SEQUENTIAL METHODS IN STATISTICS
BANACH CENTER PUBLICATIONS, VOLUME 16
PWN-POLISH SCIENTIFIC PUBLISHERS
WARSAW 1985

A GENERALIZED FUNDAMENTAL IDENTITY

INGEBORG KÜCHLER

*Section of Mathematics, Technical University of Dresden,
Dresden, G.D.R.*

1. Introduction

Let $(Y_k)_{k\geq 1}$ be a sequence of independent identically distributed random variables with the distribution function F such that $\phi(u) := \ln E\exp(uY_1) < \infty$ for some $u \neq 0$ and put $X_n := \sum_{k=1}^{n} Y_k$. If τ is a stopping time, then under some conditions the so-called *Wald fundamental identity*

$$E\exp\left(uX_\tau - \phi(u)\tau\right) = 1 \tag{1}$$

holds (see, e.g., [2], [15]). This identity has turned out to be substantial for sequential statistics and has been generalized by several authors (see, e.g., [4], [6], [9], [12]). We shall mention here a generalized version of (1) proved by Bahadur ([1]). If τ is any finite stopping time for $(X_n)_{n\geq 1}$ then

$$E\exp\left(uX_\tau - \phi(u)\tau\right) = P^*(\tau < \infty),$$

where P^* is the probability measure corresponding to the sequence $(X_n)_{n\geq 1}$ and the Y_k, $k \geq 1$, are independent identically distributed with the distribution function G where G is defined by

$$dG(x) = \exp\left(ux - \phi(u)\right)dF(x).$$

Let $(W_t)_{t\geq 0}$ be a standard Wiener process on the real line and τ a stopping time with $E\exp(u^2 2^{-1}\tau) < \infty$ for some real u. Then this u satisfies the fundamental identity

$$E\left(\exp(uW_\tau - u^2 2^{-1}\tau)\right) = 1$$

This is a particular case of a result of Novikov ([11]). Shepp ([13]) proved a more general result. He considered stopping times τ of the form

$$\tau := \inf\{t > 0 \mid W_t = f(t)\},$$

where f is an arbitrary continuous function on $[0, \infty)$, and showed that without further assumptions on τ for every real u we have

$$E\left(\exp(uW_\tau - u^2 2^{-1}\tau); \tau < \infty\right) = P\left(W_t + ut = f(t) \text{ for some } t > 0\right). \quad (2)$$

In this note we shall extend Shepp's result (2) to one-dimensional processes with independent stationary increments and arbitrary stopping times. The method used below can also be partially applied to more general processes, e.g., to processes with independent nonstationary increments or to semimartingales, by using their local characteristics (see [10]). Finally, let us mention that very general results in the theory of absolute continuity of stochastic processes, which can be interpreted as fundamental identities in a certain sense, can be found in [3], [5], [14].

2. Generalized fundamental identity

Let R be the set of real numbers and $(\Omega, \mathfrak{A}, P)$ a probability space, and $(X_t)_{t\geq 0}$ a process with independent stationary increments on $(\Omega, \mathfrak{A}, P)$. Without restrictions of generality we can assume for our aims that $(\Omega, \mathfrak{A})$ is a canonical space, i.e., that Ω is the set of all right-continuous real-valued functions on $[0, \infty)$ having limits from the left, $\mathfrak{A}$ is σ-algebra generated by all cylindric sets and $X_t(\omega) = \omega_t$ is the coordinate projection. In this sense the probabilistic behaviour of $(X_t)_{t\geq 0}$ is given by P; we shall identify $(X_t)_{t\geq 0}$ and P.

Furthermore, it is known for stochastically continuous processes with independent stationary increments that the characteristic function $\psi_P(t, \lambda)\colon = E_P(\exp i\lambda X_t)$ of X_t under P has the form

$$\psi_P(t, \lambda) = \exp t\left[i\gamma\lambda - \frac{\lambda^2\sigma^2}{2} + \int\limits_{R\setminus\{0\}} \left(\exp(i\lambda y) - 1 - \frac{i\lambda y}{1+y^2}\right)\nu(dy)\right]$$

$$(t > 0,\ \lambda \in R).$$

The parameters γ, σ^2, ν, where $\gamma \in R$, $\sigma^2 \geq 0$ and ν is a σ-finite measure on $R\setminus\{0\}$ satisfying the condition $\int\limits_{R\setminus\{0\}} \frac{y^2}{1+y^2}\nu(dy) < \infty$, are called the *Lévy-characteristics of the process P*. Conversely, for every triple γ, σ^2, ν with the above properties there is a process P with independent stationary increments having γ, σ^2, ν as its Lévy-characteristics. We shall denote by $\mathfrak{A}_t$ the σ-algebra generated by $\{X_s, s \leq t\}$ and, for an arbitrary stopping time τ, by $\mathfrak{A}_\tau$ the σ-algebra generated by all sets of the form $\{A \in \mathfrak{A} | A \cap \{\tau \leq t\} \in \mathfrak{A}_t\}$.

THEOREM. *We choose a fixed $u \in R$. Then the following properties are equivalent*:

(i) $\int\limits_{R} \exp(uX_t)\,dP < \infty$ *for some* $t > 0$,

(ii) $\int\limits_{R\setminus\{0\}} \frac{y^2}{1+y^2} \exp(uy)\, \nu(dy) < \infty$.

If these properties hold, the following identity is valid for every stopping time τ:

$$\int\limits_{A\cap\{\tau<\infty\}} \exp\big(uX_\tau - v(u)\tau\big)\,dP = P^{(u)}(A \cap \{\tau < \infty\}), \qquad A \in \mathfrak{A}_\tau, \tag{3}$$

where

$$v(u) = u\gamma + \frac{u^2\sigma^2}{2} + \int\limits_{R\setminus\{0\}} \left(\exp(uy) - 1 - \frac{uy}{1+y^2}\right)\nu(dy) \tag{4}$$

and $P^{(u)}$ represents the process with independent stationary increments having the Lévy-characteristics γ_u, σ_u^2, ν_u given by the Lévy-characteristics of P and the chosen $u \in R$; in particular we have

$$\begin{aligned} \gamma_u &= \gamma + \sigma^2 u + \int\limits_{R\setminus\{0\}} \frac{y}{1+y^2}\big(\exp(uy) - 1\big)\,\nu(dy), \\ \sigma_u^2 &= \sigma^2, \\ d\nu_u(y) &= \exp(uy)\,d\nu(y) \end{aligned} \tag{5}$$

(see also [10]).

Proof. The equivalence of the two properties (i) and (ii) is known. In a note by Küchler and Küchler ([7]) about the so-called exponential class of processes with independent stationary increments it is shown that a process with independent stationary increments not being a deterministic motion fulfils properties (i) and (ii) for $u \neq 0$ iff it is a process from the exponential class of processes with independent stationary increments.

The generating function for processes from this class has the form

$$E\exp(uX_t) = \exp\big(v(u)t\big), \qquad \forall u \in R_P,$$

with $R_P := \{u \in R\colon \text{(i) holds}\}$.

Furthermore, the process

$$Z_t := \exp\big(uX_t - v(u)t\big) \tag{6}$$

is a martingale. Now let us apply a stopping theorem of martingales. τ is an arbitrary stopping time. Then, for every fixed $t > 0$, the stopping time $\tau \wedge t$:

$= \min(\tau, t)$ is bounded, and there holds a stopping theorem

$$E(Z_t | \mathfrak{A}_{\tau \wedge t}) = Z_{\tau \wedge t}.$$

The event $A \cap \{\tau \leqslant t\}$ belongs to $\mathfrak{A}_{\tau \wedge t}$ because this event is equal to the event $A \cap \{\tau \leqslant \tau \wedge t\}$. We have

$$\int_{A \cap \{\tau \leqslant t\}} Z_t \, dP = \int_{A \cap \{\tau \leqslant t\}} Z_{\tau \wedge t} \, dP = \int_{A \cap \{\tau \leqslant t\}} Z_\tau \, dP \xrightarrow[t \uparrow \infty]{} \int_{A \cap \{\tau < \infty\}} Z_\tau \, dP,$$

where the last integral is the left-hand side of (3).

On the other hand, let us show that the integral $\int_{A \cap \{\tau \leqslant t\}} Z_t \, dP$ converges to the right-hand side of (3). We first define for fixed $u \in R_P$ and arbitrary $t > 0$ a one-dimensional measure

$$P_t^{(u)}(dx) := \exp\big(ux - v(u)t\big) P_t(dx)$$

with $P_t(dx) = P(X_t \in dx)$. There exists a process $P^{(u)}$ with independent stationary increments and one-dimensional distributions

$$P_t^{(u)}(dx) = P^{(u)}(X_t \in dx).$$

The Lévy-characteristics of $P^{(u)}$ can be calculated and are given in (5). It is easy to show that we have

$$P^{(u)}(C) = \int_C Z_t \, dP, \qquad C \in \mathfrak{A}_t, \ t > 0.$$

Because of $A \cap \{\tau \leqslant t\} \in \mathfrak{A}_t$ we get

$$\int_{A \cap \{\tau \leqslant t\}} Z_t \, dP = P^{(u)}(A \cap \{\tau \leqslant t\}) \xrightarrow[t \uparrow \infty]{} P^{(u)}(A \cap \{\tau < \infty\}).$$

Thus formula (3) has been proved. ∎

EXAMPLES. 1. Assume that P is the standard Wiener process. Then $\gamma = 0$, $\sigma^2 = 1$, $\nu(\cdot) \equiv 0$. Obviously (ii) holds for every $u \in R$. We have $\gamma_u = u$, $\sigma_u^2 = 1$, $\nu_u(\cdot) \equiv 0$, i.e., $P^{(u)}$ is the Wiener process with diffusion constant 1 and drift coefficient u. The theorem implies for every stopping time

$$\int_{A \cap \{\tau < \infty\}} \exp\big(uX_\tau - (u^2/2)\tau\big) dP = P^{(u)}(A \cap \{\tau < \infty\}), \qquad A \in \mathfrak{A}_\tau.$$

In particular, for $A = \Omega$ and $\tau = \inf\{t > 0\colon X_t = f(t)$, f being an arbitrary continuous function$\}$, we get Shepp's result (2).

2. Let us consider a compounded Poisson process, in particular the difference of two independent Poisson processes with the parameters λ and μ. Then its Lévy-characteristics have the form

$$\gamma = \tfrac{1}{2}(\lambda - \mu), \qquad \sigma^2 = 0, \qquad \nu(A) = \lambda \chi_A(1) + \mu \chi_A(-1),$$

where $\chi_A(1)$ is the indicator variable of the event A. We have $E\exp(uX_t) < \infty$ $(u \in R)$, and thus $R_P = R$. The measure $P^{(u)}$ has the Lévy-characteristics $\sigma_u^2 = \sigma^2 = 0$,

$$\nu_u(A) = \chi_A(1)\lambda\exp(u) + \chi_A(-1)\mu\exp(-u),$$

$$\gamma_u = \tfrac{1}{2}(\lambda\exp u - \mu(\exp)(-u)),$$

i.e., $P^{(u)}$ is again the difference of two independent Poisson processes, but with the parameters $\lambda\exp u$ and $\mu\exp(-u)$ respectively. If we choose $\lambda \geqslant \mu$ and $\tau_k := \min\{t > 0\colon X_t = k,\ k \in N\}$, then we get $P^{(u)}(\tau_k < \infty) = 1$ for every $u \geqslant 0$. Therefore the generalized fundamental identity (3) has the form

$$E\exp(uk - v(u)\tau_k) = 1$$

with $v(u) = \lambda[\exp u - 1] + \mu[\exp(-u) - 1]$.

From this identity we are able to calculate the generating function $E\exp(v\tau_k)$ of the stopping time τ_k, where v is a parameter. Thus we have to determine the parameter u as a function of v. Since $\lambda \geqslant \mu$, we have $v'(u) > 0$ for every $u \geqslant 0$, and thus we can construct the inverse function $u(v)$ of the function $v(u)$. We obtain

$$u(v) = \ln\frac{1}{2\lambda}\left(\lambda + \mu + v + \sqrt{(\lambda + \mu + v)^2 - 4\lambda\mu}\right).$$

Now we get the generating function of the stopping time τ_k

$$E\exp(v\tau_k) = \exp(u(v))^k, \qquad v \geqslant 0.$$

3. We look at an arbitrary process $(X_t)_{t \geqslant 0}$ of the exponential class of processes with independent stationary increments belonging to the probability measure P_ϑ, $\vartheta \in \Theta_P := \{\vartheta(u),\ u \in R_P\}$ (see also [10], [16]). Let the parameter ϑ be unknown; we want to test the hypothesis H_0: $\vartheta = \vartheta_0$ against the hypothesis H_1: $\vartheta = \vartheta_1$ $(\vartheta_0 \neq \vartheta_1;\ \vartheta_0, \vartheta_1 \in \Theta_P)$. Then the sequential probability ratio test (SPRT), first investigated by Wald ([15]) for the unknown parameter ϑ of a distribution function F, has the following form. We observe the likelihood ratio process

$$\Lambda_t := \frac{dP_{\vartheta_1|\mathfrak{A}_t}}{dP_{\vartheta_0|\mathfrak{A}_t}}, \qquad t > 0,$$

as long as the process $S_t := \ln\Lambda_t$ first leaves a certain interval (a, b), where a and b depend on the chosen values of the errors of the first and second kind. Then we decide for ϑ_1 or ϑ_0 if $S_t \geqslant b$ or $S_t \leqslant a$ respectively. The stopping time τ of the SPRT is defined by

$$\tau := \inf\{t\colon S_t \notin (a, b)\} \qquad \text{with} \quad \inf\emptyset := \infty.$$

By means of the generalized fundamental identity, the characteristics of this test, the operating characteristic function $L(\vartheta) := P_\vartheta(S_\tau \leqslant a)$ and the average sample time $E_\vartheta(\tau)$ can be calculated. But we still have to replace the process $(X_t)_{t\geq 0}$ by the process $(S_t)_{t\geq 0}$, also belonging to the exponential class. We get $L(\vartheta)$ if we divide the left-hand side of the identity into conditional expected values with the conditions $S_\tau \leqslant a$ and $S_\tau \geqslant b$. By differentiating the identity with respect to ϑ at $\vartheta = 0$ we get Wald's equations and from those the average sample time function $E_\vartheta(\tau)$.

References

[1] R. R. Bahadur, *A note on the fundamental identity of sequential analysis*, Ann. Math. Statist. **29** (1958), 534–543.

[2] J. L. Doob, *Stochastic Processes*, Wiley, New York 1953.

[3] H. Föllmer, *The exit measure of a supermartingale*, Z. Wahrsch. Verw. Gebiete **21** (1972), 154 166.

[4] J. Franz and W. Winkler, *Über Stoppzeiten bei statistischen Problemen für homogene Prozesse mit unabhängigen Zuwächsen*, Math. Nachr. **70** (1976), 37–53.

[5] Ju. M. Kabanov, P. S. Liptser, A. N. Shiryayev, *K voprosu ob absolutnoi niepiererywnosti i singularnosti verojatnostnych mer*, Mat. (N.S.) **104** (1977), 227–247.

[6] I. Küchler, *Der Sequentielle Quotiententest bei irreduziblen homogenen Markovschen Ketten mit endlichem Zustandsraum*, Math. Operationsforsch. Statist. Ser. Statist. **9** (1978), 227–239.

[7] I. Küchler and U. Küchler, *An analytical treatment of exponential families of stochastic processes with independent stationary increments*, Math. Nachr. **103** (1981),21-30.

[8] –, –, *A generalized fundamental identity for processes with independent stationary increments*, Preprint 07–33–80, Technical University of Dresden.

[9] I. Küchler und A. Semjonov, *Die Waldsche Fundamentalidentität und ein Sequentieller Quotiententest für eine zufällige Irrfahrt über einer homogenen irreduziblen Markovschen Ketten mit endlichem Zustandsraum*, Math. Operationsforsch. Statist. Ser. Statist. **10** (1979).

[10] U. Küchler, *Exponential families of Markov processes, I*, Math. Operationsforsch. Stat. Ser. Statist. **13** (1982), 57–69.

[11] A. A. Novikov, *On an identity for stochastic integrals*, Teor. Verojatnost. i Primenen. **XVII** (1972), 761–765, in Russian.

[12] R. M. Phatarfod, *Sequential analysis of dependent observations, I*, Biometrika **52** (1965), 157–165.

[13] L. A. Shepp, *Explicit solutions to some problems of optimal stopping*, Ann. Math. Statist. **40** (1969), 993–1010.

[14] V. N. Sudakov, *On measures, determined by Markov times*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) vol. 12, Issledovanija po teorii slučainych processov, Nauka, Moskva 1969, 157–164, in Russian.

[15] A. Wald, *Sequential Analysis*. Wiley, New York 1947.

[16] W. Winkler, *A survey on sequential estimation in processes with independent increments*, this volume.

Presented to the semester
Sequential Methods in Statistics
September 7–December 11, 1981