cf. Selberg [20], Hejhal [9], [10], Venkov [24], Elstrodt [6]). The reason for the absence of trivial zeros is explained in Gangolli [8]. The Selberg zeta-function for cocompact discrete subgroups of $\mathrm{PSL}(2, \mathbf{C})$ without elliptic elements was also introduced by Vishik [27] in different notations. The present approach was suggested by Elstrodt [6], Section 10.

There are difficult open questions connected with the eigenvalues λ_n. Apart from $\lambda_0 = 0$ not a single eigenvalue is explicitly known not even for a particular group. In the case of the hyperbolic plane it is known that for every hyperbolic area of the quotient of the hyperbolic plane modulo a fixed point free cocompact discrete group of orientation preserving hyperbolic motions arbitrarily small positive eigenvalues can exist. (For a list of references see Elstrodt [6], Section 9.) Contrarily, in the case of dimension $\geqslant 3$, Schoen [19] has shown that λ_1 is bounded below by a positive constant depending only on the volume of $M = \Gamma \backslash \mathbf{H}$.

It is well known that ξ is an entire function of order one and that the

Selberg zeta-function for a cocompact discrete subgroup of $\mathrm{PSL}(2, \boldsymbol{R})$ is an entire function of order 2 (see [10]). We shall prove in the next section that in the case of dimension 3, Z is an entire function of order 3 (cf. Gangolli [8]).

5. Weyl's asymptotic law and the canonical factorization of the Selberg zeta-function

It follows from (4.4) that

$$Z(s) > 0 \quad \text{for real } s > 1. \tag{5.1}$$

Since the zeros of Z are known from Theorem 4.4 we infer that there exists a unique holomorphic logarithm $\log Z$ of Z in the region

$$G := \boldsymbol{C} \setminus (]-\infty, 1] \cup \bigcup_{n=N}^{\infty} \{x \pm it_n\colon\ x \leqslant 0\}) \tag{5.2}$$

such that $\log Z(s) \in \boldsymbol{R}$ for real $s > 1$. Imitating the well-established notation for the Riemann zeta-function we put

$$\arg Z(s) := \operatorname{Im}(\log Z(s)) \quad \text{for} \quad s \in G. \tag{5.3}$$

We want to investigate the asymptotic behaviour of the number of eigenvalues less than T as T tends to infinity. This problem is equivalent to the asymptotic analysis of the function

$$A(T) := \#\{n\colon\ n \geqslant N+1,\ t_n \leqslant T\} \tag{5.4}$$

for $T \to \infty$. The argument principle relates $A(T)$ with $\arg Z(iT)$ in a simple way.

THEOREM 5.1. *Suppose that* $T > 0$, $T \neq t_n$ *for all* $n \geqslant N+1$. *Then*

$$A(T) = \frac{v(\mathscr{F})}{6\pi^2} T^3 + \frac{E}{\pi} T + \frac{1}{\pi} \arg Z(iT) - N. \tag{5.5.}$$

Proof. Theorem 4.4 and the argument principle yield

$$2(A(T)+N) = \frac{1}{2\pi i} \int\limits_{\partial R(T)} \frac{Z'}{Z}(s)\, ds,$$

where $\partial R(T)$ is the positively oriented boundary of the rectangle with the vertices $2+iT$, $-2+iT$, $-2-iT$, $2-iT$. $\partial R(T)$ splits into two parts, $R^+(T)$, $R^-(T)$, situated in the half-planes $\operatorname{Re} s \geqslant 0$ and $\operatorname{Re} s \leqslant 0$, respectively. The map $s \mapsto -s$ maps $R^-(T)$ onto $R^+(T)$ such that the orientation is preserved.

Hence the functional equation (4.12) yields

$$\frac{1}{2\pi i}\int\limits_{\partial R(T)}\frac{Z'}{Z}(s)\,ds=\frac{1}{2\pi i}\int\limits_{R^+(T)}\frac{Z'}{Z}(s)\,ds-\frac{1}{2\pi i}\int\limits_{R^+(T)}\frac{Z'}{Z}(-s)\,ds$$

$$=\frac{1}{2\pi i}\int\limits_{R^+(T)}\left(-\frac{v(\mathscr{F})}{\pi}s^2+2E\right)ds+\frac{1}{\pi i}\int\limits_{R^+(T)}\frac{Z'}{Z}(s)\,ds$$

$$=\frac{v(\mathscr{F})}{3\pi^2}T^3+\frac{2E}{\pi}T+\frac{2}{\pi}\arg Z(iT)$$

since $Z(\bar{s})=\overline{Z(s)}$. This proves (5.5). ∎

We want to estimate the growth of the error term $\arg Z(iT)$ for $T\to\infty$. For this we need the following Lemma 5.2 which is a preliminary result only since we show in Theorem 5.7 that Z is an entire function of order precisely 3.

LEMMA 5.2. *Z is an entire function of order at most 4.*

Proof. Let $p\in\{0, 1, 2, 3\}$ be the minimal integer such that

$$\sum_{n=0}^{\infty}{}' |s_n|^{-p-1}<\infty,$$

where the prime indicates that all terms with $s_n=0$ (if any) must be omitted (cf. (2.2)). Let $k\geqslant 0$ be the multiplicity of the eigenvalue 1 of $-\Delta\colon \mathscr{D}\to L^2(\Gamma\backslash H)$. Then the canonical product

$$\Phi(s):=s^{2k}\prod_{n=0}^{\infty}{}'\left(1-\frac{s}{s_n}\right)\exp\left\{\frac{s}{s_n}+\ldots+\frac{1}{p}\left(\frac{s}{s_n}\right)^p\right\}\times \tag{5.6}$$

$$\times\prod_{n=0}^{\infty}{}'\left(1+\frac{s}{s_n}\right)\exp\left\{-\frac{s}{s_n}+\ldots+\frac{1}{p}\left(-\frac{s}{s_n}\right)^p\right\}$$

is an entire function of order equal to the exponent of convergence of the series

$$\sum_{n=0}^{\infty}{}' |s_n|^{-\alpha}$$

which is at most equal to 4 (see Titchmarsh [21], Sections 8.23 and 8.25). Formula (4.10) yields that $Z'/Z-\Phi'/\Phi$ is a polynomial of degree at most 2 whence

$$Z(s)=\Phi(s)\,e^{q(s)} \tag{5.7}$$

with some polynomial q of degree at most 3. It ensues that Z is an entire function of order at most 4. ∎

COROLLARY 5.3. *For all a, $b \in \mathbf{R}$, $a < b$ there exists a constant $C > 0$ such that*

$$Z(\sigma+it) = O(e^{Ct^2}) \quad \textit{for} \quad |t| \to \infty \tag{5.8}$$

uniformly with respect to $\sigma \in [a, b]$.

Proof. It is obvious from the functional equation (4.12) that for every fixed $\sigma < 0$ and all $t \in \mathbf{R}$

$$Z(\sigma+it) = f_\sigma(t)\exp\left(\frac{v(\mathscr{F})}{\pi}|\sigma|\, t^2\right)Z(-\sigma-it), \tag{5.9}$$

where $f_\sigma\colon \mathbf{R} \to \mathbf{C}$ is a bounded function. For the proof of (5.8) we may assume from the outset that $a < -1 < 1 < b$. Then (5.8) is trivially true for $\sigma = b$, and (5.9) yields that (5.8) is also true for $\sigma = a$. Since Z is an entire function of finite order, the Phragmén–Lindelöf theorem implies our assertion. ∎

Imitating a classical method of reasoning for the Riemann zeta-function based on Jensen's formula (cf. Titchmarsh [22], p. 180–181), we obtain the following basic estimate for $\arg Z(iT)$.

THEOREM 5.4. *Suppose that $T > 0$, $T \neq t_n$ for all $n \geqslant N+1$. Then*

$$\arg Z(iT) = O(T^2) \quad \textit{for} \quad T \to \infty. \tag{5.10}$$

Proof. Let $L(T)$ be the line segment from $2+iT$ to iT. Then

$$\arg Z(iT) = \arg Z(2+iT) + \operatorname{Im} \int\limits_{L(T)} \frac{Z'}{Z}(s)\,ds. \tag{5.11}$$

Computing $\log Z(\sigma+it)$ for fixed $\sigma > 1$ and $t \in \mathbf{R}$ from (4.4) by means of the power series for $\log(1+z)$ one easily finds that

$$\arg Z(2+iT) = O(1) \quad \text{for} \quad T \to \infty$$

(compare (4.8), (4.9)). We now prove that the second term on the right-hand side of (5.11) is $O(T^2)$ for $T \to \infty$. Note that

$$\operatorname{Im} \int\limits_{L(T)} \frac{Z'}{Z}(s)\,ds$$

is the increment of the argument of $Z(s)$ as s runs on $L(T)$ from $2+iT$ to iT. Each time the argument of $Z(s)$ changes by a quantity of absolute value at least π, the real part of $Z(s)$ undergoes a change of sign. Let $c(T)$ denote the number of changes of sign of the function $\operatorname{Re} Z(\sigma+iT)$ as σ decreases from 2 to 0. Then it ensues that

$$\left|\operatorname{Im} \int\limits_{L(T)} \frac{Z'}{Z}(s)\,ds\right| \leqslant (2+c(T))\pi.$$

Since $Z(\bar{s}) = \overline{Z(s)}$, we conclude that $c(T)$ equals the number $n(T)$ of zeros of

$$\varphi_T(w) := Z(w+iT)+Z(w-iT)$$

in the interval [0, 2] up to an error term not exceeding 2 due to the possible zeros at 0 and 2. (Here the zeros are counted only simply; no multiplicities are taken into account.) We estimate $n(T)$ by means of an application of Jensen's formula to the disc of centre 0 and radius 3. If 0 is not a zero of φ_T, we find

$$n(T)\log\tfrac{3}{2} \leqslant \frac{1}{2\pi}\int\limits_0^{2\pi} \log|\varphi_T(3e^{i\vartheta})|\,d\vartheta - \log|\varphi_T(0)|,$$

and (5.8) yields

$$n(T) = O(T^2) \quad \text{for} \quad T \to \infty.$$

If 0 is a zero of φ_T, a slight move of the centre of our disc leads to the same conclusion. ∎

Corollary 5.5 (Weyl's Asymptotic Law). *Suppose that* $T > 0$, $T \neq t_n$ *for all* $n \geqslant N+1$. *Then*

$$A(T) = \frac{v(\mathscr{F})}{6\pi^2}T^3 + O(T^2) \quad \textit{for} \quad T \to \infty. \tag{5.12}$$

The proof is obvious from (5.5) and (5.10). We add some remarks on the sharpness of the error estimate (5.12). The asymptotic law (5.12) coincides with the corresponding result for compact Riemannian manifolds. This means that the elliptic elements of Γ (if any) basically cause no deviation from the usual eigenvalue asymptotics. In the case of the hyperbolic plane, Hejhal [10], p. 119 et seq. and Randol [15] proved that a term $\log T$ can be introduced in the denominator of the error term. The same improvement is actually possible in much greater generality. This was shown by Bérard [1], Kolk [12], [13], and Duistermaat, Kolk and Varadarajan [4], p. 89, Theorem 9.1. The same improvement will without doubt be possible in (5.12).

Corollary 5.6. (a) *The series*

$$\sum_{n=0}^{\infty}{}' |s_n|^{-\alpha}, \qquad \alpha \in \mathbf{R}$$

converges if and only if $\alpha > 3$.

(b)

$$\sum_{0<t_n\leqslant T} |s_n|^{-3} = \frac{v(\mathscr{F})}{2\pi^2}\log T + O(1) \quad \textit{for} \quad T \to \infty.$$

Proof. By partial summation we find for $n > m \geqslant N+1$, $t_m \neq 0$, $t_n < T < t_{n+1}$:

$$\sum_{k=m}^{n} |s_k|^{-\alpha} = [x^{-\alpha} A(x)]_{t_m}^{T} + \alpha \int_{t_m}^{T} x^{-\alpha-1} A(x)\, dx.$$

Weyl's asymptotic law (5.12) now implies both assertions. ∎

THEOREM 5.7. *Z and Ξ are entire functions of order three.*

Proof. Repeat the proof of Lemma 5.2. We now know from Corollary 5.6 that we must choose $p = 3$. Then the canonical product (5.6) is an entire function of order 3, and we have the representation (5.7), where q is a polynomial of degree at most 3. Hence Z and Ξ are entire functions of order at most 3. But the order of these functions is greater than or equal to the abscissa of convergence for the sequence of zeros for Z or Ξ, respectively, which is precisely equal to 3 by Corollary 5.6. Hence Z and Ξ are entire functions of order precisely 3. ∎

COROLLARY 5.8. *Let*

$$\Phi(s) = s^{2k} \prod_{n=0}^{\infty}{}' \left(1 - \left(\frac{s}{s_n}\right)^2\right) e^{(s/s_n)^2} \tag{5.13}$$

(cf. (5.6)) *be the canonical product for the sequence of zeros of Z (or Ξ). Then there exist real constants α, β such that Ξ and Z have canonical factorizations of the form*

$$\Xi(s) = \alpha e^{\beta s^2} \Phi(s), \tag{5.14}$$

$$Z(s) = \alpha \exp\left(\frac{v(\mathscr{F})}{6\pi} s^3 + \beta s^2 - Es\right) \Phi(s). \tag{5.15}$$

Proof. Since $p = 3$, the canonical product (5.6) has the form (5.13). An application of (4.10) yields that

$$\frac{1}{2s} \frac{\Xi'}{\Xi}(s) - \frac{1}{2s} \frac{\Phi'}{\Phi}(s) = \frac{1}{2t} \frac{\Xi'}{\Xi}(t) - \frac{1}{2t} \frac{\Phi'}{\Phi}(t)$$

($s, t \neq \pm s_n$ for all $n \geqslant 0$). Hence (5.14) follows, and since $s_n^2 \in \boldsymbol{R}$ for all n, we even see that α and β are real numbers. The factorization (5.15) is now trivial from (4.5). ∎

The constants α, β in (5.14) have obvious expressions in terms of certain Taylor coefficients of Ξ. It would be interesting to know if α and β are connected with other data of Γ or $-\Delta$: $\mathscr{D} \to L^2(\Gamma \backslash H)$, but we have no results in this direction.

6. Analogue of the Lindelöf hypothesis

This section is motivated by the remarkable analogy between the Riemann zeta-function and the Selberg zeta-function. We briefly recall some relevant facts from Titchmarsh [22], pp. 81–82 and p. 276 et seq. The Riemann zeta-function is of polynomial growth in vertical strips. This implies that the function

$$\mu(\sigma) := \inf\{\gamma \in \boldsymbol{R}\colon\ \zeta(\sigma+it) = O(|t|^{\gamma}) \text{ for } |t| \to \infty\}$$

is a well-defined, non-negative, convex downwards, monotonically decreasing function of $\sigma \in \boldsymbol{R}$ (cf. [21], sect. 9.41). Since $\zeta(\sigma+it)$ $(t \in \boldsymbol{R})$ is bounded for every fixed $\sigma > 1$, we have $\mu(\sigma) = 0$ for all $\sigma > 1$, and the functional equation of ζ yields $\mu(\sigma) = \frac{1}{2} - \sigma$ for all $\sigma < 0$. These equations hold by continuity also for $\sigma = 1$ and $\sigma = 0$, respectively. The precise value of $\mu(\sigma)$ is unknown for any value $\sigma \in]0, 1[$. The simplest possible hypothesis is that

$$\mu(\sigma) = \begin{cases} 0 & \text{for } \sigma \geqslant \frac{1}{2}, \\ \frac{1}{2} - \sigma & \text{for } \sigma \leqslant \frac{1}{2}, \end{cases} \tag{6.1}$$

since the function on the right-hand side of (6.1) has all the properties mentioned above. Conjecture (6.1) is known as the *Lindelöf hypothesis*. It is still unknown if the Lindelöf hypothesis is true, but it is known that the truth of the Lindelöf hypothesis follows from that of the Riemann hypothesis ([22], p. 283).

We now turn to the analogous problem for the Selberg zeta-function. Since we know that the analogue of the Riemann hypothesis is true for Z we expect the truth of an analogue of the Lindelöf hypothesis to be true as well. It is known that many estimates for the Selberg zeta-function of a fixed-point free cocompact discrete subgroup of $\mathrm{PSL}(2, \boldsymbol{R})$ are formally equal to corresponding estimates for ζ if one replaces $\log t$ by t in the estimates for ζ (see [9], [10]). In the case of dimension 3 we have to replace $\log t$ by t^2. Hence, in view of (5.8), (5.9) we introduce

$$M(\sigma) := \inf\left\{\gamma \in \boldsymbol{R}\colon\ Z(\sigma+it) = O\left(\exp\left(\gamma\,\frac{v(\mathscr{F})}{\pi}\,t^2\right)\right) \text{ for } t \to \infty\right\}$$

as a natural analogue for $\mu(\sigma)$. Since Z is of finite order, the Phragmén–Lindelöf theorem may be applied to Z, and the methods employed by Titchmarsh [21], sect. 9.41 yield the following general result.

Lemma 6.1. *The function M is a real-valued non-negative, convex downwards and monotonically decreasing function of $\sigma \in \boldsymbol{R}$.*

Obviously we have

$$M(\sigma) = 0 \quad \text{for} \quad \sigma \geqslant 1 \tag{6.2}$$

and hence by the functional equation for Z

$$M(\sigma) = |\sigma| \quad \text{for} \quad \sigma \leqslant -1. \tag{6.3}$$

The natural analogue of the Lindelöf hypothesis for the Selberg zeta-function now is

$$M(\sigma) = \begin{cases} 0 & \text{for } \sigma \geqslant 0, \\ |\sigma| & \text{for } \sigma \leqslant 0. \end{cases} \tag{6.4}$$

This is actually true and will be stated in Corollary 6.3. Adapting Littlewood's proof ([22], pp. 282–283) we can even prove the following sharper result.

THEOREM 6.2. *The estimate*

$$\log Z(\sigma+it) = O(|t|^{2(1-\sigma)} \log|t|) \quad \textit{for} \quad |t| \to \infty \tag{6.5}$$

holds uniformly with respect to σ, $(\log|t|)^{-1} \leqslant \sigma \leqslant 1$.

Proof. The function $Z(s)$ is bounded for $\operatorname{Re} s \geqslant 2$ and satisfies (5.8) uniformly for $\sigma \in [0, 1]$. Hence there exists a constant $C > 0$ such that

$$\log|Z(u+iv)| \leqslant Cv^2 \quad \text{for all } u \geqslant 0, |v| \geqslant 1. \tag{6.6}$$

In order to estimate $Z(w)$ for $\operatorname{Re} w \geqslant \delta > 0$, we apply the Borel–Carathéodory theorem ([21], sect. 5.5) to the circles with centre $2+it$ and with the radii $R = 2-\frac{1}{2}\delta$, $r = 2-\delta$ $(0 < \delta < 1)$. Then (6.6) yields

$$|\log Z(w)| \leqslant \frac{4(2-\delta)}{\delta} C(|t|+2)^2 + \frac{8-3\delta}{\delta} |\log Z(2+it)|$$

for $|t| \geqslant 3$, $|w-(2+it)| \leqslant r$. Hence there exists some constant $A > 0$, independent of δ, such that

$$|\log Z(u+iv)| \leqslant \frac{A}{\delta} v^2 \quad \text{for all } u \geqslant \delta, |v| \geqslant 3. \tag{6.7}$$

Now let $0 < \delta < \sigma < 1$, $s = \sigma+it$, $|t| \geqslant 3$ and put $\alpha := \delta^{-1}$. Let C_j denote the circle of radius r_j and centre $\alpha+it$, where

$$r_1 := \alpha-1-\delta, \quad r_2 := \alpha-\sigma, \quad r_3 := \alpha-\delta,$$

and

$$M_j := \max_{w \in C_j} |\log Z(w)| \quad (j = 1, 2, 3).$$

Then Hadamard's three-circles theorem gives

$$M_2 \leqslant M_1^{1-\lambda} M_3^{\lambda} \tag{6.8}$$

where

$$(6.9)\quad \lambda = \frac{\log(r_2/r_1)}{\log(r_3/r_1)} = \frac{\log\left(1+\dfrac{1+\delta-\sigma}{\alpha-1-\delta}\right)}{\log\left(1+\dfrac{1}{\alpha-1-\delta}\right)} = 1-\sigma+O(\delta) \quad \text{for} \quad \delta\to 0$$

uniformly with respect to σ. Now we obtain from (6.7)

$$(6.10)\qquad M_3 \leqslant \frac{A}{\delta}(|t|+\alpha)^2 \quad \text{for} \quad |t|-\alpha \geqslant 3$$

whereas M_1 is estimated as follows. Employing the notations of the proof of Lemma 4.3 we have for $w = u+iv$ with $u > 1$

$$\begin{aligned}
\log Z(w) &= -\sum_{\substack{T_0\in\mathscr{R}\\ k,l\geqslant 0,n\geqslant 1\\ k\equiv l \bmod m(T_0)}} \frac{1}{n} a(T_0)^{-2nk}\overline{a(T_0)}^{-2nl} N(T_0)^{-n(w+1)}\\
&= -\sum_{\substack{T_0\in\mathscr{R}\\ k,l\geqslant 0,n\geqslant 1\\ 0\leqslant \nu < m(T_0)}} \frac{1}{n}\frac{1}{m(T_0)} \zeta(T_0)^{-2\nu(k-l)} a(T_0)^{-2nk}\overline{a(T_0)}^{-2nl} N(T_0)^{-n(w+1)}\\
&= -\sum_{\substack{T_0\in\mathscr{R}\\ n\geqslant 1\\ 0\leqslant \nu < m(T_0)}} \frac{1}{n}\frac{1}{m(T_0)} \frac{N(T_0)^{-n(w+1)}}{|1-a(T_0)^{-2n}\zeta(T_0)^{-2\nu}|^2}
\end{aligned}$$

and hence

$$\begin{aligned}
|\log Z(w)| &\leqslant \sum_{\substack{T_0\in\mathscr{R}\\ n\geqslant 1}} \frac{1}{n}\frac{N(T_0)^{-n(u+1)}}{(1-N(T_0)^{-n})^2}\\
&\leqslant \sum_{T_0\in\mathscr{R}} \frac{-\log(1-N(T_0)^{-(u+1)})}{(1-N(T_0)^{-1})^2}\\
&\leqslant \eta \sum_{T_0\in\mathscr{R}} N(T_0)^{-u-1}
\end{aligned}$$

where $\eta > 0$ is such that $(1-N(T)^{-1})^{-2} \leqslant \eta$ for all hyperbolic or loxodromic elements $T\in\Gamma$. A partial summation based on the estimate (4.6) now yields

$$(6.11)\qquad |\log Z(w)| \leqslant \eta \sum_{\{T\}\ \text{lox.}} N(T)^{-u-1} \leqslant \frac{B}{u-1}$$

for some constant $B > 0$.

Choosing the constant $A \geqslant B$, we deduce from (6.11) that

$$M_1 \leqslant \frac{A}{\delta}. \tag{6.12}$$

Plugging (6.10) and (6.12) into (6.8) and using (6.9) we finally arrive at

$$|Z(\sigma+it)| \leqslant M_2 \leqslant \frac{A}{\delta}(|t|+\alpha)^{2(1-\sigma+O(\delta))} \tag{6.13}$$

where A is some absolute constant and where the O-constant is independent of σ and t. Hence we are free to choose $\alpha = \delta^{-1} = \log|t|$, where $|t|$ is sufficiently large. Then $(|t|+\alpha)^{O(\delta)} = O(1)$ for $|t| \to \infty$, and (6.5) follows from (6.13). ∎

COROLLARY 6.3. *Z satisfies the analogue of the Lindelöf hypothesis, i.e.,*

$$M(\sigma) = \begin{cases} 0 & \text{for } \sigma \geqslant 0, \\ |\sigma| & \text{for } \sigma \leqslant 0. \end{cases}$$

The proof is an immediate consequence of Lemma 6.1 and Theorem 6.2. Note that (6.5) yields in particular that for fixed σ, $0 < \sigma < 1$ and $\varepsilon > 0$ and for all $|t| > t_0(\varepsilon, \sigma)$

$$\big|\log|Z(\sigma+it)|\big| \leqslant \varepsilon|t|^{2(1-\sigma)+\varepsilon}.$$

This means that we have both

$$|Z(\sigma+it)| \leqslant \exp(\varepsilon|t|^{2(1-\sigma)+\varepsilon})$$

and

$$\frac{1}{|Z(\sigma+it)|} \leqslant \exp(\varepsilon|t|^{2(1-\sigma)+\varepsilon})$$

$(0 < \sigma < 1,\ \varepsilon > 0,\ |t| > t_0(\varepsilon, \sigma))$.

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