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ON THE GEOMETRIC TOPOLOGY OF LOCALLY LINEAR ACTIONS OF FINITE GROUPS*

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Let G be a finite group acting on the n-manifold M as a group of homeomorphisms. A convenient and widely studied class of such actions is the locally linear (= locally smooth = locally smoothable) ones, i.e., those such that each point x of M has a neighborhood U which is invariant under the isotropy subgroup $G_x = \{g\colon g(x) = x\}$ and is G_x-equivariantly homeomorphic with some orthogonal representation of G_x on a Euclidean space R^n_ϱ. (This hypothesis eliminates many local pathologies of a topological nature, guaranteeing, for example, that the components M^H_α of the fixed point sets M^H of the subgroups H of G are always locally flat submanifolds of M and that the intersection of two such components is a locally flat submanifold of each.) The category of locally smooth actions of compact Lie groups on n-manifolds is analysed in the fundamental reference [Bre2]. It is even a highly desirable condition (in its piecewise linear form; for, without it, there is no equivariant regular neighborhood theorem [Ro; Section 5].)

A considerable amount of work has been done to relate locally linear actions to the smooth and p.l. actions, e.g., [LRo], [Ro], [HsPa], and [I14] and to produce equivariant analogs of the basic theorems of inequivariant geometric topology, including the Deformation Principle of Edwards and Kirby [Si2], Siebenmann's Cell-Like Mapping Theorem [Han], basic p.l. theory [I12, 3, 4], the Whitehead group and s-cobordism theorem [Bre1], [I11, 5], [Ro], [Hau], [An], the obstruction to finiteness [Bag], [An], [K], [Q2] surgery [BroQ], [DoPe], [DoRo], triangulation and smoothing [I12], [AnHs1, 2, 3], [Ro], [Q2], and the "controlled topology" which we shall

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refer to as CFQ-theory (after Chapman, Ferry, and Quinn) [Ch1–10], [ChFe1,2], [Q1,2].

One of the most striking aspects of finite group actions, leading to a standard inductive strategy for argumentation, is the stratification of the space acted upon by the fixed point components of the subgroups of the acting group, since if $H \subset K$, then $M^H \supset M^K$. This stratification is particularly nice in the case of locally linear actions because the fixed point components are equivariantly locally flat submanifolds. Passing to the orbit space and filtering by the dimension of the images of these submanifolds yields the structure of a "cone-like" stratified set in the sense of [Si2], in which each point has a neighborhood U which is the product $V \times W$ of a Euclidean neighborhood V in the stratum of the point with a cone W on a stratified set.

Many inequivariant results virtually prove themselves in the equivariant context by induction on the "depth" of the stratification, an example being the s-cobordism theorem in the smooth or p.l. locally linear categories (once the appropriate definition of the Whitehead group, dimension restrictions, and hypotheses preventing knotting of fixed point sets in each other (or of detecting it) are made (cf. [BroQ], [Ro]). In connection with this, it is particularly pleasant to note that an equivariant map between two G-ANR's (ANR's in the category of spaces with G actions and equivariant maps) is an equivariant homotopy equivalence if it restricts to a homotopy equivalence between the H-fixed point sets of domain and range for each subgroup H of G, cf. [Bre1].

For the topological category of manifolds with locally linear actions, two of the most fundamental results of the inequivariant theory fail equivariantly: the topological invariance of Whitehead torsion and the finiteness of homotopy type for compact manifolds. These failures are of the greatest philosophical and technical import, as the first deprives us of the s-cobordism theorem, one of the most important tools for the construction and classification of manifolds, and the second deprives us of even the existence of handlebody decompositions, a yet more fundamental tool. Both these results fail magnificently and quite generally. For example, if the classical Whitehead group $Wh(Z[G])$ or reduced projective class group $\tilde{K}_0(Z[G])$ is non-zero, then there is a smooth action of G on an h-cobordism that is topologically a product but is not a smooth or p.l. product [Ro], [BHs].

The failure of the topological invariance of equivariant Whitehead torsion was to be anticipated after Milnor's disproof of the Hauptvermutung for polyhedra [Mi1], even though a definition of the equivariant Whitehead group was thirteen years away [Il1], [Ro]. It was discovered in the mid-1970's, cf. [Il6], [Ro], [BroHs].

The discovery that compact, locally linear equivariant manifolds need not be equivariantly homotopy equivalent to finite G-CW-complexes (CW-complexes on which G acts by permuting the cells) was made by Quinn in 1982 [Q2]. It came as a shock, although it was foreshadowed by some of Siebenmann's examples of non-triangulable spaces.

Both of these failures were discovered as a consequence of "topological Eilenberg swindles" converging to a single fixed point of the action and employing elements of the Whitehead group or the projective class group, with equivariant engulfing employed in the complement of the point to show that local linearity was preserved. (Rothenberg and Browder and Hsiang used engulfing [ConMoY], [Sta]; Quinn used his controlled end theory to achieve essentially the same thing.) It is crucial that the equivariant situation is inherently stratified, topologically for the failure of the s-cobordism theorem and homotopically for the failure of compact manifolds to have finite homotopy type, for these swindles do not apply disruptively without stratification consideratons.

This state of affairs brings many problems and questions to mind. Among them are the following:

1. Construct the appropriate Whitehead group and prove an equivariant topological s-cobordism theorem for p.l. G-manifolds. (Note that the equivariant s-cobordism theorem in the p.l. locally linear category gives a bijection between the equivariant p.l. isomorphism classes, rel. M, of equivariant p.l. h-cobordisms and the appropriate subgroup of the equivariant Whitehead group Wh_G^{PL}.

2. The kernel of the map from the p.l. equivariant Whitehead group to the topological one of (1) above will correspond to the subgroup of the p.l. isomorphism classes of h-cobordisms that are topological products, i.e., the p.l. structures on the product h-cobordism. Describe this group algebraically, or at least give means for its algebraic computation.

3. Give a topological definition of an equivariant "Whitehead" group and prove an equivariant topological s-cobordism theorem for the locally linear topological category.

4. Give examples of compact equivariant manifolds realizing all possible obstructions to (equivariant) finiteness.

5. Define the obstruction to controlled equivariant finiteness and give examples of compact locally linear manifolds realizing all possible obstructions.

6. Do there exist equivariant h-cobordisms of locally linear p.l. manifolds that do not admit handle decompositions on them?

7. Do there exist locally linear manifolds without handle decompositions but with vanishing equivariant finiteness obstruction, i.e., is the finiteness obstruction the obstruction to the existence of a handle decomposition?

8. Is the controlled equivariant finiteness obstruction the obstruction to equivariant handle decomposition?

9. If the answer to 7 is "No", what is the obstruction to equivariant handle decomposition?

10. Both [AnHs] and [Q2] give sequences of obstructions, the vanishing of which ensures the triangulability of M as a G-combinatorial manifold; however, no general realizability theorems are given. Sharpen these schemes by calculating precisely which are realizable. (Cf. [DoRo].)

11. What is the situation with respect to lower dimensional manifolds for any of the above?

12. What is the situation with respect to Hilbert cube manifolds for any of the above?

13. Has every compact G-ANR the equivariant homotopy type of a compact, locally linear manifold? What about the finite-dimensional compact G-ANR's?

14. What is true for any of the above for locally linear actions of compact Lie groups? What about proper actions of non-compact Lie groups (each isotropy subgroup compact)?

We have made an analysis (cf. [SteWes1] that solves 1 and 2, gives a candidate for 3, solves 4 and 5, answers 7 positively with a calculation of D. Webb [Web], and produces a relatively mature theory of "Q_G-manifolds" where Q_G is a universal linear action on a Hilbert cube, thus going a long way toward 12.

Our fundamental result is our solution of 1. It is motivated by Chapman's proof [Ch1, 2, 3, 5] that the inequivariant Whitehead group may be regarded as (roughly) "stable homeomorphism classes of finite CW-pairs" or "finite CW-pairs mod cell-like maps", which are clearly topologically invariant, the point being that while inequivariantly these and Whitehead's "stable algebra" description coincide, they diverge equivariantly, with the stable algebra being precisely the appropriate setting for the proof of the equivariant s-cobordism theorem in the smooth and p.l. locally linear categories and the stable homeomorphism class/cell-like map definition the right one for the topological s-cobordism theorem.

In pursuit of this idea, we have developed a theory of equivariant cell-like mappings which generally parallels the inequivariant theory because controlled topological engulfing works perfectly well equivariantly. Thus, we have equivariant versions of Siebenmann's Cell-like Mapping Theorem [Si3], of Edwards' Approximation Theorem [E], and the Chapman–Ferry α-Approximation Theorem [ChFe2], together with its corollary unobstructed Thin h-Cobordism Theorem and "Fibrations are Bundles" theorem. We have also, as mentioned above, been led to develop a theory of locally "universal-linear" G-actions on Hilbert cube manifolds.

Our techniques, in general terms, are controlled (equivariant) engulfing and equivariant versions of the controlled simple-homotopy theory of

Chapman, Ferry and Quinn [Ch 1, 2, 3, 5, 6, 7, 8, 9, 10], [F1, 2, 3], [ChFe1, 2], [Q1, 2].

The last two sections deal with subsequent refinements. Section 2, due to P. Kahn and the first author, generalizes work of Burghelea, Lashof, and Rothenberg to the equivariant setting, giving a structure set interpretation of $Wh_G^{\mathrm{PL},\mathscr{G}}(M)_c$ and showing that stably this leads to an equivariant homology theory the spectrum of which has vanishing positive homotopy. Section 10, due to the first author, uses the structure sets in a fiber sequence for manifolds of the form $M = N \times I$ to compare the $\xi = \mathrm{PL}$ and $\xi = \mathrm{TOP}$ theories with an exact sequence on the π_0-level that determines, for example when a p.l. G-manifold admits G-h-cobordisms without equivariant handle decompositions and the order of the set of equivariant homeomorphism classes of them, rel. M, which is not always 0 by a calculation of D. Webb [Web].

We have until now dealt in generalities, deliberately slighting detail for readability. The next section begins a more precise discussion.

1. Definitions and conventions

The finite group G is fixed throughout the discussion. A G-space is a space equipped with a G-action. We freely use the symbol "G-" to mean "equivariant". Representations of G are always orthogonal. If "XXX" is a familiar concept from inequivariant topology, then "G-XXX" or "GXXX" denotes its equivariant analogue.

We generally follow the terminology of [Bre2] for isotropy subgroups fixed point sets, etc. Thus, M_α^H is a component of the fixed point set M^H of the subgroup H of G and M_{H_α} is a component of the set M_H of points x with isotropy subgroup G_x equaling H.

One restriction that we generally employ is the term G-manifold. In this paper, it means a locally linear n-manifold M satisfying the codimension $\geqslant 3$ condition that the intersection of a fixed point set component with another, if it is not equal to the first, has codimension at least three in it. The purpose of this is to ensure that there is no knotting of the fixed point components in each other. The effects of this restriction are twofold. Firstly, it allows the use of a very straightforward generalization [Il1] of the inequivariant Whitehead group, which cannot detect knotting, in the smooth and p.l. s-cobordism theorems. Secondly, it provides the 1-LC complements that we need for our engulfing moves.

A second definition that requires some precision is that of equivariant h-cobordism. Here, this term means a G-manifold W with boundary $\partial W = M_0 \cup M_1 \cup E$, where E is a closed equivariant collar between ∂M_0 and ∂M_1, such that the inclusions $M_i \to W$ are equivariant homotopy equivalences and such that the union of the fixed point components of W of dimension less than 6 is equivariantly homeomorphic with the product of its intersection with M_i and the unit interval.

2. G-ANR's and manifolds modelled on Q_G

Many of the generalities of the theory of ANR's go over unchanged to the category of G-spaces. We discuss only those points that are relevant to our considerations.

We say that a compact metric G-space A is equivariantly cell-like (G-CE) if whenever it is equivariantly embedded in a G-ANR X, it equivariantly deforms to an orbit in each of its neighborhoods. This is a familiar concept. Note that because equivariant maps can only alter the isotropy subgroup by enlarging it, $A \cap X^H$ will be $N(H)$-CE. Thus, G-CE sets are stratified cell-like in a way that is vital to induction arguments using the stratification by orbit types.

We say that an equivariant map $f\colon X \to Y$ between G-ANR's is equivariantly cell-like (G-CE) if it is proper and if $f^{-1}(Gy)$ is G-CE for each $y \in Y$.

Let α be an invariant (G-) cover of a G-space Y, i.e., one whose elements are permuted by G. A proper equivariant map $f\colon X \to Y$ is an equivariant α-equivalence ((G, α)-equivalence) if there is an equivariant proper homotopy inverse $h\colon Y \to X$ such that fh is equivariantly α-homotopic to the identity and hf is equivariantly $f^{-1}(\alpha)$-homotopic to the identity. If f is a (G, α)-equivalence for all open G-covers of Y, then it is an equivariant fine homotopy equivalence.

Now we can state an equivariant version of Haver's theorem [Hav], which holds equivariantly: for separable metric G-ANR's the G-CE maps coincide with the fine homotopy equivalences.

We now turn our attention to Hilbert cube manifolds. Let Q_G be the countably infinite product of unit discs of the regular real representation $R[G]$ with the diagonal action by left translation. It enjoys in the category of G-spaces all of the familiar universality properties of the Hilbert cube. A Q_G-manifold is a metrizable G-space locally equivariantly homeomorphic with Q_G.

In [SteWes4], we show that the basic topological properties of Hilbert cube manifolds extend to our equivariant setting. (See [Ch4], [Fe1], [FaV], [ChFe1], [T1,2] for basic references.) In particular, we are interested in Edward's Stabilization Theorem [Ch4] that locally compact G-ANR's become Hilbert cube manifolds upon stabilizaion by Cartesian product with the Hilbert cube, for that is how we relate the polyhedra defining our topological Whitehead group with the manifolds we wish to classify.

The most basic tools of Hilbert cube manifold theory are Z-set unknotting and stability; from them the rest of the theory flows almost formally. By an equivariant Z-set A in a locally compact G-ANR X we mean a closed, invariant subspace such that for every open cover α of X there is an equivariant map $F: X \to X - A$ that is α-close to the identity.

We show (G-Z set unknotting) that any pair of equivariantly homotopic embeddings $f_i: X \to M$, $i = 0, 1$, of a locally compact G-space onto closed, equivariant Z-sets of a Q_G-manifold M are equivariantly ambiently isotopic. This has the corollary that closed G-Z submanifolds in Q_G-manifolds are equivariantly collared.

Stability holds for Q_G-manifolds: the projection $M \times Q_G \to M$ may be approximated arbitrarily closely by equivariant homeomorphisms.

With these results, together with the equivariant version of the Deformation Principle of [FaV] (= Edwards–Kirby [EK] for Hilbert cube manifolds), we establish equivariant analogs of virtually all the results in the foundations of the theory of Hilbert cube manifolds that are independent of handle straightening and thus do not require the vanishing of Whitehead and K-theory obstructions, for it is a feature of the equivariant K-groups that they are nonzero even for points (we think of them as functors of G-spaces).

First, we generalize Edwards' theorem: if X is a locally compact metric G-ANR, then $X \times Q_G$-manifold.

Next, we have an equivariant version of Ferry's [Fe1] α-Approximation Theorem: for every open cover α of a Q_G-manifold M there is another open cover β of M such that every (G, β)-equivalence $f: N \to M$ from another Q_G-manifold is equivariantly α-homotopic to a homeomorphism.

From the α-approximation theorem we also obtain a Q_G-manifold fibrations are bundles theorem: an equivariant fibration $p: E \to B$ between locally compact metric G-spaces is an equivariantly locally trivial bundle provided that the base space is locally finite-dimensional and each fiber $p^{-1}(b)$ is a Q_{G_b}-manifold. (Here, an equivariant fibration is a G-map with the covering homotopy property in the category of G-spaces.) A locally trivial G-bundle $p: E \to B$ is a bundle on which G acts by fiber-preserving maps such that each $b \in B$ has a G_b-invariant neighborhood U and a G_b-equivariant trivialization $P^{-1}(U) \approx U \times F$, where G_b acts diagonally on $U \times F$. Such bundles may be classified by the methods of [LRo]. Note that because of the bundle theorem and the stabilization theorem above, more general notions of equivariant bundles will not satisfy the equivariant covering homotopy property.

Finally, we obtain an equivariant version of Toruńczyk's characterization of Hilbert cube manifolds [T1]. By a *generalized n-disc* we mean a balanced product $G \times_H D^n_\varrho$ where D^n_ϱ is an n-dimensional representation disc of the subgroup H of G. A G-space Y has the equivariant disjoint n-discs property ($DD^n_\varrho F$) if each pair of G-maps into Y of (possibly distinct) generalized n-discs may be approximated, arbitrarily closely, by equivariant maps with disjoint images.

The characterization of Q_G-manifolds is: the Q_G-manifolds are the locally compact, metrizable G-ANR's with the equivariant disjoint n-discs property for all n. (We could, of course, use Q_G instead of the generalized n-discs.)

The reader familiar with Hilbert cube manifold theory will notice that handle straightening is the only ingredient in Chapman's first proof [Ch1] of the topological invariance of Whitehead torsion which has not been lifted to the equivariant setting above. Since equivariant torsion is not a topological invariant, this handle straightening is non-trivially obstructed. Moreover, the existence of compact G-manifolds M ([Q2], see Section 8 below) with non-vanishing equivariant finiteness obstruction produces by stabilization non-triangulable Q_G-manifolds, i.e., manifolds not equivariantly homeomorphic to $K \times Q_G$ for some simplicial G-complex K.

We show that any locally compact G-ANR X has a well-defined controlled "finiteness" (or "propriety") obstruction $\sigma_c(X)$ in an equivariant controlled K-group $\tilde{K}_{0,G}(X)_c$ (see Section 6, below) such that $\sigma_c(X)$ vanishes if and only if for every open cover α of x there is an equivariant proper α-equivalence $f\colon K \to X$ for some locally finite equivariant simplicial complex K. The equivariant α-Approximation Theorem now shows that if X is a Q_G-manifold, $\sigma_c(X)$ is the obstruction to equivariantly triangulating X.

Using this, we show that the obstruction to straightening a k-handle $Gx_H(R^k \times Q_H) \subset X$ lies in the H-equivariant lower K-group $K_{(1-k),H}(*)$ and hence vanishes for $k \geqslant 3$.

3. Equivariant Whitehead groups and s-cobordism theorems

Illman [Il1, 5] gave an equivariant Whitehead group modelled on [Co2], i.e., equivalence classes of equivariantly proper homotopy equivalent pairs of finite G-CW complexes (Y, X) modulo equivariant cellular expansion and collapse, rel. X. (An equivariant n-cell of type H is $G/H \times e^n$.) We denote it by $Wh_G{}^{\mathrm{PL}}(X)$; the class of a pair is also called its torsion $\tau(Y, X)$ or $\tau(Y)$. If we use the restricted subgroups $Wh_G{}^{\mathrm{PL},\varrho}(X)$, in which Y is only allowed cells of types appearing in X with dimension at least five, then an equivariant s-cobordism theorem holds [BroQ], [Ro] in the smooth and p.l. categories of locally linear G-manifolds (see Section 1 for definitions). It even holds in the topological category for G-manifolds with equivariant handle decompositions if one asks for equivalence up to finite handle manipulation, i.e., isomorphism classes, rel. M, of G-h cobordisms (W, M) are in bijective correspondence, via $\tau(W, M)$, with $Wh_G^{\mathrm{PL},\sigma}(M)$.

With this, we can establish an equivariant version of Siebenmann's Cell-Like Mapping Theorem [Si3]: Let $F: M \to N$ be a G-CE map of G-manifolds that is a homeomorphism of boundaries and of fixed point components M_α^H of dimension less than five. Then f is a limit of equivariant homeomorphisms agreeing with it on the boundary and fixed point components of dimension less than five.

The argument is by induction on orbit types, beginning with the inequivariant result and utilizing the fact that the G-CE maps are NH-CE maps on M^H for each H. Note that the Bing Shrinking Criterion works equivariantly.

A remark should be made here concerning the relation of this theorem to Handel's Stratified Cell-Like Mapping Theorem [Han]. Handel assumes that the point inverses lie in single "pure" strata of the stratified sets. Equivariantly, this is equivalent to the hypothesis that $f^{-1}(y) \subset M_H$, where $H = G_y$, and hence to the hypothesis that the map is isovariant. Our situation allows the point inverses to traverse several or all strata. It is this that allows us to make the connection between equivariant homeomorphisms of manifolds and the geometrically defined Whitehead group: the collapse to K of the regular neighborhood of K in a representation disc, for example, is only isovariant for free actions.

Equipped with the Equivariant Cell-Like Mapping Theorem and controlled engulfing, we can now manipulate Chapman's shuffling machine of [Ch8], which is basically a torus argument reminiscent of Connell's shuffle in [Con] that avoids any use of Whitehead groups or surgery results, to prove the α-Approximation Theorem for G-manifolds: Let M^n be a G-manifold. For each open cover α of M there is an open cover ϱ of M such that every equivariant ϱ-equivalence f: $N^n \to M$ that is a homeomorphism of boundaries and on fixed point components N_δ^H of dimension less than five is equivariantly α-homotopic to a homeomorphism. (Again it should be emphasized that isovariance is not hypothesized.)

With the Equivariant α-Approximation Theorem, we can now obtain our primary technical goal in this material, a fully equivariant version of an unobstructed controlled h-cobordism theorem of Chapman and Ferry (cf. [Q1], [Ch9]). To state it, we need two more definitions.

An equivariant map $f: X \to Y$ is equivariantly $(\alpha, 1)$-connected (cf. [Q1]), where α is an open cover of Y, if for each equivariant relative 2-complex (R, S) and equivariant maps h: $S \to X$ and k: $R \to Y$ with $fh = k$, there is a G-map k': $R \to X$ extending h such that fk' is α-close to k.

If α is an open cover of the G-manifold M, then a G-h cobordism (W, M) is a (G, α, h)-cobordism provided that there is an α-strong deformation retraction r of W to M for which the inclusion $\partial W\text{-}M \to W$ is equivariantly $(r^{-1}(\alpha), 1)$-connected.

The equivariant controlled h-cobordism theorem we need is now the following: For each G-manifold M there is an open cover α such that all equivariant (α, h)-cobordisms (W, M) are equivariant topological products. It is an easy corollary to the Equivariant α-Approximation Theorem.

A second corollary of the Equivariant α-Approximation Theorem is an equivariant "fibrations are bundles" result quite analogous to the one in [ChFe2]: a G-fibration of locally compact, locally finite-dimensional metric G-spaces $p: E \to B$ with B equivariantly path connected is an equivariant bundle if each $p^{-1}(b)$ is an n-dimensional compact G_b-manifold and p restricts to an equivariant bundle on the union of the boundaries of the fibers $p^{-1}(b)$ as well as the union of their fixed point components which are of dimension less than five.

Finally, there is an equivariant version of Edwards' Approximation Theorem that CE-resolvable ANR homology manifolds with the Disjoint Discs Property are manifolds: each equivariant CE map $f: M \to X$ from a G-manifold to a G-ANR with the equivariant disjoint 2-discs property is a limit of equivariant homeomorphisms if it restricts to a homeomorphism of ∂M and each fixed point component M_α^H of dimension less than five. We suppose it will be useful in identifying locally linear actions on known manifolds.

5. Equivariant controlled-simple homotopy theory and a controlled equivariant s-cobordism theorem

In this section and the next, we sketch the outlines of some equivariant versions of Chapman's controlled simple-homotopy theory [Ch10], which is the end-product of a decade of research by him and Ferry. (Cf. [Ch 1, 2, 3, 5, 6, 7, 8, 9, 10], [Fe1, 2, 3], [ChFe1, 2].) It plays a major role in our study. We follow Chapman [Ch10] closely. In particular, Chapters 1–11 of it go over verbatim, the primary difference being that we get several distinct but parallel equivariant theories which all coincide inequivariantly. We use them for our analysis of G-manifolds.

The notation is as follows. B is a fixed, finite-dimensional metrizable G-space. The G-space in which we are interested is X, for the moment a locally finite simplicial G-complex, which is equipped with an equivariant control map $p: X \to B$. (For simplicity we assume p is proper. The general case is given by passage to direct limits from subspaces of X mapping properly to B.) We fix an open G-cover α of B and let $\mathrm{DR}(X)_\alpha$ be the equivariant proper $p^{-1}(\alpha)$ strong deformation retractions $r: Y \to X$ of locally finite simplicial G-complex pairs (Y, X).

Let ξ denote one of the following classes of equivariant surjections: PL simple (p.l. with each point inverse $f^{-1}(y)$ G_y-equivariantly contractible and representing the trivial element of $Wh_{G_y}^{PL}(*)$), CE PL (p.l. with each point inverse $f^{-1}(y)$ G_y-equivariantly contractible), G-CE. Now let $\mathrm{DR}^{\xi}(X)_{\alpha}$ be the equivalence classes of members of $\mathrm{DR}(X)_{\alpha}$ under the equivalence relation generated by equivariantly $p^{-1}(\alpha)$-homotopy commutative diagrams

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ Y & & Y' \\ & X \arrow[ul] \arrow[ur] \arrow[uu] \arrow[d, "p"] & \\ & B & \end{tikzcd}$$

where f and f' are in ξ and the identity on X. Define $Wh_G^{\xi}(X)_{\alpha}$ to be the invertible elements of $\mathrm{DR}^{\xi}(X)_{\alpha}$ under the operation of union over X (pushout). Let $Wh_G^{\mathrm{CEPL}}(X) = Wh_G^{\mathrm{CEPL}}(X)_{\{B\}}$, and note that $Wh_G^{\mathrm{CE}}(X)_{\{B\}} = Wh_G^{\mathrm{htop}}(X)$ by the shuffle argument of [SteWes2].

There are naturality properties of these groups [Ch10] that we forbear to detail, but note the relaxation (of control) homomorphisms $Wh_G^{\xi}(X)_{\alpha} \to Wh_G^{\xi}(X)_{\beta}$ when α refines β and the quotients $Wh_G^{\xi}(X)_{\alpha} \to Wh_G^{\psi}(X)_{\alpha}$ when $\xi \subset \psi$.

The kernel of the quotient

$$Wh_G^{\mathrm{PL}}(X)_{\alpha} \to Wh_G^{\mathrm{CEPL}}(X)_{\alpha}$$

is seen, with the help of [Ste] (cf. [Co3]) to be the subgroup generated by the inclusion induced images of the groups $Wh_G^{\mathrm{PL}}(G(x))$, $x \in X$.

We set $Wh_G^{\xi}(X)_c = \lim_{\alpha} Wh_G^{\xi}(X)_{\alpha}$, the inverse limit. The above remark holds here, too. If p is simplicial or, more generally, is properly dominated by a simplicial map, $Wh_G^{\xi}(X)_c$ is stable in the sense of being equivalent to a constant system.

Unlike the inequivariant case [Ch10, Section 12], if $B = X$ and p is the identity, $Wh_G^{\mathrm{PL}}(X)_c$ and $Wh_G^{\mathrm{CEPL}}(X)_c$ are usually nonzero.

An immediate application of these limits is that the Equivariant α-Approximation Theorem for Q_G-manifolds shows immediately that the following sequence is exact for $\xi = \mathrm{PL}$ or CEPL:

$$Wh_G^{\xi}(X)_c \to Wh_G^{\xi}(X) \to Wh_G^{\mathrm{htop}}(X) \to 0.$$

(Here $B = X$ and $p = \mathrm{id}$.)

Following Ferry [Fe2] and Chapman [Ch10] 9cf. [AnHs2], [K], [Ra]), we define $\tilde{K}_{0,G}^{\xi}(X)_{\alpha}$ to be the subgroup of $Wh_G^{\xi}(X \times S^1)_{\alpha}$ comprised of those elements invariant under the standard geometric transfers (pull-backs via $\mathrm{id}_X \times v$, v being a standard orientation preserving self-covering of S^1). Here S^1 has the trivial G-action, and the control is $p(\mathrm{proj})$: $X \times S^1 \to X \to B$. Analogously, we set $K_{-i,G}^{\xi}(X)_{\alpha}$ to be those elements of $Wh_G^{\xi}(X \times T^{i+1})_{\alpha}$ that are invariant under all the geometric transfers of the standard S^1 factors of the $(i+1)$-torus T^{i+1}. $\tilde{K}_{0,G}^{\xi}(X)_c$ and $K_{-i,G}^{\xi}(X)_c$ is the inverse limit. Again, if X

is compact and p is properly dominated by a simplical map then the inverse systems are stable.

When we come to studying equivariant finiteness obstructions we need non-compact X but relatively finite (Y, X). These groups are $Wh_G^{\xi}(X)_{\alpha}^F$ etc.

We can now obtain a strict equivariant analog of Chapman's controlled Whitehead theorem. We consider G-h cobordisms (W, M) in the smooth, p.l., or topological category with given equivariant handlebody decompositions and ask for isomorphisms that are smooth, p.l., or obtained by handle manipulations.

The choice of a G-$p^{-1}(\alpha)$ homotopy inverse of i: $M \to W$ gives us an element of $\mathrm{DR}(M)_{\alpha}$ which determines a unique torsion $\tau(W, M) \in Wh_G(M)_{\beta}$, where β is an iterated star of α, the number of iterations being a structural constant of the theory. The effect of handle manipulations and cell trading is generally to lose control with each step, but a key observation in the theory [Q1, 2], [Ch10] shows that this loss is well-regulated. Following [Ch10], we have a function φ: $Z_+ \times Z_+ \to Z_+$, preserving the product partial order, describing this loss.

The equivariant analog of Chapman's Controlled h-Cobordism Theorem (cf. [Q2]) is now that the equivariant $p^{-1}(\alpha)$-h cobordism (W, M) is equivariantly p.l. $p^{-1}(\gamma)$-isomorphic with $(M \times I, M \times \{0\})$ if and only if its torsion $\tau(W)$ vanishes in $Wh_G{}^{\mathrm{PL},e}(M)_{\beta(\alpha)}$, where γ is the $\varphi(\dim B, \dim M)^{\text{th}}$ star of α; moreover, for any $x \in Wh_G{}^{\mathrm{PL},e}(M)_{\alpha}$ there is an equivariant (smooth, p.l., or top-with-handles) $p^{-1}(\gamma)$-h cobordism (W, M), the torsion of which in $Wh_G{}^{\mathrm{FL}}(M)_{\beta(\gamma)}$ equals the image of x (under relaxation).

We conclude this section by showing how to extend these definitions to locally compact G-ANR's. For a compact G-ANR X with control map p: $X \to B$ and open G-cover α of B, let $\beta_1 < \beta_2 < \dots < p^{-1}(\alpha)$ be a sequence of precise shrunk open G-refinements [Du] of $p^{-1}(\alpha)$ such that if A_i denotes the element of γ_i corresponding to $A \in p^{-1}(\alpha)$, then $A = \bigcup A_i$. Let $\gamma_i < \gamma_{i+1} < \dots$ be a sequence of open G-covers of X such that $\mathrm{St}^2(\beta_i, \gamma_{i+1})$ refines β_{i+1}. Now choose equivariant finite γ_i-dominations u_i: $X \to L_i$, d_i: $L_i \to X$ and let $\delta_i = d_i^{-1}(\beta_i)$. Then define $Wh_G^{\xi}(X)_{\alpha} = \lim_{\to} Wh_G^{\xi}(L_i)_{\delta_i}$ where the bonding homomorphisms are $(u_{i+1} d_i)_*$: $Wh_G^{\xi}(L_i)_{\delta_i} \to Wh_G^{\xi}(L_{i+1})_{\delta_{i+1}}$, $\xi = $ PL, CEPL, or CE. (We need to use equivariant simplicial approximation here.) To extend to locally compact G-ANR's, we need to use more than a sequence of β's and γ's, but the idea is the same. Altering the above choices results in isomorphic limit groups.

From this definition, the $Wh_G^{\xi}(X)_c$, $\tilde{K}_{0,G}^{\xi}(X)_{\alpha}$, etc., may be defined as before.

If the control map p: $X \to B$ is the identity then it is properly dominated by a simplicial map so that the inverse system defining $Wh_G^{\xi}(X)_c$ is stable. This extends to the equivariant controlled lower K-groups.

6. Controlled equivariant finiteness and splitting obstructions

An equivariant version of Wall's finiteness obstruction [Wa] was given by Anderson [An] (cf. [Bag], [K]) in the form of a direct sum of inequivariant obstructions exploiting the theorem of Bredon that equivariant maps that are inequivariant homotopy equivalences on all fixed point sets are equivariant homotopy equivalences. Geometrically defined inequivariant controlled finiteness obstructions have been given by Ferry [Fe4], Chapman [Ch10] and Quinn [Q2]. We present here equivariant versions of the Chapman–Ferry obstruction to (controlled) finiteness, applying them to obtain an equivariant "End Theorem" modelled on those of [Ch10] and [Q2]. The homotopical parts are taken from [Ch10], sections 7–11].

Let X be a G-space with a control map P: $X \to B$ (equivariant). For simplicity, assume B is compact. We give a generalization to controlled propriety obstruction in [SteWes3]. If (u, d): $X \to K \to X$ is a $p^{-1}(\alpha)$-domination of X by a simplicial G-complex K, then X is equivariantly $p^{-1}(\beta)$-homotopy equivalent to the infinite mapping cylinder D_e of the homotopy idempotent $e = ud$ (D_e is the infinite cyclic cover of the mapping torus $T(e)$ of e).

Here β is an iterated star of α, the number of iterations being a structural constant of the theory [Ch10] (cf. [Fe2]). Thus, we may restrict ourselves to the case where X is a simplicial G-complex. If K is finite, then by taking the simplicial mapping cylinder of d we may presume K to be a subcomplex of X.

Now $T_{(e)}$ is equivariantly $p^{-1}(\beta)$-homotopy equivalent to $X \times S^1$ [Ch10]. Let μ be such a homotopy equivalence with inverse v, and let $\varkappa$ be the reflection of S^1 through some line through the origin in R^2. Then, for an equivariant simplicial approximation ψ to $v(\mathrm{id}_X \times \varkappa)u$, the simplicial mapping cylinder $M(\psi)$ of ψ equivariantly $p^{-1}(\gamma)$-deformation retracts to its "domain" end for γ sufficiently larger than α. This yields an element of $\mathrm{DR}(X \times S^1)_\gamma$ by μ_*, which is represented by adding $M(\mu)$ to $M(\psi)$ along their domain ends. Again for some iterated star δ of α, the number of iterations being a structural constant of the theory, this produces a unique element $\sigma_\alpha(X) \in \tilde{K}_{0,G}{}^{\mathrm{PL}}(X)_\delta^f$, which is zero if X is $p^{-1}\alpha$-finite. (See [Ch10].) If $\sigma_\alpha(X) = 0$, then X is equivariantly $p^{-1}(\varepsilon)$-homotopy equivalent to a finite simplicial G-complex, with ε again an iterated star of α, the number of times being a universal constant.

Similarly, if (X, A) is a G-ANR pair with A closed in X there is a controlled relative finiteness obstruction $\sigma_\alpha(X \text{ rel } A) \in \tilde{K}_{0,G}{}^{\mathrm{PL}}(X)_\alpha^f$ whose vanishing implies that X is $p^{-1}\beta$-equivalent rel A to a relatively finite relative G-CW complex built on A. Moreover, if c: $A \subset X$, then $\sigma_\alpha(X) = \sigma_\alpha(X \text{ rel } A) + i_* \sigma_\alpha(A)$ modulo slight relaxation of control.

This leads to an equivariant controlled end theorem, the statement of which is exactly like Chapman's but again needs two definitions. Let M be a noncompact smooth or p.l. G-manifold with equivariant control $p: M \to B$, B finite dimensional and metrizable. Neighborhoods U of the end (s) $\mathscr{E}$ of M are the complements of compact sets. M is equivariantly $p^{-1}(\alpha)$ 1-movable at α if for each neighborhood U of $\mathscr{E}$ and open G-cover α of E there is a smaller neighborhood V of $\mathscr{E}$ such that for each neighborhood W of $\mathscr{E}$ there is a smaller one Z such that each equivariant map f: $(K, L) \to (V, Z)$ of a relative 2-complex may be $p^{-1}(\alpha)$-approximated, rel. L, by another f': $(K, L) \to (W, Z)$ that is equivariantly $p^{-1}(\alpha)$-homotopic to it in U and rel. L.

The end(s) is equivariantly tame over B if for every neighborhood U of $\mathscr{E}$ and open G-cover α of B there is a smaller neighborhood V of $\mathscr{E}$ so that U equivariantly $p^{-1}(\alpha)$-deforms into $U - V$.

Both of the above conditions are implied, of course, by the existence of an equivariant boundary for M over B (i.e., a manifold $N \supset M$, with boundary, such that $N - M \subset \partial N$ and such that p extends to N).

Controlled equivariant tameness ensures that each neighborhood U of $\mathscr{E}$ with bicollared frontier in M is equivariantly $p^{-1}(\alpha)$-finitely dominated for each α, so we get $\sigma_\alpha(U) \in \tilde{K}_{0,G}{}^{\mathrm{PL}}(U)^f_\beta$, and these give an element $\sigma_c(\mathscr{E}) \in \lim_{U,\alpha} \tilde{K}_{0,G}{}^{\mathrm{PL}}(U)_\alpha = \tilde{K}_{0,G}(\mathscr{E})^f_c$. This element is independent of all choices in the construction. The vanishing of $\sigma_c(\mathscr{E})$ is as in [Ch10] necessary and sufficient that $\mathscr{E}$ have neighborhoods that may be split as a sequence of controlled G-h cobordisms with ever-increasing control over E. To adjust them to controlled equivariant s-cobordisms requires that the sequence of controlled torsions represent 0 in the $\lim^1$ group of the sequence of their controlled equivariant Whitehead groups (with bonding maps derived from inclusion of neighborhoods). This provides a second obstruction $\tau'_c(\mathscr{E}) \in \lim^1_{U,\alpha} Wh_G{}^{\mathrm{PL}}\ (U)_\alpha = Wh'_G{}^{\mathrm{PL}}(\mathscr{E})^f_c$, it is defined when $\sigma_c(\mathscr{E}) = 0$ and is independent of all choices.

Now the equivariant controlled end theorem can be stated. Except for the by now familiar hypothesis on low-dimensional fixed point sets, it reads exactly as does Chapman's [Ch10]: a smooth or p.l. G-manifold controlled over a finite-dimensional metrizable G-space E admits and equivariant controlled boundary over E if and only if the five conditions below hold.

1. $\partial M = \phi$ and all noncompact fixed point components M^H_α of M are of dimension at least five.
2. M is equivariantly 1-movable at α over B.
3. M is equivariantly controlled ame at ∞ over E.
4. The controlled equivariant finiteness obstruction $\sigma_c(\mathscr{E}) \in \tilde{K}_{0,G}{}^{\mathrm{PL}}(\mathscr{E})^f_c$ of M at ∞ over B vanishes.
5. The $\lim^1$ torsion obstruction $\tau'_c(\mathscr{E}) \in Wh'_G{}^{\mathrm{PL}}(\mathscr{E})^f_c$ vanishes.

(If the control is equivariantly simplicial or an equivariant simplicial p-NDR, then $\tau'_c(\mathscr{E}) = 0$.)

There is a relatively straight-forward analog for smooth and p.l.-G-manifolds with ends that are "proper" over B. For this theorem we define a neighborhood U of $\mathscr{E}$ to be an open set of M such that p restricts to a proper map of $M-U$ to B and proceed as above, obtaining obstructions $\sigma_c(\mathscr{E})\in \tilde{K}_{0,G}{}^{\mathrm{PL}}(\mathscr{E})_c$ and $\tau'_c(\mathscr{E})\in Wh'_G{}^{\mathrm{PL}}(\mathscr{E})_c$.

An additional use of the controlled equivariant finiteness obstruction (in a relative form) is as a controlled splitting obstruction for elements of $Wh^{\xi}_G(X)_\alpha$. Chapman's treatment in [Ch10] again goes over verbatim. We summarize as follows.

Suppose that $B=C\times \boldsymbol{R}$ for C compact and that $p\colon X\to B$ is proper For $r\colon Y\to X$ in $\mathrm{DR}(X)_\alpha$, choose a $p^{-1}(\alpha)$-deformation retraction r_t and an invariant Urysohn function

$$\mathscr{T}\colon \big(Y, r^{-1}p^{-1}(\mathrm{st}(C\times(-\infty,0]=A),\alpha), r^{-1}p^{-1}(\mathrm{st}^2(A,\alpha))\big)\to(I,0,1).$$

(Here set $(F,\beta)=\bigcup\{U\in\beta\colon U\cap F\neq\emptyset\}$, and $\mathrm{st}^2(D,\beta)=\mathrm{st}(\mathrm{st}(F,\beta),B)$.)

Then $y\to r_{\mathscr{T}(y)}(y)=u(y)$ is an equivariant domination of Y by $Z=r^{-1}p^{-1}(A)\cup X\cup F$, where F is a finite subcomplex of (a subdivision of) Y containing the closure of $r^{-1}p^{-1}(\mathrm{st}^3(A,\alpha)-A)$. Thus, it may be regarded as a relatively finite domination of the pair $(Y, X\cup r^{-1}p^{-1}(A))$, and we can ask it whether the pair is equivariantly p^{-1} (β)-homotopy equivalent to a relatively finite pair $(K, X\cup r^{-1}p^{-1}(A))$ rel. $X\cup r^{-1}p^{-1}(A)$.

If this is so then, as in [Si1], modulo an equivariant expansion, we may assume that F is so chosen that there is a $p^{-1}(\beta)$ strong deformation retraction F of Y to Z. Now Fr_t is an equivariant $p^{-1}(\mathrm{st}(\alpha,\beta))$-strong deformation retraction of Z to X, which shows as in [Si1] that r is equivalent in $\mathrm{DR}(X)_{\mathrm{st}(\alpha,\beta)}$ to $W=Z\cup_\varphi Y-Z$, where $\varphi=r|\colon D\to Z$, with $D=Z\cap Y-Z$ the site of attachment of $Y-Z$ to Z. Observe that now W is split into the two pieces Z and $X\cup_\varphi Y-Z$ united along X and so equals the sum of the two classes they determine in $Wh^{\xi}_G(X)_{\mathrm{st}(\alpha,\beta)}$.

This argument shows that the controlled equivariant splitting obstruction may be identified with the obstruction to controlled relative finiteness given by the idempotent $e=u|\colon Z\to Z$. This obstruction may be handled in exactly the same way that the absolute finiteness obstruction was by using the mapping torus construction $(M(\psi))$ above and injecting it into $\tilde{K}_{0,G}{}^{\mathrm{PL}}(X)^f_c$ by first truncating Z to $Z'=Z\cap r^{-1}p^{-1}(\mathrm{st}^n(A,\alpha)\cap C\times[0,\alpha))$ for n sufficiently large that Z' contains F and then taking $X\cup_\eta M(\psi)$, where $\eta\colon t(e|Z')\to X\times S^1$ maps the domain end of $M(\psi)$ to $X\times S^1$ and is induced by r.

Hence, there exists an integer k, a structural constant of the theory, such that for $[r]\in Wh^{\xi}_G(X)^f_\alpha$ or $Wh^{\xi}_G(X)_\alpha$ there is a unique element $s^0_\alpha([r])\in \tilde{K}^{\xi}_{0,G}\big(p^{-1}(s^{2s}(C\times 0,\alpha))\big)^f_\gamma$, where $\gamma=\mathrm{st}^s(\alpha)$, the vanishing of which ensures that $[r]$ splits near $p^{-1}(0)$ in $Wh_G{}^{\mathrm{PL}}(X)_\gamma$ (or $Wh_G{}^{\mathrm{PL}}(X)^f_\gamma$).

This function s^0_α is Chapman's splitting homomorphism and enjoys equivariantly all the properties established in [Ch10]. Moreover, his realization theorem of Section 11 of that work holds equivariantly in all three of the theories we are discussing. We reiterate, however, that Chapman's Theorem 12.1 and its two corollaries (one of which is that compact ANR's have finite ε-homotopy type for all $\varepsilon > 0$) fail equivariantly, although Theorem 12.2 and its implications hold.

7. Calculations

Let X be a finite G-CW complex with fundamental group π. The Bass–Heller–Swan splitting [BHS] of $Wh(Z[\pi \times Z])$ as $Wh(Z[\pi]) \oplus \tilde{K}_0(Z[\pi]) \oplus \mathrm{Nil}$ extends to the isomorphism

$$Wh_G{}^{\mathrm{PL}}(X \times S^1) \cong Wh_G{}^{\mathrm{PL}}(X) \oplus \tilde{K}_{0,G}{}^{\mathrm{PL}}(X) \oplus \mathrm{Nil}$$

with each summand itself decomposed into a direct sum over conjugacy classes of subgroups (H) and component classes (α) of fixed point components. This also applies to $Wh_G{}^{\mathrm{CEPL}}(X)$ and to the restricted groups $Wh_G^{\xi,\varrho}(X)$. However, with our definition of $\tilde{K}^{\xi}_{0,G}(X)$, the injection into $Wh_G^{\xi}(X \times S^1)$ is in general slightly different from that of Bass, Heller, and Swan. (See [Ra].)

Our first order of business is to analyse the kernel of q: $Wh_G{}^{\mathrm{PL}}(X) \to Wh_G{}^{\mathrm{htop}}(X)$. We have already seen in Section 5 an exact sequence

$$Wh_G^{\xi}(X)_c \to Wh_G^{\xi}(X) \to Wh_G{}^{\mathrm{htop}}(X) \to 0$$

for $\xi = $ PL, CEPL or CE. This sequence also holds for the restricted groups $Wh_G^{\xi,\varrho}(X)$. We also have the kernel of $Wh_G{}^{\mathrm{PL}}(X) \to Wh_G{}^{\mathrm{CEPL}}(X)$ being the subgroup generated by the PL groups of orbits $Wh_G{}^{\mathrm{PL}}(G(x))$, $x \in X$. If $X = Z \times S^1$, this kernel contains no transfer-invariant element, so $\tilde{K}_{0,G}{}^{\mathrm{PL}}(X)_\alpha = \tilde{K}_{0,G}{}^{\mathrm{CEPL}}(X)_\alpha$ and similarly for the lower K-groups, the inverse limit groups and the restricted groups and all their extensions to locally compact G-ANR's X. We drop use of the ξ notation for them.

If the control map is simplicial, then we use the splitting material of Section 6 together with Carter's Vanishing Theorem [Car] that $K_{-i}(Z[\pi]) = 0$ for finite π if $i > 1$ in an induction over dual cells and a Leray spectral sequence calculation with coefficients in the Whitehead and K_{-i}-groups of p^{-1} (orbits) to calculate the $Wh_G{}^{\mathrm{CEPL}}(X)_c$ and $K_{-i,G}{}^{\mathrm{CEPL}}(X)_c$ groups. For X a finite simplicial G-complex and simplicial (or equivariant simplicial p-NDR) control map or a locally compact G-ANR and control the identity of X, this yields the following results. For

$$i > 1, \quad K_{-i,G}(X)_c = 0; \quad K_{-1,G}(X)_c \cong H_0^{G,1F}(X; \mathscr{K}_{-1,G});$$

and there is an exact sequence

$$H_3^{G,lf}(X; \mathscr{K}_{-1,G}) \to H_1^{G,lf}(X; \mathscr{K}_{0,G}) \to Wh_G{}^{\mathrm{CEPL}}(X)_c \to H_2^{G,lf}(X; \mathscr{K}_{-1,G})$$
$$\to H_0^{G,lf}(X; \mathscr{K}_{0,G}) \to \tilde{K}_{0,G}(X)_c \to H_1^{G,lf}(X; \mathscr{K}_{-1,G}) \to 0.$$

(Here, $H_*^{G,1f}(X; \mathscr{K}_{i,G})$ denotes Bredon homology [Brel] with locally finite chains and coefficient system given by restriction of $K_{i,G}$ to orbits.) This sequence is reminiscent of the one Quinn obtains in [Q2] from an Atiyah–Hirzebruch spectral seequence and can in fact be obtained from one using the material of Section 9.

With this sequence, we can prove that for compact G-ANR X, $Wh_G^{\mathrm{htop}}(X)$ and $\tilde{K}_{0,G}(X)_c$, as well as the restricted groups, are functors of the "π_1-system" of X, by which we mean that an equivariant map inducting an isomorphism of the system $\Pi = \{\pi_1(X^H, gx_H)$: H a subgroup of G, $x_H \in X^H$ a base point, and $g \in NH/H\}$, induces an isomorphism of $Wh_G^{\mathrm{htop}}(X)$ and of $\tilde{K}_{0,G}(X)_c$. From the above sequence, it is seen that $Wh_G^{\mathrm{CEPL}}(X)_c$ depends on the 3-skeleton.

For an informative example, suppose that X is a compact G-ANR with X^H nonvoid and simply connected for each subgroup of G. Then with the trivial action on T^i we have

$$\begin{aligned}(1)\qquad Wh_G^{\mathrm{PL}}(X \times T^i) &\cong Wh_G^{\mathrm{PL}}(X) \oplus i\tilde{K}_{0,G}(X) \oplus \binom{i}{2} K_{-1,G}(X) \oplus \mathrm{Nil\ terms}\\ &\cong Wh_G^{\mathrm{PL}}(*) \oplus i\tilde{K}_{0,G}(*) \oplus \binom{i}{2} K_{-1,G}(*) \oplus \mathrm{Nil\ terms}\\ &\cong \oplus_{(H)} Wh(Z[NH/H]) \oplus_{(H)} \tilde{K}_0(Z[NH/H])\\ &\qquad \oplus_{(H)} \binom{i}{2} K_{-1}(Z[NH/H]) \oplus \mathrm{Nil\ terms};\end{aligned}$$

(2)

$$Wh_G^{\mathrm{CEPL}}(X \times T^i) \cong \oplus_{(H)} \tilde{K}_0(Z[NH/H]) \oplus_{(H)} \binom{i}{2} K_{-1}(Z[NH/H]) \oplus \mathrm{Nil\ terms};$$

$$(3)\qquad Wh_G^{\mathrm{htop}}(X \times T^i) \cong \mathrm{Nil\ terms}.$$

8. Obstructions to finiteness in compact G-manifolds

Since Quinn [Q2] gave the first explicit examples of compact G-manifolds not equivariantly homotopy equivalent to finite G-complexes, several others have given examples, e.g., [DoRo]. We apply the preceeding material to give examples of compact G-manifolds M (with boundary) exhibiting all controlled finiteness obstructions of compact G-ANR's and hence, as control relaxation $\tilde{K}_{0,G}(X)_c \to \tilde{K}_{0,G}(X)$ sends $\sigma_c(X)$ to $\sigma(X)$, of all finiteness obstructions of compact G-manifolds. As may be seen from the example of Section 7, this picks up distinct new classes of obstructions.

Our examples are easy to construct: let $x \in \tilde{K}_{0,G}(X)_c$, for X a compact G-ANR and control the identity of X. From Section 7, we know that $\tilde{K}_{0,G}(X)_c$ is a π_1-system functor, so we may assume that X is a compact p.l. G-manifold. Now let $\varepsilon > 0$ be so small that all equivariant ε-h cobordisms on $X \times S^1$ are topologically trivial, using our controlled equivariant h-cobordism theorem of Section 5. Choose an element f: $Y \to X \times S^1$ of $\tilde{K}_{0,G}(X)_\varepsilon$ (for sufficiently small ε). By our version of Chapman's realization theorem of [Ch10], we may choose Y to be an equivariant ε-h cobordism on $X \times S^1$. Next, let $(\tilde{Y}, X \times \boldsymbol{R})$ be the indicated infinite cyclic cover of $(Y, X \times S^1)$ and construct a G-p.l. bicollared splitting submanifold N of $\tilde{Y}$ with $N \cap (X \times \boldsymbol{R}) = X \times (0)$ such that inclusion $N \to \tilde{Y}$ is a π_1-system isomorphism. Let Z be the closure of the "positive" component of $\tilde{Y} - N$. Using an equivariant topological ε-trivialization of $\tilde{Y} \cong X \times \boldsymbol{R} \times I$, we can attach a copy $X \times (\infty) \times I$ of $X \times I$ to the end of Z to compactify it into a G-manifold M.

The restriction of the projection $X \times \boldsymbol{R} \times I \to X \times \{0\}$ now induces a retraction r: $M \to X$ which is a π_1-system isomorphism and thus a $\tilde{K}_{0,G}(\cdot)_c$-isomorphism. A simple definition chase verifies that $\sigma_c(M) = i_* \sigma_c(\varepsilon)$, where ε is the end of Z over M controlled by inclusion i: $Z \to M$, so naturality gives $r_* \sigma_c(M) = r_* i_* \sigma_c(\varepsilon) = r_* \sigma_c(Z) = \eta$, where Z is controlled over M by inclusion. Note that $\eta = r_*(\zeta)$, where ζ is the controlled finiteness obstruction $\sigma_c(Z) \in \tilde{K}_{0,G}(Z)_c$ of Z with control over X by r, but $r_*(c)$, suitably relaxed, equals the splitting obstruction $s_\delta^0(\tilde{Y})$. By an equivariant version of a result in [Ch10], this is up to sign conventions the torsion x of f: $Y \to X \times S^1$.

Our examples have the property that even though they represent all (identity-controlled) finiteness obstructions of compact G-ANR's, they all stabilize to equivariantly triangulable G-manifolds by product with $\boldsymbol{R}$ (trivial action).

If one desires closed manifolds, one cannot realize all of the above obstructions because of Poincaré duality. A version of Milnor's duality formula [Mi2] shows that for $\boldsymbol{R}$-stably triangulable closed G-manifolds M, $\sigma_c(M) = \tau + \bar{\tau}$ for some element τ, where $\bar{\tau}$ is an analogous conjugate of τ. All of the obstructions satisfying this symmetry property can be obtained from the above examples M by taking $\partial(M \times I^n)$, $n > 2$.

If the relaxation $\tilde{K}_{0,G}(X)_c \to \tilde{K}_{0,G}(X)$ is not injective, then the realization of an element of the kernel will be a compact G-manifold M with vanishing equivariant finiteness obstruction that does not admit a handle structure because its controlled finiteness obstruction does not vanish. D. Webb [Web] has shown this to indeed be the case.

9. Structure spaces

This material is due to P. Kahn and the first author, generalizing inequivariant work of Burghelea, Lashof, and Rothenberg [BuLRo], [BuL1, 2, 3].

Fix a category ξ = DIFF of PL of locally linear G-manifolds, and let M be a G-manifold in ξ. Define $\mathscr{S}^\xi(M \times I$, rel. $M \times \{0\})$ to be the simplicial space of ξ structures on $M \times I$, relative to $M \times \{0\}$ with the naturally induced G-action. A typical k-simplex of type H is an H-equivariant sliced ξ structure on $M \times I \times \Delta^k$ that is equivariantly fiber-preserving (over Δ^k) homeomorphic to the product structure, rel. $M \times \{0\} \times \Delta^k$. Face and degeneracy operators are induced from Δ.

The first theorem is that $Wh_G{}^{PL,\varrho}(M)_c \simeq \pi_0^{G,\mathscr{S}^\xi}(M \times I, \text{ rel. } M \times \{0\})$. Here $\pi_*^G(X)$ denotes $\pi_*(X^G)$.

Now, for a locally finite finite dimensional simplicial G-complex K, let M be a regular G-neighborhood of K in some representation space and set $P^\xi(K) = \lim_{\to\varrho} \mathscr{S}^\xi(M \times D_\varrho \times I, \text{ rel. } M \times D_\varrho \times \{0\})$, where ϱ ranges over the isomorphism classes of orthogonal G-representations and D_ϱ is its unit disc.

The next theorem is that $P_*^G P^\xi(K)$ is the non-negative half of the integrally graded portion of an RO_G-graded.

Let E^ξ be the representing G-spectrum of this homology theory. Then

$$\pi_0^H E^\xi \cong Wh_H{}^{PL}(*),$$

$$\pi_{-1}^H E^\xi \cong \tilde{K}_{0,H}{}^{PL}(*),$$

$$\pi_{-2}^H E^\xi \cong K_{-1,H}{}^{PL}(*).$$

For this homology we have seen that the 0-th space of $E^\xi \wedge M_+$ is G-homotopy equivalent to $P^\xi(M)$. The higher homotopy of $E^{PL} = 0$.

10. *h*-Cobordisms without equivariant handle decompositions

This material is due to the first author. Fix a category ξ = DIFF, PL, or TOP, and let M be a G-manifold in ξ. Let $\text{hCob}^\xi(M)$ denote the classifying space for G-h cobordisms (on M) bundles $E \to B$ (no action on B), that is, the classifying space for bundles P: $E \to B$ containing trivial sub-bundles $B \times M \subset E$ with each inclusion $\{b\} \times M \to p^{-1}(b)$ an equivariant h-cobordism.

For $M = N \times I$ there is a fiber sequence

$$\mathscr{S}^\xi(M \times I, \text{ rel. } M \times \{0\}) \to \text{hCob}^\xi(M) \to \text{hCob}^{TOP}(M))$$
$$\to \mathscr{S}^\xi(N \times R \times I, \text{ rel. } N \times \boldsymbol{R} \times \{0\}) \to \text{hCob}^\xi(N \times R).$$

The above fiber sequence induces an exact sequence

$$Wh_G{}^{PL,\varrho}(M)_c \to Wh_G{}^{PL,\varrho}(M) \to \pi_0^G \,\text{hCob}^{TOP}(M) \to \tilde{K}_{0,G}^\varrho(M)_c \to \tilde{K}_{0,G}^\varrho(M).$$

As a corollary, we have from Webb's calculation [Web] that $\tilde{K}_{0,G}^\varrho(M)_c \to \tilde{K}_{0,G}^\varrho(M)$ is not always injective that there are G-h cobordisms on compact PL G-manifolds without equivariant handlebody decompositions. The exact sequence, of course, is so designed that the image of the middle term gives precisely the equivariant homeomorphism classes, rel. M of such h-cobordisms.

A second corollary is that

$$\pi_0^G \,\text{hCob}^{TOP}(M) \cong Wh^{top,\varrho}(M),$$

where the latter is the Q_G-stable homeomorphism classes (equivariant) of restricted compact G-ANR pairs (Y, M), restricted meaning as ever that $Y_H = \emptyset$ if $M_H = \emptyset$. This is probably true without the I-stability hypothesis on M.

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