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LQ-OPTIMAL CONTROL WITH STABILIZATION CONSTRAINTS

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1. Introduction

The LQ-optimal control problem is concerned with regulating the *linear* time-invariant system

$$\dot{x}(t) = Ax(t)+Bu(t) \tag{1}$$

in such a way that the *quadratic* cost function

$$\eta = \int_{t_0}^{t_1} [x'(t)Qx(t)+u'(t)Ru(t)]dt+x'(t_1)Sx(t_1) \tag{2}$$

is minimized. In these expressions x and u are respectively the state and control vectors, A and B are matrices of appropriate dimensions, Q and S are positive semi-definite matrices, and R is a positive definite matrix. In this paper A, B, Q and R are assumed to be time-invariant. The above problem statement is a mathematical expression of the fact that one wants to design a control to drive the state or part of the state of the system to zero, without using excessive input energy. Moreover, at the final time t_1 the deviation of the state from the zero state has particular importance; it is therefore penalized separately. In many applications the penalization of the state has the following aspects:

(i) Some state variables should be kept small during the transient period; they are penalized by $x'(t)Qx(t)$.

(ii) Some state variables are not important during the transient period, but should be controlled towards zero; they are penalized by $x'(t_1)Sx(t_1)$, but not by $x'(t)Qx(t)$ in the integral term of η.

(iii) It may also occur that some state variables are not to be controlled; for example, for a machine speed control system the angular position may not be important. Such state variables are penalized neither in $x'(t)Qx(t)$, nor in $x'(t_1)Sx(t_1)$. It should also be pointed out that in many practical applications the transient interval $(t_1 - t_0)$ is rather artificial. To avoid this problem, an infinite time interval is considered: $t_1 \to \infty$. Without loss of generality, t_0 can be taken equal to zero. Then the problem statement is to control system (1), minimizing

$$\eta = \int_0^\infty [x'(t)Qx(t) + u'(t)Ru(t)]dt \tag{3}$$

with stabilization constraint:

$$\lim_{t_1 \to \infty} Fx(t_1) = 0, \tag{4}$$

where F is defined by

$$S = F'F.$$

Without loss of generality it can be assumed that F has full rank. Hence if the rank of S is r, then F is a matrix with r rows and n columns, where n is the dimension of the state x and F has rank r.

If the pair (A, B) is controllable or stabilizable, and if $Q = C'C$, with the pair (A, C) observable, or even detectable, then the solution to the above problem is the *standard optimal regulator*. The optimal control is the feedback strategy (Kwakernaak and Sivan [5]):

$$u(t) = -R^{-1}B'Kx(t), \tag{5}$$

where K is the unique positive semi-definite solution of the algebraic Riccati equation

$$A'K + KA - KBR^{-1}B'K = -Q. \tag{6}$$

Then the closed loop system is asymptotically stable, i.e., all eigenvalues of $(A - BR^{-1}B'K)$ have negative real parts. Moreover, the unique positive semi-definite solution is the only solution of (6) with the stabilization property. The optimal regulator is hence the optimal control strategy with respect to the cost function (3) even without the stabilization constraint (4).

If (A, C) is an undetectable pair, then the optimal control of (1) with respect to cost function (3) is also given by (5). However, the algebraic Riccati equation (6) has now several positive semi-definite solutions. This set of solutions forms a lattice with a largest element, the stabilizing solution, and a smallest element, as was pointed out by Kucera ([4]). The smallest solution yields the optimal control, but it does not stabilize the system. This control hence does not necessarily satisfy constraint (4).

In the present paper the solution of the LQ-control with stabilization constraints is discussed. In Section 2 it is shown that incorrect results may be obtained by considering the problem as the limiting case of an optimal control problem over a finite time interval with terminal constraints. The correct solution is derived in Section 3 and discussed in Section 4. Throughout the paper stabilizability of system (1) is assumed; this means that system (1) can be made asymptotically stable by linear time-invariant feedback. It is also supposed that the Hamiltonian matrix

$$H = \begin{bmatrix} A & -BB' \\ -C'C & -A' \end{bmatrix}$$

has no imaginary axis eigenvalues. These conditions are necessary and sufficient for the existence of a ***stabilizing*** solution of (6), i.e., a solution K of (6) such that system (1) with feedback control (5) is asymptotically stable. This is proved by Kucera [4]. For simplicity we only explicitly discuss the case that A has distinct undetectable eigenvalues.

2. Limiting case of a finite time interval problem

To solve the problem formulated in the previous section the limiting case of a finite time interval control problem is considered. The optimal control of system (1) is derived with respect to the cost function

$$\eta = \int_0^{t_1} (x'Qx + u'Ru)\,dt \tag{7}$$

with terminal constraints

$$Fx(t_1) = 0. \tag{8}$$

Afterwards the limiting behavior for $t_1 \to \infty$ is discussed. This control problem can be solved in two ways: a direct solution procedure leads to the optimal feedback (Bryson and Ho [2])

$$u(t) = -R^{-1}B'[K(t)+H(t)]x(t) \tag{9}$$

for $0 \leqslant t < t_1$, where $K(t)$ is the solution of the Riccati differential equation

$$\dot{K}(t) = -A'K(t)+K(t)A-K(t)BR^{-1}B'K(t)+Q \tag{10}$$

with terminal condition

$$K(t_1) = F'F. \tag{11}$$

The matrix $H(t)$ is given by

$$\Phi'(t_1, t)F'M(t)^{-1}F\Phi(t_1, t),$$

where $\Phi(t_1, t)$ is the transition matrix of the system

$$\dot{x}(t) = [A - BR^{-1}B'K(t)]x(t), \tag{12}$$

and where

$$M(t) := F\Big[\int_0^{t_1} \Phi(t_1, \tau)BR^{-1}B'\Phi(t_1, \tau)\,d\tau\Big]F'.$$

The invertibility of $M(t)$ for $t < t_1$ is implied by the controllability of system (1). Actually, the weaker property of *output controllability* should only be required; this is the property that there exists a control which drives $Fx(t)$ to zero at $t = t_1$. Note that $H(t)$ tends to infinity as $t \to t_1$. The behavior of $M(t)$ and $M(t)^{-1}$ is discussed in much detail by Brunovsky and Komornik [1].

An alternative procedure for deriving the optimal control of (1) with respect to the cost function (7) and constraint (8) is to consider the optimal control of (1) with respect to the cost function

$$\eta = \int_0^{t_1} (x'Qx + u'Ru)\,dt + ax'(t_1)F'Fx(t_1)$$

and to let the scalar a tend to $(+\infty)$. This algorithm is often called the *penalty function method*, because the state constraint is replaced by adding a term in the cost function. The constraint (8) is indeed replaced by a strong penalization of a non-zero $Fx(t_1)$. The method of solving the linear-quadratic optimal control problem with terminal condition as the limiting case of an unconstrained problem is analysed rigorously in a recent paper by Brunovsky and Komornik [1]. The following optimal control law, equivalent to (9), is obtained:

$$u(t) = -R^{-1}B'P_\infty(t)x(t),$$

where

$$P_\infty(t) = \lim_{a\to\infty} P_a(t) = K(t) + H(t).$$

The matrix $P_a(t)$ is the solution of

$$\dot{P}_a(t) = -A'P_a(t) - P_a(t)A + P_a(t)BR^{-1}B'P_a(t) - Q$$

with terminal condition

$$P_a(t_1) = aF'F.$$

It can easily be shown that $P_\infty(t)$ exists for all $t < t_1$.

Next the limiting behavior for $t_1 \to \infty$ of the obtained feedback law is considered. The results can be derived from the criteria developed by Callier and Willems [3]. Earlier useful results on the limiting behavior of the solution of the Riccati equation for increasing time interval have been obtained by Kucera [4]. In particular, it has been shown that the limiting behavior of $P_a(t)$, for $t_1 \to \infty$, is independent of a for all positive a. The main conclusions are:

(i) Depending on A, C, and F, the matrix $P_\infty(t)$ may or may not converge to a constant matrix as $t_1 \to \infty$.

(ii) If A, C, and F, satisfy the criterion derived by Callier and Willems [3], then $P_a(t)$, and hence also $P_\infty(t)$, tends to a constant matrix P_0 as $(t_1 - t) \to \infty$. This constant matrix is such that all modes, detectable with respect to the pair (A, C), are stabilized in the closed loop. Moreover, some undetectable modes are also stabilized, but some are not. A procedure has been developed by Callier and Willems [3] to identify the modes that are stabilized in

$$A - BR^{-1}B'P_0.$$

The number of unstable closed loop modes is the dimension of

$$\mathcal{N}\mathcal{D}(C, A) \cap \mathcal{N}(F),$$

where $\mathcal{N}$ denotes the null space, and where $\mathcal{N}\mathcal{D}(C, A)$ is the space spanned by the undetectable (generalized) eigenvectors of A, or the undetectable subspace of the system $\dot{x} = Ax$ with output $y = Cx$.

EXAMPLE 1. Consider the second-order system $(a > b > 0)$

$$\dot{x}_1 = ax_1 + u_1,$$
$$\dot{x}_2 = bx_2 + u_2$$

with cost function

$$\eta = \int_0^{t_1} (u_1^2 + u_2^2)\,dt$$

and constraint

$$x_1(t_1) + x_2(t_1) = 0.$$

Then the limiting value for $t_1 \to \infty$ of the optimal feedback is

$$u_1(t) = -2ax_1(t),$$
$$u_2(t) = 0.$$

The most unstable mode is stabilized; the least unstable mode is not stabilized. Hence this feedback does not yield

$$\lim_{t_1 \to \infty} [x_1(t_1) + x_2(t_1)] = 0.$$

EXAMPLE 2. The second-order system

$$\dot{x}_1 = x_1 + x_2 + u_1,$$
$$\dot{x}_2 = -x_1 + x_2 + u_2$$

has two unstable complex conjugate open loop modes. Consider the cost function

$$\eta = \int_0^{t_1} (u_1^2 + u_2^2)\, dt$$

and the constraint

$$x_1(t_1) = 0.$$

For $t_1 \to \infty$, the optimal control strategy tends to the feedback strategy

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} 2\cos^2(t-t_1) & -2\sin(t-t_1)\cos(t-t_1) \\ -2\sin(t-t_1)\cos(t-t_1) & 2\sin^2(t-t_1) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

where the feedback matrix is periodic.

Both examples show that the feedback

$$u(t) = -R^{-1}B'P_0 x(t)$$

obtained by letting $t_1 \to \infty$ in the problem statement (1), (7), and (8), is not always the solution of the infinite time control problem (1), (3), and (4) since the stabilization constraint is not necessarily satisfied. In the next section a correct solution procedure is developed. In the remainder of the present section an interpretation is given of the fact that the constraint

$$Fx(t_1) = 0$$

for $t_1 \to \infty$ does not imply stabilization of all system modes which are present in Fx. The reason is that stabilization is not equivalent to having Fx zero at some very distant instant of time.

For Example 1 consider an initial condition

$$x_1(0) = 0$$

such that, without control, only the least unstable mode is excited. Hence the optimal feedback derived above is zero for that initial condition. To explain this phenomenon, consider the trajectories of the uncontrolled system equations; one readily obtains

$$x_1(t_1) + x_2(t_1) = \exp(at_1)x_1(0) + \exp(bt_1)x_2(0).$$

This expression may be made zero by only using a control which transfers the state from

$$x_1(0) = 0, \quad x_2(0) = \beta$$

to

$$x_1(0+) = \exp[(b-a)t_1]\beta, \quad x_2(0+) = \beta.$$

As $t_1 \to \infty$, the control required to realize this transfer tends to zero.

In Example 2 the evolution of $x_1(t)$ for the uncontrolled system is

$$x_1(t_1) = [x_1(0)\cos(t_1) + x_2(0)\sin(t_1)]\exp(t_1).$$

For a given initial state the *uncontrolled* system state satisfies

$$x_1(t_1) = 0$$

for periodic values of t_1; for other values of t_1, a control is necessary to satisfy the constraint. The limit of the finite time interval control problem obviously does not lead to the solution of the infinite time interval control problem, because it can easily be seen that the optimal control for problem statement (1), (3), and (4), is time-invariant.

Brunovsky and Komornik [1] also analyse the limit of the linear-quadratic optimal control problem for increasing time interval. However, their approach is different; they consider a finite time problem without final state penalty and without terminal constraints. Then they consider the limit for infinite interval, and afterwards they analyse the stability properties of the obtained closed loop system; for time-invariant problems the limit always equals the smallest positive semi-definite solution of the Riccati equation, as is pointed out by Callier and Willems [3], such that no undetectable modes are stabilized. However, Brunovsky and Komornik's analysis is more general since they also consider non-stationary systems which are not necessarily stabilizable.

3. Alternative solution procedure

The problem of the optimal control of (1) with respect to the cost function (3) and constraint (4) can be solved by associating it with an infinite time LQ-problem *without stabilization constraint* in the following way. Consider the optimal control of (1) with respect to the cost function

$$\eta_\mu = \int_0^\infty [x'Qx + \mu x'Sx + u'Ru]\,dt,$$

where μ is a positive parameter. The optimal control is the feedback strategy

$$u(t) = -R^{-1}B'K_\mu x(t),$$

where K_μ is the *smallest* positive semi-definite solution of the algebraic Riccati equation

$$A'K_\mu + K_\mu A - K_\mu BR^{-1}B'K_\mu + Q = 0. \tag{13}$$

It follows from the structural properties of the algebraic Riccati equation that, as $\mu \to 0$, the matrix K_μ tends to a limiting value, which is denoted by K_0. This matrix K_0 satisfies equation (6), but it is not necessarily its smallest positive-definite solution. It is straightforward to conclude that the control

(14) $$u(t) = -R^{-1}B'K_0x(t)$$

is the optimal control of (1) with respect to cost function (3) and with the stabilization constraint (4). This feedback control exactly stabilizes these modes of (1) which are detectable from the output

$$y = \begin{bmatrix} C \\ F \end{bmatrix} x.$$

Hence K_0 is that solution of (6) which is supported, in the sense discussed by Molinari [6], by the invariant subspace spanned by the eigenvectors of A which are undetectable from the output which consists of the output $y_1 = Cx$ which appears in the cost function and the output $y_2 = Fx$ which is to be stabilized. For the examples considered in the previous section, the results are:

EXAMPLE 1. The optimal control of the system of this example with stabilization constraint

$$\lim_{t\to\infty}[x_1(t)+x_2(t)] = 0$$

and cost function

$$\eta = \int_0^\infty (u_1^2+u_2^2)\,dt$$

is the feedback strategy

$$u_1(t) = -2ax_1(t), \quad u_2(t) = -2bx_2(t).$$

The minimum cost equals

$$\eta_{\min} = 2ax_1(0)^2+2bx_2(0)^2.$$

This solution is different from the solution obtained by means of the limiting procedure of Section 2. The present solution is the correct one since the stabilization constraint is satisfied.

EXAMPLE 2. The optimal control of the system of the second example, with stabilization constraint

$$\lim_{t\to\infty} x_1(t) = 0$$

and with cost function

$$\eta = \int_0^\infty (u_1^2+u_2^2)\,dt$$

is the feedback strategy

$$u_1(t) = -2x_1(t), \quad u_2(t) = -2x_2(t).$$

The corresponding minimum cost is

$$\eta_{\min} = 2[x_1(0)^2 + x_2(0)^2].$$

4. Discussion

4.1. The limiting procedure discussed in Section 2 leads to the exact solution if and only if

$$K_0 = \lim_{t \to -\infty} P_\infty(t).$$

A necessary and sufficient condition is that the subspace

$$\mathcal{N}\mathcal{D}(C, A) \cap \mathcal{N}(F)$$

is A-invariant. A particular case occurs when F is the identity matrix. Then the correct solution is obtained by considering the limiting case of the finite time problem.

4.2. The invariant subspace supporting the solution K_0 of (6) is the largest A-invariant subspace contained in

$$\mathcal{N}\mathcal{D}(C, A) \cap \mathcal{N}(F).$$

It is the invariant subspace

$$\mathcal{N}\mathcal{D}\left(\begin{bmatrix} C \\ F \end{bmatrix}, A\right).$$

This is the undetectable subspace of the system $\dot{x} = Ax$ with respect to the output consisting of Cx and Fx. The optimal control stabilizes all modes except those which are at the same time undetectable with respect to Cx_1 and contained in $\mathcal{N}(F)$. Hence, if F is the identity matrix, then the optimal control stabilizes all modes. Then K_0 is the largest positive semi-definite solution of (6). As was pointed out in 4.1, it is also the limiting case of the finite time problem. These results are equivalent with Theorem 4 of Brunovsky and Komornik [1].

4.3. The solution of problem statement (1), (3), and (4), leads to an asymptotically stable closed loop system if and only if system (1) with output

$$y = \begin{bmatrix} C \\ F \end{bmatrix} x$$

is detectable. Then the largest solution of (6) should be used (Kucera [4]). Hence, *as far as stabilization is concerned*, penalization of an output in the cost function or a stabilization constraint with respect to that output are equivalent.

4.4. The stabilization constraint (4) does not affect the solution of the optimal control problem if and only if

$$\mathscr{N}\mathscr{D}(C, A) \subset \mathscr{N}(F)$$

or, equivalenty, if

$$\mathscr{N}\mathscr{D}(C, A) = \mathscr{N}\mathscr{D}\left(\begin{bmatrix} C \\ F \end{bmatrix}, A\right).$$

Then no eigenvalues of A which are undetectable from the output Cx are stabilized by the control strategy (14). The solution K_0 of (6) which should be used here is its smallest positive semi-definite solution (Kucera [4]). This corresponds to Theorem 2 of Brunovsky and Komornik [1] for the time-invariant case.

4.5. The solution of the optimal control problem (1), (3), and (4) can always be derived as the limiting case of the following finite time interval problem where the time interval tends to infinity, provided appropriate terminal state constraints are introduced. The finite interval problem consists of optimally controlling system (1) with cost function (7) and *with terminal constraint*

$$Wx(t_1) = 0,$$

where

$$W := F'F + A'F'FA + \ldots + (A')^{n-1}F'F(A)^{n-1}.$$

The optimal feedback control strategy can hence be derived from the solution of the Riccati differential equation (10) with

$$K(t_1) = W$$

and $t_1 \to \infty$. Note that W has been constructed in such a way that

$$\mathscr{N}(W) \cap \mathscr{N}\mathscr{D}(C, A)$$

is A-invariant since $\mathscr{N}(W)$ is A-invariant.

4.6. Intuitively one might expect that the introduction of a final state penalization in (2) may represent the stabilization constraint for large time intervals. The question thus arises whether the cost function (2) leads to the stabilization

$$\lim_{t \to \infty} Sx(t) = 0$$

when the control interval $(t_1 - t_0)$ tends to infinity. Here again the answer is that this is the case if and only if

$$\mathscr{N}(S) \cap \mathscr{N}\mathscr{D}(C, A)$$

is an A-invariant subspace. In particular, the final state penalization leads to an asymptotically stable closed loop system iff

$$\mathscr{N}(S) \cap \mathscr{N}\mathscr{D}(C, A) = \{0\},$$

as is shown by Kucera [4] or Callier and Willems [3]. The stabilization constraint (4), however, leads to an asymptotically stable closed loop system iff the largest A-invariant subspace contained in

$$\mathcal{N}(S) \cap \mathcal{N}\mathcal{D}(C, A)$$

is $\{0\}$. To ensure a stabilization constraint by means of final state penalization the procedure of 4.5 should hence be used.

5. Conclusion

In this paper the optimal control of a linear system with respect to a quadratic cost function and stabilization constraints has been analysed. It was shown that incorrect results may be obtained by considering the limiting solution of a finite time interval control problem with terminal state constraint. The correct solution was derived; that solution of the algebraic Riccati equation should be selected which exactly stabilizes the modes in the stabilization constraint.

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