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PRIME NUMBERS IN SHORT INTERVALS

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One of the most interesting problems in analytic number theory is to determine a possibly slowly increasing function $y(x)$ such that the interval

$$[x-y(x),\, x] \tag{1.1}$$

would contain a prime number for $x > x_0(y)$. The result of Chebyshev,

$$0.92129 < \frac{\pi(x)}{(x/\log x)} < 1.10555 \qquad (x > x_0) \tag{1.2}$$

implies (1.1) with $y(x) = 0.17x$. The prime number theorem without a remainder term yields $y(x) = \varepsilon x$ $(x > x_0(\varepsilon))$ whereas any form of it with a remainder term implies a better estimate for $y(x)$. The so-called quasi Riemann hypothesis,

$$\zeta(s) \neq 0 \quad \text{for} \quad \sigma > \vartheta,\ \vartheta < 1, \tag{1.3}$$

would imply (1.1) with

$$y(x) = x^{\theta} \tag{1.4}$$

for any $\theta > \vartheta$, even in the stronger form

$$\pi(x) - \pi(x-y) \sim y/\log x. \tag{1.5}$$

In particular, the Riemann hypothesis yields $y(x) = O(\sqrt{x}\log^2 x)$, or with a further idea of Cramér this can be improved to give $y(x) = O(\sqrt{x}\log x)$.

While (1.3) is undecided at present, Hoheisel, in 1930, succeeded in proving (1.4)–(1.5) for $\theta > 1-(33\,000)^{-1}$. He derived his result from two theorems which had been already known for nearly ten years and seemed to be of no or only very slight arithmetical consequence. The first of them was a theorem of Carlson which (with a slight modification due to Hoheisel) asserts

$$(1.6)\qquad N(\sigma, T) := \sum_{\substack{\zeta(\beta+i\gamma)=0 \\ \beta \geq \sigma, 0<\gamma\leq T}} < c_0 T^{\lambda(\sigma)(1-\sigma)} \log^6 T, \quad \lambda(\sigma) = 4\sigma.$$

The second was the zero free region proved by Littlewood,

$$(1.7)\qquad \zeta(s) \neq 0 \quad \text{for} \quad \sigma > 1 - \frac{A \log\log t}{\log t} \quad (A = \text{const.} > 0).$$

Using the shortened explicit prime number formula of Riemann and Von Mangoldt, one obtains with $T = x^{1+\varepsilon}/y$

$$(1.8)\qquad \Big| \sum_{x-y<p^m\leq x} \log p - y \Big| = \Big| \sum_{|\gamma|\leq T} \frac{x^\varrho - (x-y)^\varrho}{\varrho} \Big| + O(yx^{-\varepsilon/2})$$
$$\leq y \sum_{|\gamma|\leq T} x^{\beta-1} + o(y)$$

(where $\varrho = \beta + i\gamma$ runs through the non-trivial zeros of $\zeta(s)$) and it is clear that (1.6)–(1.7) can be used to prove Hoheisel's theorem. It is easy to see that if A can be chosen arbitrarily large (which was shown to be a consequence of Vinogradov's estimate of trigonometric sums by Chudakov in 1936), then (1.4)–(1.5) can be proved for

$$(1.9)\qquad \theta > 1 - \frac{1}{\lambda}, \quad \lambda = \max_{1/2\leq\sigma\leq 1} \lambda(\sigma).$$

In this way Chudakov's and Carlson's theorems yielded $\theta = 3/4 + \varepsilon$. In 1940 Ingham proved that if

$$(1.10)\qquad \zeta(\tfrac{1}{2} + it) = O(t^b)$$

then one may take

$$\lambda = 2 + 4b, \quad \text{and so} \quad \theta = \frac{1+4b}{2+4b} + \varepsilon.$$

The result of Hardy and Littlewood, $b = 1/6 + \varepsilon$ implied

$$(1.11)\qquad \lambda = \tfrac{8}{3} + \varepsilon, \quad \theta = \tfrac{5}{8} + \varepsilon.$$

Slight improvements in the value of b led automatically to slightly better values of λ and θ. In 1970 Montgomery proved (1.6),

$$(1.12)\qquad \lambda = \tfrac{5}{2} + \varepsilon \quad \text{and so} \quad \theta = \tfrac{3}{5} + \varepsilon,$$

without making use of any estimate of the (1.10) type. Soon after his result Huxley [4] refined his method to yield

$$(1.13)\qquad \lambda = \tfrac{12}{5} + \varepsilon, \quad \theta = \tfrac{7}{12} + \varepsilon.$$

(For exact references concerning the results mentioned earlier see, e.g., Montgomery [9].)

Using another approach, avoiding density theorems completely, Heath-Brown [2] has recently proved (1.4)–(1.5) with

$$y(x) = x^{7/12-\varepsilon(x)}, \tag{1.14}$$

for any $\varepsilon(x) \to 0$. This is the best result of this type at present. The famous density hypothesis,

$$\lambda = 2 \quad \text{or} \quad \lambda = 2+\varepsilon \tag{1.15}$$

yields

$$\theta = \tfrac{1}{2}+\varepsilon. \tag{1.16}$$

As was pointed out by Heath-Brown, Littlewood's original zero-free region (1.7) suffices, if one uses "log-free" zero-density theorems, e.g., result of Jutila [8],

$$N(\sigma, T) \ll T^{(12/5+\varepsilon)(1-\sigma)}. \tag{1.17}$$

It is even sufficient to have

$$\zeta(s) \neq 0 \quad \text{for} \quad \sigma > 1-\frac{A}{\log t} \tag{1.18}$$

for an arbitrary large A, and this, combined with (1.17), leads to Huxley's theorem $\theta > 7/12$. We may further note that to prove (1.1)–(1.4) with a $\theta < 1$ it is sufficient to work with the original zero-free domain of De la Vallée-Poussin (which corresponds to (1.18) with $A = 0.032$, $t > t_0$) if one has an estimate of type

$$N(\sigma, T) \ll T^{\lambda(1-\sigma)}. \tag{1.19}$$

Finally, we note that Cramér conjectured that (1.1) is true for

$$y(x) = (1+\varepsilon)\log^2 x, \qquad x > x_0(\varepsilon) \tag{1.20}$$

but not for

$$y(x) = (1-\varepsilon)\log^2 x. \tag{1.21}$$

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However Iwaniec and Jutila [7] showed in 1979 by a combination of sieve methods and weighted zero-density estimates that in problem (1.1)–(1.4) one can take $\theta = 13/23$. Further, they sketched what type of refinements might lead to $\theta > 5/9 = 1/2+1/18$. Their method was not capable of showing (1.5), but besides (1.1) they proved

$$\pi(x)-\pi(x-y) \gg \frac{y}{\log x}. \tag{2.1}$$

The main idea of their proof is that with a $z \in (x^{1/3}, x^{1/2})$

$$\pi(x)-\pi(x-y) = \sum_{\substack{x-y<p\leqslant x\\ x-y<pq\leqslant x\\ z<p\leqslant q}} 1 - \sum_{\substack{x-y<pq\leqslant x\\ z<p\leqslant q}} 1, \tag{2.2}$$

and the first sum can be estimated non-trivially from below by using the new, bilinear form of the remainder term of Rosser's sieve, given by Iwaniec [5]. The treatment of the second sum by an upper bound sieve alone cannot be sufficient to obtain a final positive lower estimate for $\pi(x)-\pi(x-y)$, because of the so-called parity phenomenon. But part of the second sum may be evaluated asymptotically. For any single $p > x^{1/3}$ one has, for $\theta \leqslant 7/12$

$$\frac{y}{p} < \left(\frac{x}{p}\right)^{3/8}, \tag{2.3}$$

so naturally one has no hope of evaluating

$$\pi(x/p)-\pi((x-y)/p) \quad \text{or} \quad \psi(x/p)-\psi((x-y)/p).$$

However, when one deals with

$$S_P = \sum_{P<p\leqslant 2P} \psi\left(\frac{x}{p}\right)-\psi\left(\frac{x-y}{p}\right) \quad \left(\psi(x) = \sum_{p^m\leqslant x} \log p\right) \tag{2.4}$$

the situation changes favourably. Namely, analogously to (1.8), this leads to the estimation of

$$\sum_{|\gamma|\leqslant x^{1+\varepsilon}/y} |K(\varrho)|\, x^{\beta-1}, \quad K(\varrho) = \sum_{P<p\leqslant 2P} \frac{1}{p^{\varrho}}. \tag{2.5}$$

This makes it possible to use the mean-value theorem and the Halász–Montgomery inequality in proving that the weights $K(\varrho)$ are much less than 1 on the average, and this enables us to treat (2.4) for a θ with $6/11 < \theta$. As regards the sieve part, it is interesting to note that by the classical, trivial treatment of the remainder term of the sieve any positive lower bound for the first sum would need the requirement $y > z^2 > x^{2/3}$ (even slightly more), which case was already solved by Ingham's result of 1940.

The linear sieve gives the following lower bound for the first sum Σ_1 in (2.2):

$$\Sigma_1 \geqslant \frac{y}{\log D}\left\{2 \log\left(\frac{\log D}{\log z}-1\right)-c\varepsilon\right\}-R(D) \tag{2.5}$$

where c is an absolute constant and ε is arbitrarily small positive constant. $R(D)$ may be expressed by

$$(2.6)\qquad r(d) = \left[\frac{x}{d}\right] - \left[\frac{x-y}{d}\right] - \frac{y}{d}, \quad 1 \leqslant d \leqslant D$$

and D is an important parameter for which $z^4 > D > z^2$ is required in the present case. The bilinear form of the remainder term can be written as

$$(2.7)\qquad R(D) = \sum_{l < c(\varepsilon)} \sum_{m < M} \sum_{\substack{n < N \\ mn \mid \prod_{p < z} p}} a_m(l)\, b_n(l)\, r(mn),$$

for any M, N with $M, N > 1$, $MN = D$, where $|a_m(l)|$, $|b_n(l)| \leqslant 1$. Thus, in order to have a negligible remainder term in (2.5), it is sufficient to show that

$$(2.8)\qquad R(M, N) = \sum_{M < m \leqslant 2M} \sum_{N < n \leqslant 2N} a_m b_n r(mn) \ll yx^{-\varepsilon}$$

(where, in the sequel, we allow the symbol $\ll$ to depend on ε) for $M, N \leqslant \sqrt{D}$ and arbitrary $|a_m|$, $|b_n| \leqslant 1$. Iwaniec and Jutila [7] have proved this for

$$(2.9)\qquad \theta > 5/9 \quad \text{and} \quad D < x^{\min(1,(5\theta-1)/2)-\varepsilon}.$$

This was extended by Heath-Brown and Iwaniec [3] for

$$(2.10)\qquad \theta > 11/20 \quad \text{and} \quad D < x^{\min(1,(12\theta-2)/5)-\varepsilon}.$$

In addition, they were able to evaluate some terms in the linear sieve, which had been ignored by Iwaniec and Jutila. In this way they succeeded in extending the range of validity of (2.1) to the value $\theta > 11/20 = 1/2 + 1/20$ ([3]), given by (2.10), and this was the limit of their method. We shall briefly sketch the method in proving (2.8)–(2.10) and point out how the constant 11/20 appears.

Introducing the notation

$$(2.11)\qquad M(s) = \sum_{M < m \leqslant 2M} \frac{a_m}{m^s}, \quad N(s) = \sum_{N < n \leqslant 2N} \frac{b_n}{n^s}, \quad L(s) = \sum_{L/5 < l \leqslant 2L} \frac{1}{l^s}$$

where $L = x/MN$, we have with $T = x^{1+2\varepsilon}/y$

$$(2.12)\qquad \sum_{\substack{L/5 < l \leqslant 2L \\ x-y < mnl \leqslant x}} a_m b_n \cdot l = \frac{1}{2\pi i} \int_{1+1/\log x - iT}^{1+1/\log x + iT} f(s) \frac{x^s - (x-y)^s}{s}\, ds + O(yx^{-\varepsilon})$$

with $f(s) = M(s)N(s)L(s)$. By non-trivial but standard methods one can show that the main term of (2.12), $y \sum a_m b_n/mn$, is given with an admissible error by

$$\int_{1+1/\log x - iT_0}^{1+1/\log x + iT_0} f(s) \frac{x^s - (x-y)^s}{s} ds, \tag{2.13}$$

where T_0 can be chosen as $L^{1/2}$. Since one can shift the path of integration onto the line $\sigma = 1/2$ the proof of (2.8) reduces to showing

$$\int_{T_0}^{T} |f(\tfrac{1}{2}+it)| dt \ll x^{1/2-\varepsilon}. \tag{2.14}$$

Let us denote by $\Omega(U, V, W)$ the measure of those points $t \in [T_0, T]$ for which

$$\begin{aligned} U &\leqslant |L(\tfrac{1}{2}+it)| < 2U, \\ V &\leqslant |M(\tfrac{1}{2}+it)| < 2V, \\ W &\leqslant |N(\tfrac{1}{2}+it)| < 2W, \end{aligned} \tag{2.15}$$

where

$$x^{-1} \leqslant U, V, W \leqslant x. \tag{2.16}$$

In order to prove (2.14) it is clearly sufficient to show, for any U, V, W with (2.15)–(2.16), that

$$UVW\Omega(U, V, W) \ll x^{1/2-2\varepsilon}. \tag{2.17}$$

Introducing the notation

$$F = F(U, V, W) = \frac{\Omega(U, V, W)}{\log^{10} x}, \tag{2.18}$$

we have to show

$$UVWF \ll x^{1/2-3\varepsilon}. \tag{2.19}$$

Using the mean-value theorem, the Halász–Huxley–Montgomery inequality and the fourth power moment of $\zeta(s)$, one can show

$$\begin{aligned} F \leqslant \min\{&V^{-2}(M+T), W^{-2}(N+T), V^{-2}M + V^{-6}MT, \\ &W^{-2}N + W^{-6}NT, U^{-4}L^2 + U^{-12}L^2T, U^{-4}T\}. \end{aligned} \tag{2.20}$$

We consider four cases.

Case 1: $F \leqslant 2V^{-2}M,\ 2W^{-2}N$;
Case 2: $F > 2V^{-2}M,\ 2W^{-2}N$;
Case 3: $F > 2V^{-2}M,\ F \leqslant 2W^{-2}N$;
Case 4: $F \leqslant 2V^{-2}M,\ F > 2W^{-2}N$.

In the most critical Case 2 we distinguish
Case 2a: $F > 2U^{-12}L^2 T,\ 2V^{-2}M,\ 2W^{-2}N$;
Case 2b: $F \leqslant 2U^{-12}L^2 T,\ F > 2V^{-2}M,\ 2W^{-2}N$.

Since the crucial requirement $\theta > 11/20$ is only needed in Case 2b, we shall consider only this case. (The treatment of the other cases is similar, and it needs only $\theta > 17/31$.) In case 2b we have

$$
\begin{aligned}
(2.21)\quad F &\leqslant 2T\min\{V^{-2}, W^{-2}, V^{-6}M, W^{-6}N, U^{-12}L^2, U^{-4}\}\\
&\leqslant 2T(V^{-2})^{7/20}(W^{-2})^{7/20}(V^{-6}M)^{1/20}(W^{-6}N)^{1/20}(U^{-12}L^2)^{1/40}(U^{-4})^{7/40}\\
&= 2(UVW)^{-1}T(MNL)^{1/20},
\end{aligned}
$$

which yields (2.19) if $T \leqslant x^{9/20-3\varepsilon}$, i.e., $\theta \geqslant 11/20+5\varepsilon$. It is easy to see from (2.21) that if $M \neq N$ or

$$MN = x^{0.8+\varphi}, \quad \varphi \neq 0 \tag{2.22}$$

then the minimum in (2.21) is strictly less than the weighted geometric mean value in the second line of (2.21); by some calculations one can even show

$$F \ll (UVW)^{-1}Tx^{1/20-|\varphi|/16}. \tag{2.23}$$

Thus if we want to improve $\theta = 11/20+\varepsilon$ we are entitled to assume $\log(M/N) = o(\log x)$ and $\varphi = o(1)$ in (2.22).

3

Very recently Iwaniec [6] and independently the present author [10] were able to improve the result $\theta > 11/20$ to

$$\theta = 17/31 = 1/2+1/(20+2/3) \tag{3.1}$$

and somewhat later we jointly proved

$$\theta = 23/42 = 1/2+1/21. \tag{3.2}$$

In all these proofs a crucial role was played by the following deep theorem of Deshouillers and Iwaniec [1]: If $0 < L_1 \leqslant L \leqslant T_1 < T$, $K(s) = \sum_{n\leqslant K} k_n n^{-s}$, $\sum_{n\leqslant K} |k_n|^2/n \ll 1$, then

$$\int_{T_1}^{T} \Big|\sum_{L_1<l\leqslant L} \frac{1}{l^{1/2+it}}\Big|^4 \Big|\sum_{n\leqslant K}\frac{k_n}{n^{1/2+it}}\Big|^2 dt \ll \mathscr{J}T^{1+\varepsilon}, \tag{3.3}$$

$$\mathscr{J} = \max(1, T^{-1/2}K^2, T^{-1/2}K^{5/4}L^{1/2}).$$

This theorem itself does not imply an improvement of (2.10), but one might use the multilinear form of the remainder term, of which the bilinear form (2.8) is a consequence. In fact, Theorem 4 of Iwaniec [5] shows that $R(D)$ can be written in the form

$$R(D) = \sum_{1 \leqslant r \leqslant x^{\varepsilon}} \sum_{(P_1, \ldots, P_r) \in \mathscr{D}} c_v(P_1, \ldots, P_r) R_v(P_1, \ldots, P_r), \tag{3.4}$$

where all elements (r-tuples) of the set $\mathscr{D}$ satisfy

$$D^{\varepsilon^2} \leqslant P_r \leqslant \ldots \leqslant P_1 \leqslant z, \tag{3.5}$$

$$P_1 \ldots P_{l-1} P_l^2 \leqslant D = x^{(12\theta-2)/5-\varepsilon} < x^{0.92} \qquad (1 \leqslant l \leqslant r) \tag{3.6}$$

and $c_v(P_1, \ldots, P_r)$ are bounded by 1 in absolute value, further

$$R_v(P_1, \ldots, P_r) = \sum_{P_1 < p_1 \leqslant Q_1} \cdots \sum_{P_r < p_r < Q_r} r(v p_1 p_2 \ldots p_r) \tag{3.7}$$

with some $P_i < Q_i \leqslant 2P_i$. By using the fact that $\mathscr{D}$ contains at most $O((\log x)^{c(\varepsilon)})$ elements it is sufficient to show

$$R_v(P_1, \ldots, P_r) \ll v^{-1} y x^{-2\varepsilon}. \tag{3.8}$$

In proving (3.8) one can assume without loss of generality that $v = 1$, since otherwise the interval $[x-y, x]$ is transformed into $[v^{-1}(x-y), v^{-1}x]$. As in Section 2, it is sufficient to prove (2.14) (with $x^{1/2-\delta_0}$), where $\mathscr{M}$ and $\mathscr{N}$ is a suitable subdivision of $(P_1, \ldots, P_r)$

$$M(s) = \prod_{P_l \in \mathscr{M}} \sum_{P_l < p_l \leqslant Q_l} \frac{1}{p_l^s}, \qquad N(s) = \prod_{P_l \in \mathscr{N}} \sum_{P_l < p_l \leqslant Q_l} \frac{1}{p_l^s}, \tag{3.9}$$

$$M = \prod_{P_l \in \mathscr{M}} P_l \leqslant \sqrt{D}, \qquad N = \prod_{P_l \in \mathscr{N}} P_l \leqslant \sqrt{D}. \tag{3.10}$$

It is rather complicated to prove that the range of validity of (2.14) can be extended to $\theta = 23/42$ or to $\theta = 17/31$. Our aim here is only to show that for every kind of functions (3.9)–(3.10) type, where (3.5)–(3.6) are satisfied, (2.14) can be proved, e.g., for

$$\theta = \frac{11}{20} - \frac{1}{2 \cdot 10^4}, \quad \text{i.e.,} \quad T = x^{(9/20 + 1/2 \cdot 10^4) + 2\varepsilon} \tag{3.11}$$

by using the crucial result (3.3).

Since Cases 1, 2a, 3 and 4 are treated suitably for every $\theta > 17/31 = 11/20 - 1/620$ in Heath-Brown and Iwaniec [3], we can assume that Case 2b holds and, in view of (2.23),

$$MN = \prod_{l=1}^{r} P_l = x^{0.8+\varphi}, \qquad L = x^{0.2-\varphi}, \qquad |\varphi| \leqslant 10^{-3}. \tag{3.12}$$

Further, we can restrict ourselves to the interval $[T_1, T]$ where $T_1 = x^{0.45-3\varepsilon}$ in view of the work of Heath-Brown and Iwaniec [3]. If we succeed in finding a relatively short factor of $M(s)$ or $N(s)$, then we can attach this to the zeta-factor $L(s)$, and the application of (3.3) furnishes an improvement. If one would be contented with a very slight improvement. one could take the shortest factor, corresponding to P_r. But if, say, $P_r \ll D^{\varepsilon^2}$, then this would lead to an improvement of θ with $\varepsilon^2/16$ only, where, however, ε can be chosen as a fixed positive number. This, however, would need explicit calculation of c in (2.5), and so we would probably have a very slight improvement only. (In that case we would have to work in (2.8)–(2.10), (2.14) etc. with another, much smaller value η, instead of ε, but this would cause no difficulties at all.)

So we define the index α, $1 \leqslant \alpha \leqslant r$, by

$$\prod_{l=\alpha}^{r} P_l \geqslant x^{0.004} > \prod_{l=\alpha+1}^{r} P_l \tag{3.13}$$

(where the empty product means 1) and by symmetry we can suppose

$$K := \prod_{\substack{l=\alpha \\ P_l \in \mathcal{M}}}^{r} P_l \geqslant x^{0.002} \tag{3.14}$$

and let $K(s)$ be the product of factors corresponding to $P_l \in \mathcal{M}$, $l \geqslant \alpha$. Now K is not too large, since by (3.6), (3.12), (3.13) we have

$$P_\alpha \leqslant \frac{D}{\prod_{l=1}^{\alpha} P_l} \leqslant \frac{D}{(MN/x^{0.004})} < x^{0.124-\varphi}, \tag{3.15}$$

and thus by (3.13)–(3.14)

$$x^{0.002} \leqslant K < x^{0.128-\varphi}. \tag{3.16}$$

Let us denote by $\Omega(U, V, W, Z)$ the measure of those points $t \in [T_1, T]$ for which besides (2.15)–(2.16) (with an integer m)

$$Z \leqslant |K(\tfrac{1}{2}+it)| < 2Z, \quad x^{-1} \leqslant Z \leqslant x, \ Z = 2^m \tag{3.17}$$

and let us define $\Omega(U, V, W)$ and F as in (2.15) for the shorter interval $t \in [T_1, T]$. Then, clearly,

$$\begin{aligned} \log^{10} x \cdot F &= \Omega(U, V, W) \\ &< 3 \log x \max_Z \Omega(U, V, W, Z) =: F_1 \log^{10} x. \end{aligned} \tag{3.18}$$

Then in Case 2

$$F_1 > F > 2V^{-2} M > 2 \left(\frac{V}{Z_0}\right)^{-2} \frac{M}{K}, \tag{3.19}$$

if the maximum in (3.6) is attained for $Z = Z_0$, since $Z_0 \ll \log^{-\nu} x\sqrt{K} = o(\sqrt{K})$ if $K(s)$ consists of ν factors of the shape $\sum_{P_l < p_l \leqslant Q_l} p_l^{-s}$.

So in this case we can use the Halász–Montgomery inequality for $M(s)/K(s)$ instead of $M(s)$, and further the Deshouillers–Iwaniec theorem; thus by (3.19) we obtain for Case 2 the inequality

$$
\begin{aligned}
(3.20)\quad F &\ll T^{1+\varepsilon} \times \\
&\times \min\{V^{-2}, W^{-2}, V^{-6} Z_0^6 M K^{-1}, W^{-6} N, U^{-4}, U^{-4} Z_0^{-2} \mathcal{J}\} \\
&\leqslant T^{1+\varepsilon} (V^{-2})^{5/16} (W^{-2})^{5/16} (V^{-6} Z_0^6 M K^{-1})^{1/16} \times \\
&\qquad \times (W^{-6} N)^{1/16} (U^{-4})^{1/16} (U^{-4} Z_0^{-2} \mathcal{J})^{3/16} \\
&= (UVW)^{-1} T^{1+\varepsilon} (MN)^{1/16} (\mathcal{J}^3 K^{-1})^{1/16} \\
&= (UVW)^{-1} T^{1+\varepsilon} x^{1/20} (\mathcal{J}^3 x^{\varphi} K^{-1})^{1/16}.
\end{aligned}
$$

Now by (3.3), (3.11), (3.12) and (3.16) we have

$$
\begin{aligned}
(3.21)\qquad \mathcal{J}^3 x^{\varphi} K^{-1} &\leqslant \max\left(\frac{x^{\varphi}}{x^{0.002}}, \frac{K^5 x^{\varphi}}{x^{27/40}}, \frac{K^{11/4} L^{3/2} x^{\varphi}}{x^{27/40}}\right) \\
&\leqslant x^{\max(\varphi - 2\cdot 10^{-3}, -0.035 - 4\varphi, -0.023 - 13\varphi/4)} \\
&\leqslant x^{-0.001}.
\end{aligned}
$$

Thus in view of the choice of T in (3.11)

$$
(3.22)\qquad UVWF \ll x^{1/2 - 10^{-5} + 3\varepsilon}.
$$

This shows that the remainder term of the sieve is of a smaller order of magnitude than the main term for $\theta = 11/20 - 1/(2\cdot 10^4)$. Naturally, the whole method works only if in (2.2) the final lower estimate remains positive. Thus, to obtain the improvements (3.1) or (3.2) in the original problem (1.1) or (2.1), it is also necessary to extend the range where the second sum of (2.2) can be evaluated asymptotically. Some further ideas are needed also in the treatment of those terms which were ignored in the form given by (2.2). We do not want to discuss these problems here. They are treated in [6] and [10].

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