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CLASSIFICATION PROBLEMS IN LOW-DIMENSIONAL TOPOLOGY *

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I. Introduction

This is a survey on classification results in low-dimensional topology.

The Greeks were the first ever to whom the idea of classifying mathematical objects occured. From those early days on their classification of platonic bodies, in particular, became a prototype of what should be considered a "classification". However, the classification of platonic bodies deals with a *finite* set of mathematical objects, whereas most of the mathematical theories today have to face *infinitely* many objects, and in the latter case it often remains obscure what is really meant by their classification. One reason for this is that the Greeks' classification has actually two aspects: one is that it offers a *complete* list of objects in question, and the other is that it offers *models* for these objects. Hence, in general, one is prompted again to ask for a *complete list*, i.e., more precisely, for a complete enumeration without repetition, of *models* for the (equivalence classes of the) objects in question. This is sometimes possible to achieve (see e.g. the classification of finite dimensional vector spaces, up to isomorphy). In the other case, however, either a complete list does not exist at all, if e.g. the set of objects is not countable, or the meaning of the notion "model" is not clear, for a model should reflect "many" of the properties shared by the objects which it represents.

Here we deal with low-dimensional manifolds, up to homeomorphy, and homotopy equivalences (especially homeomorphisms) between them, up to homotopy. It is known that in dimensions 2 and 3 these sets of objects are countable, and this fact led Papakyriakopoulos [Pa] to the definition: "*Classification* means to define an infinite sequence of closed n-manifolds $M_1, M_2, \ldots$ such that any two of these are not homeomorphic, but any closed n-manifold M is homeomorphic with one of them".

* This paper is in final form and no version of it will be submitted for publication elsewhere.

In II we outline in some detail the classification of Haken 3-manifolds in the previous sense, as given by Haken and Hemion. Furthermore, we discuss the possibility of models for homotopy equivalences between 2- and 3-manifolds, and we describe Hemion's solution of the conjugacy problem in the mapping class group of surfaces.

The material is organized under the following headlines:

II. Surfaces

Surfaces are classified. Models for closed, orientable surfaces, say, are provided by connected sums of the 2-dimensional sphere, S^2, with tori. This result is classical and well known. The next step is the study of homotopy equivalences, especially diffeomorphisms, of surfaces. There are several approaches to this problem, but I think it is fair to say that two of them proved to be of particular interest and power: the Nielsen theory and the Teichmüller–Thurston approach. Both theories were begun around 1930 and have eventually led (after almost half a century) to a number of classification results in surface theory, which include the classification of surface-diffeomorphisms, up to isotopy, and the conjugacy problem in the mapping class group. The aim of this section is to outline Nielsen's approach (for the Teichmüller–Thurston approach see [Ber 1, 2] and [FLP]).

To begin with let M^2 be any closed, orientable surface of genus $\geqslant 2$ (the non-closed case is similar), and fix some basepoint in M^2. Let h be any (basepoint preserving) diffeomorphism of M^2. Then h induces an automorphism, h_*, of $\Pi_1 M^2$ which is well-defined, up to (basepoint preserving) isotopies of h. Now, call h a *pseudo-Anosov diffeomorphism*, if no power of h_* maps any element of $\Pi_1 M^2$ to itself (or some conjugate of it in the fundamental group). In §§ 1–3 we only consider this particular type of diffeomorphisms and defer to § 4 the discussion of how to reduce the general study of homotopy equivalences between surfaces (closed or not) to that one of pseudo-Anosov diffeomorphisms.

§ 1. **Circle-homeomorphisms given by surface-diffeomorphisms.** Let $\mathrm{Diff}_*(M^2)$ denote the space of all (basepoint preserving) diffeomorphisms of M^2. Then the assignment $[h] \to h_*$ defines a map $\Pi_0 \mathrm{Diff}_*(M^2) \to \mathrm{Aut}\, \Pi_1 M^2$ and Nielsen shows that this map is an isomorphism [Nie 1]. Hence the group $\Pi_0 \mathrm{Diff}_*(M^2)$ can be considered as acting on $\Pi_1 M^2$, and Nielsen studies the elements of $\Pi_0 \mathrm{Diff}_*(M^2)$ via their action on $\Pi_1 M^2$. For this purpose Nielsen first turns $\Pi_1 M^2$ into a metric space with the remarkable property that the iterates of any point in this metric space under the action of a pseudo-Anosov automorphism defines a Cauchy sequence.

In order to describe *Nielsen's metric for* $\Pi_1 M^2$, let us first, once and for all, fix some hyperbolic structure on M. Then the universal cover, $\tilde{M}$, of M is the hyperbolic plane. This hyperbolic plane is equipped with a natural compactification as a disc, especially equipped with a boundary, $\partial\tilde{M}$, if we think of $\tilde{M}$ as being the interior of the unit disc in the complex plane. W.l.o.g. the origin, P, in $\tilde{M}$ lies above the basepoint in M.

With these preliminaries in mind we have

LEMMA. *There is a natural map* $\Pi_1 M^2 \to \partial\tilde{M} = S^1$ *which is almost injective.*

More precisely, the preimage of any point in $\partial\tilde{M}$ under this map is given by the set of all (positive) multiples of some element in $\Pi_1 M^2$.

Proof. This lemma is immediate from the well-known fact that $\Pi_1 M^2$ always can be identified with some discrete subgroup of $\mathrm{PSL}_2 \boldsymbol{R}$ which consists of hyperbolic elements alone. The required map is then obtained by simply mapping an element of $\Pi_1 M^2$ to the positive fundamental-point (= end-point) of the axis of the corresponding element in $\mathrm{PSL}_2 \boldsymbol{R}$.

Alternatively, start with some based loop, α, in M^2. Such a loop gives rise to a sequence, $\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots$, of lifted arcs in $\tilde{M}$ with the property that the starting point of $\tilde{\alpha}_1$ is the origin, P, and that the starting point of any other $\tilde{\alpha}_{i+1}$ is the end-point of $\tilde{\alpha}_i$. The union of all these arcs defines a ray, $\tilde{\alpha}$, (the ray corresponding to α) which converges (in the euclidean metric) to some unique point in $\partial\tilde{M}$, and we get the required map by mapping each $[\alpha] \in \Pi_1 M^2$ to the far end-point of $\tilde{\alpha}$. (Indeed the far end-point of $\tilde{\alpha}$ equals the positive fundamental point corresponding to $[\alpha] \in \Pi_1 M^2 < \mathrm{PSL}_2 \boldsymbol{R}$.)

The *Nielsen-metric*, d_N, for $\Pi_1 M^2$ is now given by the induced metric on $S^1 \subset \boldsymbol{C}$ via the previous map $\Pi_1 M^2 \to S^1$. In particular, $d_N(\alpha, \beta) = 0$ iff $\alpha^n = \beta^m$, for some $n, m \geqslant 1$.

We note that, given a presentation of $\Pi_1 M^2$, one may conclude, using rays as above, that the points of $\partial\tilde{M}$ are in correspondence with sequences (or rather products) in the given generators of $\Pi_1 M^2$. But while the *rational points of* $\partial\tilde{M}$ (:= the image points of the map $\Pi_1 M \to \partial\tilde{M}$) correspond to finite sequences, the *irrational points* (i.e. the remaining points) correspond to infinite sequences [Nie 1, p. 220].

In particular, the set of rational points is countable, and, moreover, it is dense in $\partial\tilde{M}$ (i.e. the rational set in $\partial\tilde{M}$ behaves like $\boldsymbol{Q}$ in $\boldsymbol{R}$, hence the name) [Nie 1, p. 210]. Therefore any homeomorphism of the rational set extends to a unique circle-homeomorphism. This gives an injection $\operatorname{Aut} \Pi_1 M \to H(\partial\tilde{M}) = H(S^1)$, $\varphi \to \varphi_*$, where $H(S^1)$ denotes the group of all circle-homeomorphisms. Thus to any (basepoint preserving) diffeomorphism, h, of M^2 there is assigned the circle-homeomorphism $h_{**} := (h_*)_*$ which preserves the rational set. Furthermore, any covering translation, $d \in \Pi_1 M^2$, extends continuously to $\partial\tilde{M}$ and so we may note that $d \circ h_{**}$ is again a well-defined circle-homeomorphism.

Now, let h be any (basepoint preserving) pseudo-Anosov diffeomorphism of M^2. Furthermore, let $g = dh^m_{**}$ and $g' = d' h^n_{**}$, for some d, $d' \in \Pi_1 M^2$ and $m, n \in \boldsymbol{Z}$. Then the following properties of the induced circle-homeomorphisms, g, g', lie in the core of Nielsen's theory:

(1) $\operatorname{Fix}(g)$ consists of irrational points alone (here $\operatorname{Fix}(\cdot)$ denotes the fixpoint set).

(2) $\operatorname{Fix}(g)$ is a finite set and consists of an even number of points.

(3) The fixpoints of g in $\partial\tilde{M}$ are alternating attracting and repelling. Now, let $\operatorname{Fix}^+(g)$, resp. $\operatorname{Fix}^-(g)$, denote the set of all contracting, resp. repelling fixpoints of g. Then we have

(4) $\operatorname{Fix}^+(g')$ either equals $\operatorname{Fix}^+(g)$ or is entirely contained in one *component* of the complement of $\operatorname{Fix}^+(g)$ in $\partial\tilde{M}$ (the same with $\operatorname{Fix}^-(g')$).

(5) $\bigcup \operatorname{Fix}^{\pm}(g)$ is countable and dense in $\partial\tilde{M}$, where the union is taken over all $g = dh^m_{**}$ with $d \in \Pi_1 M^2$ and $m \in \boldsymbol{Z}$ (see [Mil, Prop. 5]).

The previous properties (except (5)) are classical and contained in [Nie 1]. Nielsen discovered that these results can be fruitfully used in a study of the fixpoint problem for surface-diffeomorphisms (solving this problem in two special cases). However, far more is true. Not only are these properties very useful in the study of special problems, but, moreover, they describe a pseudo-Anosov diffeomorphism completely. We will see soon how easily indeed, Nielsen's results lead to a full classification of pseudo-Anosov diffeomorphisms (and so also to a full solution of the fixpoint problem for pseudo-Anosov diffeomorphisms). As a matter of fact it is somewhat astonishing that this possibility escaped attention for such a long time.

To continue observe that, by the above properties, every rational point in $\partial\tilde{M}$ is contained in some expanding interval of h_{**}, i.e. in some interval of $\partial\tilde{M}$ bounded by two neighboring points of $\operatorname{Fix}^+(h_{**})$. In particular, the iterates of elements from $\Pi_1 M^2$ under (pseudo-Anosov) automorphisms of $\Pi_1 M^2$ define indeed a Cauchy sequence in Nielsen's metric of $\Pi_1 M^2$. Therefore the question arises naturally, how fast such a Cauchy sequence might converge to its limit point in the completion of $\Pi_1 M^2$. A more recent argument of Hemion [He] suggests that the velocity cannot be very fast. Since we need this result later on, we give the argument below.

To get a first insight into the *velocity of convergence* we fix, following Hemion, a tesselation, Δ, of $\tilde{M}$ by geodesic fundamental regions of $\Pi_1 M^2$. This tesselation is chosen such that the origin, P, of $\tilde{M}$ appears as lattice point of Δ. Let $\alpha \in \Pi_1 M^2$ and let $\tilde{h}$ be that lifting of h fixing the origin. Then $\tilde{\alpha}$ denotes the ray corresponding to α (see above) and $\tilde{\alpha}_i$ denotes the ray $\tilde{\alpha}_i := \tilde{h}^i \tilde{\alpha}$. Then the set $\{\tilde{\alpha}_i\}$ is a bunch of rays whose far end-points *all* lie in *one* expanding interval of h_{**}. Each two rays $\tilde{\alpha}_i$ and $\tilde{\alpha}_{i+1}$ separate a uniquely given region, A_i, from the closure $\tilde{M}^-$ which contains no fixpoint from $\mathrm{Fix}^+(h_{**})$. A *branching point*, z_i, of the pair $(\tilde{\alpha}_i, \tilde{\alpha}_{i+1})$ is, by definition, one of those lattice points of Δ different from P, contained in the region A_i and with minimal distance, $d_\Delta(P, z_i)$, to the origin. Furthermore, denote by $z_{\min}$ one of those points from the set of all branching points whose distance, $d_\Delta(P, z_{\min})$, to P is minimal. Here the distance, $d_\Delta(x_1, x_2)$, between two points (or sets) x_1, x_2 of $\tilde{M}$ is defined to be the minimal number of fundamental regions one has to meet while travelling from x_1 to x_2.

The distance $d_\Delta(P, z_i)$ (or rather its reciprocal) of the branching point z_i of $(\tilde{\alpha}_i, \tilde{\alpha}_{i+1})$ to the origin also is a measure for the distance between the two elements of $\Pi_1 M^2$ covered by $\tilde{\alpha}_i$ and $\tilde{\alpha}_{i+1}$. This new distance actually "approximates" the distance of the same elements in the Nielsen metric.

Our aim is now to estimate the distance, $d_\Delta(z_i, z_{\min})$, of the branching points to the minimal branching points. This can be done via the *distortion of* $\tilde{h}$. To define this distortion define first the *Δ-measure*, $m_\Delta(A)$, of any set, A, in $\tilde{M}$ to be the number of all fundamental regions of Δ which meet A. Then the distortion of $[h]$, or $\tilde{h}$, is defined to be

$$d_\Delta(h) := \min\{m_\Delta(\tilde{g}\Omega)\},$$

where Ω is some fundamental region of Δ and where the minimum is taken over all homeomorphisms $\tilde{g}\colon \tilde{M} \to \tilde{M}$ which are Π_1-equivariantly isotopic to $\tilde{h}$, fixing lattice points.

Proposition (Hemion). $d_\Delta(z_i, z_{\min}) \leqslant d_\Delta(h)^i$.

Proof. By our choice of $z_{\min}$ as minimal branching point, this point is a vertice of at least one fundamental region, Ω, of Δ which meets all rays, $\tilde{\alpha}_i$. If the proposition is false, then $d_\Delta(z_i, z_{\min}) > d_\Delta(h^i)$ and so $\Omega \cap \tilde{h}^i \Omega = \emptyset$. Since $\tilde{h}$ acts transitively on the bunch of rays, $\tilde{h}\Omega$ separates the region $A_{\min}$, too, and the point $z_{\min}$ has to lie below $\tilde{h}^i \Omega$, i.e. in that component of $(A_{\min} - \tilde{h}\Omega)$ containing P. But then $\tilde{h}^{-i} z_{\min}$ is below Ω, contradicting our minimal choice of $z_{\min}$.

This ends the analysis of the circle-homeomorphisms given by surface-diffeomorphisms. Next we translate the information about the circle homeomorphisms into informations about the original surface-diffeomorphisms themselves.

§ 2. **Extremal surface-diffeomorphisms.** Given any isotopy class, $[h]$, of (based) surface-diffeomorphisms, the question arises whether there is a way of picking a "typical" member in this class. Teichmüller and Nielsen were the first to consider such a question and Thurston has given the final solution.

Nielsen e.g. studied isotopy classes, $[h]$, which are periodic elements in $\Pi_0 \operatorname{Diff}_*(M^2)$, or in the mapping class group, $\Pi_0 \operatorname{Diff}(M^2)$. He showed in a long paper that in this case one always can pick a *periodic* diffeomorphism in $[h]$ (see [Nie 2] and [Zi] for filling a gap of that paper). The same result can be obtained from Teichmüller theory as well [Kra].

Moreover, Teichmüller shows that there always exists a certain extremal diffeomorphism in the isotopy class of each diffeomorphism between Riemannian surfaces. Today this unique extremal diffeomorphism is known as *Teichmüller map.* The precise definition of a Teichmüller map involves the notion of (holomorphic) quadratic differentials. For our purpose, however, the essential feature of a quadratic differential $\varphi(z)\,dz^2$ on a Riemannian surface M^2 is given by the fact that it describes (similar to differential forms) a foliation on M^2 with singularities. In fact, it describes a pair of transversal codim. 1 foliations – the horizontal and the vertical foliation (given by the real and the imaginary part of $\varphi(z)\,dz^2$, respectively). These foliations have singularities of a very special type only, however, they are not necessarily carried by vector fields. A Teichmüller map now (1) maps the pair of foliations of one (the initial) quadratic differential to that one of another (the terminal) quadratic differential, and (2) is expanding (by a factor λ, say) in the horizontal direction and contracting (by the factor λ^{-1}) in the vertical direction. The existence of such Teichmüller maps in each isotopy class of diffeomorphisms between Riemannian surfaces follows from Teichmüller space considerations [Tei 1, 2] (see also [Ber 1] for a modern proof). Thurston shows, by means of an appropriate compactification of the Teichmüller space, that a similar, but even stronger theorem is true (see [FLP], and [Ber 2] for a different proof). He also noticed that this stronger result follows from Nielsen's analysis of induced circle-homeomorphisms, too. The latter will be described below.

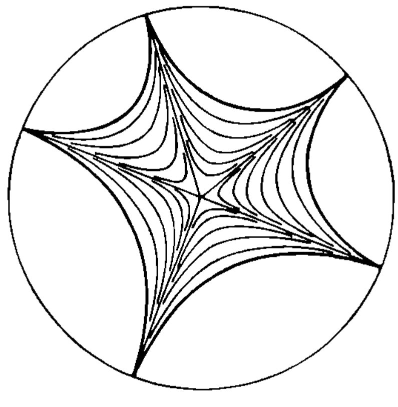
To reach our goal we first have to construct the *extremal extension* of the circle-homeomorphism $h_{**}\colon \partial\tilde{M} \to \partial\tilde{M}$ given by $[h]$. For this first associate to each set $\operatorname{Fix}^+(dh^n_{**})$ the ideal polygon, $\Omega^+(dh^n_{**})$, in $\tilde{M}$ generated by this set, i.e. the convex region in $\tilde{M}$ spanning the points of $\operatorname{Fix}^+(dh^n_{**})$. Let $\mathfrak{D}^-$ denote the union over all these ideal polygons and define $\mathfrak{D}^-$ similarly, using $\operatorname{Fix}^-(dh^n_{**})$ instead. Then note that, by properties (4) and (5) of h_{**}, the set $\mathfrak{D}^+$ (and $\mathfrak{D}^-$ as well) is dense in $\tilde{M}$ and that it consists of pairwise disjoint ideal polygons.

Now, observe that the circle-homeomorphism h_{**} maps all the ideal vertices of an ideal polygon, $\Omega(dh^n_{**})$, from $\mathfrak{D}^+$, resp. $\mathfrak{D}^-$, onto those of another ideal polygon from $\mathfrak{D}^+$, resp. $\mathfrak{D}^-$. Thus, given h_{**}, there is an obvious way to map the vertices of the polygons from $\mathfrak{D}^+ \cap \mathfrak{D}^-$. Applying the cone construction twice (to the edges first and then to the polygons themselves), this map extends to a diffeomorphism, $\tilde{h}_e\colon \tilde{M} \to \tilde{M}$, which permutes the polygons from $\mathfrak{D}^+ \cap \mathfrak{D}^-$ and which, moreover, extends h_{**} continuously. We call $\tilde{h}_e$ the *extremal extension of* h_{**}.

Since h_{**} is Π_1-equivariant, the same holds for $\tilde{h}_e$ and so $\tilde{h}_e$ lies above some unique (based) surface-diffeomorphism, h_e, of M^2. Since h_e is determined by h_{**} it is known that h_e actually lies in the isotopy class $[h]$ (see [Nie 1]). We call h_e an *extremal diffeomorphism from* $[h]$.

The extremal diffeomorphisms from $[h]$ are not unique, for they depend on the choice of a particular hyperbolic structure on M^2. However, it is known that any two extremal diffeomorphism within the same isotopy class are conjugate (via some surface-diffeomorphism which is isotopic to the identity) [FLP, exposé 12]. Thus invariants of the conjugacy-type of surface-diffeomorphisms are the same for all isotopic extremal diffeomorphisms. Examples of such invariants are provided by the (topological) entropy, the number of fixpoints (or fixpoint classes) and the (topological) dilatation of surface-diffeomorphisms. It turns out that extremal diffeomorphisms minimize all these invariants within their isotopy class, which justifies their name [FLP, exposé 10], [Ber 2]. Moreover, extremal diffeomorphisms behave very much like very nice Teichmüller maps.

To see the latter, foliate any ideal polygon from $\mathfrak{D}^+$ as follows:



Carrying this out for all ideal polygons from $\mathfrak{D}^+$, gives a Π_1-equivariant foliation, $\mathfrak{F}^+$, of $\tilde{M}$ (with singularities). Similarly, we obtain $\mathfrak{F}^-$, using $\mathfrak{D}^-$ instead of $\mathfrak{D}^+$. The foliations $\mathfrak{F}^+$ and $\mathfrak{F}^-$ are transversal (see property (3)) and their critical leaves are geodesics converging to attracting resp. repelling fixpoints of h_{**}. It follows that $\tilde{h}_e$ preserves these foliations and is expanding in the direction of $\tilde{\mathfrak{J}}^+$ and contracting in the direction of $\tilde{\mathfrak{J}}^-$. Since everything is Π_1-equivariant, we have a similar picture in M^2 below. In particular, $\tilde{\mathfrak{J}}^+$, $\tilde{\mathfrak{J}}^-$ lie over a pair of transversal foliations, $\mathfrak{J}^+$, $\mathfrak{J}^-$, on M^2 left invariant by the extremal diffeomorphism h_e: $M \to M$. The local structure of the pair $(\mathfrak{J}^+, \mathfrak{J}^-)$ is given by the two sets of level curves associated to the two maps $z \to \mathrm{Re}(z^k)$ and $z \to \mathrm{Im}(z^k)$. Moreover, the fixpoints of h_e are precisely the singularities of $\mathfrak{J}^+$, and h_e "rotates" the set of critical leaves near a fixpoint.

To complete the picture note that $\mathfrak{J}^+$, $\mathfrak{J}^-$ can be realized by some quadratic differential on M^2, i.e. there is a hyperbolic structure on M^2 and a quadratic differential $\varphi(z)dz^2$, on M^2 whose pair if horizontal and vertical foliations equals $\mathfrak{J}^+$, $\mathfrak{J}^-$ (for a more general statement see [Hu–Ma]). With other words, h_e is a Teichmüller map whose initial and terminal quadratic differentials coincide. In patricular, h_e appears to be (locally) affine in the flat metric given by $\varphi(z)dz^2$ on M^2 minus singularities of $\varphi(z)dz^2$.

This ends our discussion of the qualitative behaviour of extremal diffeomorphisms.

§ 3. The conjugacy problem. In this section a surface-problem is to be discussed which itself is closely related to the classification of Haken 3-manifolds (see next chapter). This is the conjugacy problem for the mapping class group, $\Pi_0 \mathrm{Diff}(M^2)$, of surfaces.

Conjugacy problem. Let M^2 be a closed orientable surface of genus $\geqslant 2$. Given two diffeomorphisms, f and g, of M^2, does there exists a diffeomorphism, h, of M^2 so that hf is isotopic to gh?

This problem was solved by Hemion in 1978 [He]. Here we outline the argument in the closed case (the bounded case being similar, requiring some technical modifications, however).

First of all, the conjugacy problem is easily reduced to the case where f and g are pseudo-Anosov. Since only this case will be needed later on, we simply suppose that f and g are both pseudo-Anosov. Moreover, we assume w.l.o.g. that f, g and h are always isotoped so that they all fix the basepoint.

The idea to solve the conjugacy problem is to show that there are "essentially" only *finitely* many diffeomorphisms, h, which possibly can conjugate f and g, and that, moreover, the set of all of them can be effectively *constructed*. To make this idea work the distortion, $d_\Delta(h)$, of a diffeomorphism, h, as introduced by Hemion (see § 1) turns out to be very helpful. The reason for this is that the isotopy class of a basepoint-preserving diffeomorphism is completely determined by the action of its lift (to the universal cover $\tilde{M}$) on the set of vertices of some fundamental region. Hence the set of all diffeomorphisms, h, of M^2 with $d_\Delta(h) \leqslant$ const. is a finitely, effectively constructable set (see definition of "distortion"). Thus the conjugacy problem follows from

THEOREM (Hemion). *Let f and g be given as above, and suppose that h is a diffeomorphism of M^2 such that fh is isotopic to gh. Then there is some diffeomorphism h^* with $h^* = f^n h$, for some $n \geq 1$, such that*

$$d_\Delta(h^*) \leq \Phi(d_\Delta(f), d_\Delta(g), \mathrm{genus}(M^2)).$$

Here Φ denotes some explicitely known function in three variable. By abuse of language, a positive integer will be called *small* if it is bound from above by some Φ.

Now, the complexity of the previous theorem can be reduced considerably by the observation that it follows from the mere existence of some "doubly small" curve. To be more precise, let Δ be the given tesselation of $\tilde{M}$ by fundamental regions, let $\Delta^* = h^* \Delta$ with $h^* := f^n h$, and let the lengths $d_\Delta(c)$, $d_{\Delta^*}(c)$ of a curve, c, in M^2 be defined to be the Δ-, resp. Δ^*-measure of some lift of c to $\tilde{M}$ (see § 1 for definitions). Then the previous theorem follows from the existence of some closed curve, c, in M^2 (singular or not) for which there is an integer n such that $d_\Delta(c)$ as well as $d_{\Delta^*}(c)$ is small (in the above sense). To see this claim recall that f is supposed to be pseudo-Anosov. In this case there exists a small integer, q, (depending on the genus of M^2 alone) such that the union $\bigcup_{1 \leq i \leq q} c_i$ splits M^2 into a system of discs, where c_i denotes the unique, closed geodesic in the homotopy class of $f^i \circ c$. Now, lift all these discs to the universal cover to obtain a new tesselation of $\tilde{M}$ and recall that c is supposed to be doubly small. It is then easily checked that the image of some (and so any) fundamental region of $\Pi_1 M^2$ under the lifting of h^* (or rather some diffeomorphism Π_1-equivariantly isotopic to it) is covered by a small number of discs of the previous tesselation and that all these discs have to have small Δ-measure. It follows that $d_\Delta(h^*)$ is small, proving the claim.

Therefore everything boils down to

The construction of some doubly small curve. The existence of a doubly small curve follows from a certain combinatorial fact about pairs of tesselations in $\tilde{M}$. In order to describe this fact first, let us be given two tesselations, say Δ and Δ^*, of $\tilde{M}$ by geodesic polygons. Suppose that the set of all lattice points of Δ^* is contained in that of Δ. Then the intersection $\Delta \cap \Delta^*$ consists of geodesic polygons again. We distinguish between "rectangles" and "non-rectangles" in $\Delta \cap \Delta^*$. Here a *rectangle* is a component of $\Delta \cap \Delta^*$ which is a quadrilateral with precisely two opposite sides in (interiors) of sides of some polygon of Δ.

Finally, let us be given two rays, $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, corresponding to two different elements, α_1, α_2, of $\Pi_1 M^2$ ($\alpha_1^n \neq \alpha_2^m$, for all $n, m \in N$) and let z be some branching point between $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ (see § 1).

LEMMA. *There is at least one non-rectangle, Σ, in $\Delta \cap \Delta^*$ such that $d_\Delta(z, \Sigma) = 1$ and $d_{\Delta^*}(P, \Sigma) \leq 2\, d_{\Delta^*}(\alpha_1) + d_{\Delta^*}(\alpha_2)$.*

(recall that P denotes the origin of $\tilde{M}$, and so P is the starting point of the rays $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$).

Proof. Let Ω be the polygon from Δ whose one vertex is z and which meets both $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$. Then one side of Ω, say σ, meets both $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ (z is a branching point). W.l.o.g. σ splits the area between $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ into two regions and we let A' denote that one of those containing the origin, P. Then A' contains no other lattice point of Δ, and so none of Δ^* except P.

Now, let x be some point from the intersection of $\tilde{\alpha}_1$ with $\partial\Omega-\sigma$. Let Σ_1, resp. Σ_2, be the non-rectangle in Ω which we first meet while travelling from x along $\partial\Omega-\sigma$ to the left, resp. to the right. Then certainly $d_\Delta(z, \Sigma_i) = 1$, for $i = 1$ and 2, and $d_{\Delta^*}(P, x) \leqslant d_{\Delta^*}(\alpha_1)$ since A' contains no lattice point besides P. Hence it remains to show that $d_{\Delta^*}(x, \Sigma_i) \leqslant d_{\Delta^*}(\alpha_1)+d_{\Delta^*}(\alpha_2)$, for $i = 1$ or $i = 2$.

To see the latter observe that we only meet rectangles in Ω while travelling from Σ_1 to Σ_2. *All* of these rectangles have to join the same two sides of Ω. It follows that either *all* rectangles which we meet on the right of x or *all* of those which we meet on the left of x, have to intersect $\tilde{\alpha}_1 \cup \tilde{\alpha}_2$, for $A'-P$ contains no lattice points. Thus the lemma follows.

In order to obtain a doubly small curve from the previous lemma, let Δ now denote the tesselation of $\tilde{M}$ given by geodesic fundamental regions of $\Pi_1 M^2$. Furthermore, let α be some non-trivial element of $\Pi_1 M^2$, and denote by $\tilde{\alpha}$ the ray corresponding to α. Denote by $\tilde{\alpha}_i$ the ray $\tilde{\alpha}_i := \tilde{f}^i \tilde{h}\tilde{\alpha}$, and let z_i be the branching point between $\tilde{\alpha}_i$ and $\tilde{\alpha}_{i+1}$. Let n be an integer with the property that $d_\Delta(P, z_n) \leqslant d_\Delta(P, z_i)$, for all $i \in \mathbf{Z}$, i.e. z_n is a minimal branching point. Finally, let $h^* := f^n h$ and $\Delta^* := h^* \Delta$. Then Δ^* is again a tesselation, and we here simply assume that it is a tesselation by geodesic polygons (see [He, § 1]). Moreover, lattice points of Δ^* are lattice points of Δ.

Observe that there is only a finite set of non-rectangles from $\Delta \cap \Delta^*$ in any given fundamental region of Δ – limited by some integer, say $m-1$, which depends on the genus of M^2 alone. Thus in any set of m pairwise different non-rectangles at least two of them are Π_1-equivalent. Consider, in particular, the non-rectangles, Σ_i, near z_i as given to us by the lemma above. Then w.l.o.g. Σ_n and Σ_{n+m} are Π_1-equivalent.

We finally estimate the distances between Σ_n and Σ_{n+m}. First of all, $d_\Delta(\Sigma_n, \Sigma_{n+m}) \leqslant (d_\Delta(f))^m$ (see prop. from § 1), i.e. $d_\Delta(\Sigma_n, \Sigma_{n+m})$ is small. To estimate $d_{\Delta^*}(\Sigma_n, \Sigma_{n+m})$ we have to utilize the fact that fh is isotopic to hg. First recall from the lemma above that $d_{\Delta^*}(\Sigma_n, \Sigma_{n+m})$ is estimated by some expression in the $d_{\Delta^*}(\alpha_{n+i})$, $0 \leqslant i \leqslant m+1$. But $d_{\Delta^*}(\alpha_{n+1}) = d_{h^*\Delta}(f^{n+i} h\alpha) = d_{h^*\Delta}(h^* g^i \alpha)$ since $fh \simeq hg$. Hence $d_{\Delta^*}(\alpha_{n+i}) \leqslant (d_\Delta(g))^i$, i.e. $d_{\Delta^*}(\alpha_{n+i})$, $0 \leqslant i \leqslant m+1$ is small, and so $d_{\Delta^*}(\Sigma_n, \Sigma_{n+m})$.

Now, altogether, projecting an arc in $\tilde{M}$ down to M^2 which joins two different Π_1-equivalent points in $\Sigma_n \cup \Sigma_{n+m}$, we obtain the required doubly small curve.

§ 4. **Homotopy equivalences.** In this section we study homotopy equivalences between surfaces. Our aim is to discuss briefly the method of splitting up homotopy equivalences into simpler pieces which then can be studied individually. This is a method which works nicely also in dimension 3 (see § 8).

The following two splitting results are relevant in our context. To describe them let M^2, N^2 denote two compact, orientable surfaces (closed or not!).

PROPOSITION. *Let f: $M^2 \to M^2$ be a diffeomorphism which is not pseudo-Anosov. Then one of the following holds*:

(1) *f is isotopic to a periodic diffeomorphism, or*

(2) *there is a system, k, of pairwise disjoint, simple closed curves in M^2 such that $f(k) = k$, up to isotopy.*

PROPOSITION. *Let f: $M^2 \to N^2$ be a homotopy equivalence. Then there is an essential surface, O_f, in N^2 such that f can be homotoped so that $f^{-1}O_f$ is an essential surface in M^2 and that, in addition,*

(1) *$f|f^{-1}O_f$: $f^{-1}O_f \to O_f$ is a homotopy equivalence,*

(2) *$f|(M-f^{-1}O_f)^-$: $(M-f^{-1}O_f)^- \to (N-O_f)^-$ is a diffeomorphism, and*

(3) *any surface, O'_f, with the above properties contains O_f, up to ambient isotopy of N^2.*

(A surface, S, in M^2 is called "essential" if the inclusion $S \subset M$ defines an injection $\Pi_1(S, \partial S) \to \Pi_1(M, \partial M)$ of the relative fundamental groups.)

Proofs of both of the previous proposition can be found in [Joh 1, 6.6 and 30.15.]. The proof of the first proposition essentially goes back to Nielsen. The proof of the second proposition is technical, but surprisingly difficult.

Observe that, by (3), the surface O_f is unique, up to ambient isotopy of N^2. By a well-known theorem of Nielsen [Nie 1], any homotopy equivalence between *closed*, orientable surfaces (genus $\geqslant 1$) is homotopic to a diffeomorphism. With other words, $O_f = \emptyset$ if $\partial N^2 = \emptyset$. On the other hand, we call f a *totally exotic* homotopy equivalence, if $O_f = N^2$.

Now, the second proposition enables us to split up any homotopy equivalence into a diffeomorphism and a totally exotic homotopy equivalence. Furthermore, by iterated application of the first proposition, any surface-diffeomorphism can be split up into periodic and pseudo-Anosov diffeomorphisms.

As discussed earlier (for the closed case) pseudo-Anosov diffeomorphisms, and periodic diffeomorphisms as well, are pretty well known. Totally exotic homotopy equivalences are not so nicely understood.

Moreover, in general, $O_f \neq f^{-1} O_f$ for self homotopy equivalences, $f\colon M^2 \to M^2$. However, self homotopy equivalences, $f\colon M^2 \to M^2$, can be analyzed to some extent by using Nielsen's presentation of the outer automorphism group of free groups [Nie 3] (note that the fundamental group of any surface with non-empty boundary is free). As a result one easily obtains that homotopy equivalences are products of Dehn flips. Here a Dehn flip is a homotopy equivalence, g, with $O_g = g^{-1} O_g$ and such that O_g is a small neighborhood of some proper arc in M^2, i.e. a Dehn flip has support in a neighborhood of some proper arc. However, the relations between Dehn flips are still unknown. Similarly, one knows that any (orientation-preserving) surface-diffeomorphism, $f\colon M^2 \to M^2$; is a product of Dehn twists ([Deh], [Lic]). Here a Dehn twist is a diffeomorphism with support in a small neighborhood of some simple closed curve in M^2. In fact, a finite set of Dehn twists can be chosen which generates the whole mapping class group of M^2. Furthermore, a finite presentation of the mapping class group has been worked out by Hatcher and Thurston [Ha–Th 2]. Finally, it has been shown by Scott [Sc] that $\Pi_i \operatorname{Diff}_*(M^2)$ is zero, for all $i \geqslant 1$ (see [Hat] for the 3-dimensional case).

This ends our discussion of surfaces.

III. 3-Manifolds

It is not known whether 3-manifolds can be classified in general. Nevertheless, certain special classes of 3-manifolds have already been classified for a while, e.g. lens spaces and sufficiently large Seifert fibre spaces. More recently, it has been shown that it can be decided whether an irreducible 3-manifold is a Haken 3-manifold [Ja–Oe], and, moreover, that the isomorphism problem for Haken 3-manifolds is solvable [Hak 2, He]. This is remarkable since Haken 3-manifolds constitute a fairly large class of 3-manifolds, including e.g. all knot spaces (see § 2 for definitions). As a matter of fact, a few years ago one actually was under the impression that "almost" all (irreducible) 3-manifolds are Haken 3-manifolds. In the meantime this opinion has changed. However, there is still strong evidence that a good understanding of Haken 3-manifolds and their geometry will eventually lead to a good understanding of all 3-manifolds as well (see e.g. Thurston's recent work on 3-manifolds).

The ultimate goal of this chapter is to discuss the classification of Haken 3-manifolds and their homotopy equivalences (see also [Wa 3]).

§ 5. Presentations. Parallel to groups in combinatorial group theory, compact 3-manifolds are explicitely given by presentations. In the theory of 3-manifolds, however, a number of different presentations are in use. In the following we give a list of them for closed (orientable) 3-manifolds (the presentations can be modified as to cover also manifolds with boundary).

Triangulations. By a well-known theorem of Moise [Moi], every compact 3-manifold, M^3, admits some triangulation, Δ. Recall that such a triangulation is given by its incidence-matrix which, by its very definition, records which two simplices are incident. In this way, each 3-manifold is presented by finitely many data.

Heegaard-presentations. A handlebody is defined to be a 3-manifold which is homeomorphic to a small neighborhood of some finite graph in the 3-sphere. Now, any (closed) 3-manifold is obtained by attaching two copies of an appropriate handlebody along their boundaries. To see this, simply observe that a small neighborhood of the 1-skeleton of the triangulation, Δ, as well as its complement in M^3 is a handlebody. Furthermore, observe that the above identification of boundaries is completely described by the isotopy class of a surface-diffeomorphism, and the latter in turn is again easily be given by finitely many data.

Polyeder-presentations. Any 3-manifold, M^3, is obtained from the pairwise identification of the faces of an appropriate polyhedron. To see this, consider the 1-skeleton of the dual triangulation of Δ, and take a spanning tree of this 1-skeleton. The required polyeder is then obtained from M by successively splitting along all those 2-simplices of Δ corresponding to those edges of the dual 1-skeleton not contained in the spanning tree.

Surgery-presentations. Let, L, be a link in S^3, i.e. a finite system of pairwise disjoint, simple closed (possibly knotted) curves in S^3. Let $N(L)$ denote a small neighborhood of L in S^3. Then $N(L)$ consists of solid tori, and taking them out from S^3 and sewing them back differently gives a new manifold. We say that this new manifold is obtained by *surgery along* L. Each such surgery is given by some link and a system of torus-diffeomorphisms–hence by some finite set of data. Moreover, a universal link, L_0, can be chosen in the sense that every 3-manifold can be obtained by surgery along L_0.

Branched coverings. Each 3-manifold is the 3-fold branched cover over S^3 branched along some link in S^3 [Mon 1, Hil, Hir]. This link may even be chosen to be a knot. Another remarkable result in this direction is the recent discovery of links (and even knots) which are universal in the sense that each 3-manifold is the n-fold, $n \geqslant 3$, branched cover over S^3 branched along this particular link (resp. knot) [Th 4] (see also [Mon 2]).

This completes our list of 3-manifold presentations. We finally note that each of these presentations can be utilized for a complete enumeration of 3-manifolds. To describe such an enumeration using the first presentation, say, simply generate all 3-complexes, by an induction on the number of 3-simplices. After each step check whether really a manifold is produced, and keep only those. For 3-complexes this check is always possible, but it involves the classification of surfaces (see [S–T, pp. 208]). Hence we get an enumeration, indeed (observe that this does not work in dimensions $\geqslant 4$).

§ **6. Hierarchies.** In the last section we saw a number of different presentations for 3-manifolds. In order to attack specific problems in the realm of 3-manifold theory it is good to have such a variety of presentations at hand. However, as far as the classification problem is concerned, one obstacle remains for all of these presentations since for none of them the isomorphism problem is solved. Hence in order to classify 3-manifolds still more internal structure seems to be needed. In the case of surfaces, M^2, this internal structure is provided by the set of all curves in M^2. Thus the idea suggests itself to consider, for any manifold M^n, the set of all codim 1 submanifolds as some sort of internal structure. However, we have to face the problem that this set might be infinite and indeed it is in dimensions 2 and 3. (even if we consider istopy classes only). Essentially two methods have been worked out to deal with this sort of infinite sets—both for surfaces, M^2, and for 3-manifolds, M^3, as well. Loosely speaking, one of these methods consists in constructing an appropriate finite base, and the other one in turning the set in question into an appropriate metric space. In chapter I we saw how nicely the second method (applied to singular submanifolds) works for the study of surface-diffeomorphisms. Here we deal with the first method. This method initially was worked out by Haken and led him eventually to a program for solving the classification problem for Haken 3-manifolds.

The starting point for Haken's program [Hak 2] is the surprising discovery that in 3-manifolds there are actually only a few possibilities to produce sub-surfaces at all. Two of these procedures are well known. They even produce an infinity of (isotopy-classes of) surfaces and are easily described as follows.

Adding handles. Let S be any surface in M^3 (orientable, but not neccessarily connected), and let k be any (non-singular) arc in M^3 which intersects the surface, S, in a finite number of points. Define $S' = (\partial N(k \cup S) - \partial M)^-$, where $N(\ .\ )$ denotes a small neighborhood in M^3. Then we say that S' (or rather its components not homeomorphic to components of S) is "obtained from S by adding handles".

Twisting surfaces. Let S be given as above, and let A be an annulus or torus in M^3 which intersects the surface, S, in curves which are neither contractible in A nor in S. Finally, let h be any diffeomorphism of M^3 with support in $N(A)$. Then, in general, $h^n(S)$, $n \geqslant 1$, defines an infinity of pairwise non-isotopic surfaces in M^3. We call these surfaces the "surfaces obtained from S by twisting along A".

It turns out that these two procedures are "essentially" the only ways of producing infinities of surfaces in M^3. This claim can and will be made precise. For this, however, we first have to introduce an internal characterization of surfaces obtained by adding handles. The following definition will provide us with such a characterization.

DEFINITION. An (orientable) surface S in M^3 with $S \cap \partial M = \partial S$ is called *essential* if it is not the 2-sphere and if the inclusion $S \subset M$ induces an injection $\Pi_1(S, \partial S) \to \Pi_1(M, \partial M)$ of relative fundamental groups.

Observe that a surface obtained by adding handles can never be essential. Vice versa, it follows from the loop-theorem [Pa 1, St] that any inessential surface definitely is obtained by adding handles.

Moreover, a 3-manifold, M^3, is called *irreducible* if every (PL or differentiable) 2-sphere in M^3 is the boundary of some 3-ball in M^3 (in a similar way it is defined when a manifold is boundary-irreducible, see [Hem]. In the following we let the notion "irreducible" include both these properties).

DEFINITION. A *Haken 3-manifold* is an orientable, irreducible 3-manifold which contains at least one essential surface.

The important feature about Haken 3-manifolds is the existence of at least one essential surface. This excludes a number of cases, e.g. lens spaces and a number of 3-manifolds obtained by surgeries along knots (see e.g. [Th 1] [Ha–Th 1]). On the other hand, every 3-manifold with non-empty boundary is a Haken 3-manifold, provided it is irreducible. This even holds, more generally, for all those irreducible 3-manifolds which have infinite first homology [Hem, 6.6.]). In particular, knot spaces of all non trivial knots are Haken 3-manifolds.

As far as the classification of Haken 3-manifolds is concerned the following finiteness result due to Haken is crucial:

THEOREM (Haken). *Let M^3 be a Haken 3-manifold without any essential annulus and torus (such 3-manifolds are called "simple 3-manifolds"). Then, given an integer $n \geqslant 1$, there are only finitely many, pairwise non-isotopic essential surfaces in M^3 whose Euler characteristic is greater than $-n$. Furthermore, this finite set can be effectively constructed.*

The proof of this theorem is contained in [Hak 1]. (See also [Ja–Oe].)

Now, the set of (isotopy classes of) all essential surfaces in the Haken 3-manifold M^3 can be considered as a geometric homeomorphy-invariant. For simple 3-manifolds this invariant can be determined effectively, for then, in light of Haken's theorem, this set is computable. Although the above geometric invariant already reflects some of the topology of Haken 3-manifolds, it does not determine their homeomorphy type completely – it only "approximates" it. A much better approximation is the set of all hierarchies. Indeed, hierarchies were used as a technical tool in the inductive proofs of almost all important theorems about Haken 3-manifolds. In the study of exotic homotopy equivalences between Haken 3-manifolds [Joh 1] a variant of Haken's hierarchy-concept was introduced, which explicitely takes into account the existence of essential annuli and tori. It resulted the concept of *great hierarchies*. Since this concept also yields a language in which Haken's classification program can be formulated more conveniently, I am going to describe the concept of great hierarchies first. For this purpose the notion of the characteristic submanifold in Haken 3-manifolds is needed. In order to give this notion observe first that small neighborhoods of essential annuli and tori in M^3 belong to the class of I-bundles and Seifert fibre spaces. Both of the latter 3-manifolds are defined by the property that they are foliated by curves – in the first case by arcs and in the second case by circles.

Let us now call a system of I-bundles and Seifert fibre spaces in M^3 a *fibered submanifold* of M^3. Then it turns out that any two essential fibered submanifolds always can be isotoped so that afterwards their union is a fibered submanifold again. Moreover, the set $\mathfrak{B}(M^3) := \{$isotopy classes of essential, fibered submanifolds in $M^3\}$, together with the partial ordering induced by the inclusion, not only has a maximal, but even a greatest element. (For the proofs of all this and the notion "essential", in particular, the reader is refered to [Joh 1], see also [Ja–Sh].) This greatest element is called the *characteristic submanifold*, V, of M^3.

The characteristic submanifold, V, of the Haken 3-manifold, M^3, is a system of I-bundles and Seifert fibre spaces and $(\partial V - \partial M)^-$ consists of essential annuli and tori. Besides a number of other pleasent properties, the characteristic submanifold is constructable (in finitely many steps) and unique, up to isotopy. Its complement in M^3 consists of simple 3-manifolds alone (besides some trivial components which we neglect in the following for simplicity – we neglect boundary-patterns as well) [Joh 1].

A *great hierarchy*, $\mathfrak{M}$, is now, by definition, a finite sequence of codim 0 submanifolds, $M = M_0 \supset M_1 \supset M_2 \supset \dots M_n$, with the following properties

(1) M_n = system of 3-balls or empty, and

(2) $(M_i - M_{i+1})^- = \begin{cases} \text{characteristic submanifold in } M_i, & \text{if } i = \text{even}, \\ \text{small neighborhood of some essential surface in } M_i \\ \text{with minimal Euler characteristic}, & \text{if } i = \text{odd} \end{cases}$

(note that this is an informal definition – for the precise version, involving boundary-patterns, I have to refer to [Joh 1] or to the survey article [Wa 3]).

Now, from Haken's finiteness theorem and the above mentioned properties of characteristic submanifolds, it follows immediately that the set, $\mathfrak{M}(M^3)$, of all great hierarchies in M^3 is finite. Moreover, this set is non-empty for Haken 3-manifolds [Hem]. In the next section we will see that with the set $\mathfrak{M}(M^3)$, together with its attaching data, we now actually have the geometric homeomorphy-invariant at hand which is constructable and which determines the homeomorphy-type of Haken 3-manifolds, M^3, completely.

Let us conclude this section by finally noting that essential surfaces in I-bundles and Seifert fibre spaces are classified [Wa 1, 2] [Joh 1, § 5]. Moreover, given any integer $n \geqslant 1$, it can be shown, by the method used in [Joh 1, § 25], that there are only finitely many essential surfaces, S, in a given I-bundle or Seifert fibre space with $\chi S \leqslant n$, up to isotopy and twisting along annuli and tori (see above). Since any essential surface, S, in M^3 can be isotoped so that afterwards $S \cap V$ as well as $S \cap (M-V)^-$ is essential, the previous statement extends to all Haken 3-manifolds, by appealing to Haken's finiteness theorem. Hence for any given irreducible 3-manifold, M, and any given integer, $n \geqslant 1$, there are finitely many surfaces in M with the property that, up to isotopy, *all* (orientable) surfaces, S, in M with $|\chi S| \leqslant n$ can be obtained from this finite set by iterated applications of adding handles and twistings. E.g. all surfaces in S^3 can be obtained from the 2-sphere by iterated handle-addings. This finally is the precise formulation concerning the variety of surfaces in 3-manifolds aspired in the beginning of this section.

§ 7. **Classification.** As already mentioned in the introduction the classification of 3-manifolds in the sense of Papakyriakopoulos [Pa] asks for a complete enumeration *without repetition*. Since we already know how to enumerate 3-manifolds (§ 5), it still remains to solve the isomorphism problem for 3-manifolds. With other words, given any two 3-manifolds, M, M', we are asked to decide whether or not M and M' are homeomorphic. In order to present the solution of this problem for *Haken 3-manifolds* we better reformulate it slightly, using great hierarchies. For this purpose let us fix great hierarchies, $\mathfrak{M}$, $\mathfrak{M}'$, in the Haken 3-manifolds, M, M', respectively. Recall that $\mathfrak{M}$ and $\mathfrak{M}'$ are given by some sequence of 3-manifolds, $M = M_0 \supset M_1 \supset \ldots M_m$, resp. $M' = M'_0 \supset M'_1 \supset \ldots M'_n$, enjoying certain additional properties. We let $\mathfrak{H}(\mathfrak{M}, \mathfrak{M}')$ denote the set of all (isotopy classes of) diffeomorphisms, $h\colon M \to M'$, which map $\mathfrak{M}$ to $\mathfrak{M}'$, i.e. $h(M_i) = M'_i$, for all $0 \leqslant i \leqslant m$. Observe that the union $\bigcup \mathfrak{H}(\mathfrak{M}, \mathfrak{M}')$, taken over all pairs, $(\mathfrak{M}, \mathfrak{M}')$, of great hierarchies, gives the set, $\mathfrak{H}(M, M')$, of all (isotopy classes of) diffeomorphisms $h\colon M \to M'$.

Now, M and M' are certainly homeomorphic iff $\mathfrak{H}(M, M')$ is non-empty and the latter holds iff $\mathfrak{H}(\mathfrak{M}, \mathfrak{M}')$ is non-empty, for some pair $(\mathfrak{M}, \mathfrak{M}')$. To continue recall from § 6 that the set of all great hierarchies in a Haken 3-manifold is finite and constructable. As a result it remains to decide whether the set $\mathfrak{H}(\mathfrak{M}, \mathfrak{M}')$ is non empty, for the given $\mathfrak{M}$, $\mathfrak{M}'$. We are going to describe the solution of this problem first for the special class of Stallings manifolds and extend this result then successively to the general case.

Simple Stallings manifolds. A Stallings manifold is, by definition, a

3-manifold which is obtained from two copies, X, X', of one I-bundle over some (orientable or non-orientable) surface by identifying ∂X with $\partial X'$ via some diffeomorphism. A Stallings manifold is therefore either the mapping torus, $T(f)$, of some surface diffeomorphism, $f\colon S \to S$, or admits a canonical 2-sheeted cover which is a mapping torus. Thus w.l.o.g. we may suppose that $M = T(f)$ and $M' = T(g)$, for some appropriate surface-diffeomorphisms, f, g. Moreover, since M and M' are supposed to be simple 3-manifolds, it follows from [Joh 1, § 6] that f and g have to be pseudo-Anosov and that the characteristic submanifolds in M and M' are empty. In particular, the great hierarchy of M, say, is simply be given by $M = M_0 = M_1 \supset M_2 \supset \emptyset$, where M_2 is a product I-bundle. M is obtained from M_2 by attaching the two components of ∂M_2 via f. Similarly with M'. It hence easily follows that $\mathfrak{H}(\mathfrak{M}, \mathfrak{M}')$ is non-empty iff f and g are conjugate, up to isotopy. Hence the isomorphism-problem for simple Stallings manifolds follows from Hemion's solution of the conjugacy problem for the mapping class group of surfaces (see Chapter II).

Simple 3-manifolds. In the case of simple 3-manifolds the isomorphism-problem immediately follows from the following

THEOREM. *Let M and M' be simple 3-manifolds. Then $\mathfrak{H}(\mathfrak{M}, \mathfrak{M}')$ is a finite and constructable set.*

Proof. Consider the great hierarchies $\mathfrak{M}$: $M = M_0 \supset M_1 \supset \ldots M_m$ and $\mathfrak{M}'$: $M' = M'_0 \supset M'_1 \supset \ldots M'_n$. We may suppose that $m = n$, for otherwise $\mathfrak{H}(\mathfrak{M}, \mathfrak{M}')$ is certainly empty and we are done. Furthermore, we may suppose that M is neither a Seifert fibre space nor a Stallings manifold for the isomorphism-problem is already solved for these manifolds. Since M is simple, it follows, moreover, that no essential Seifert fibre space or Stallings manifold is contained in M. In particular, we may suppose that M_m is not empty but a (system of) balls. Observe, however, that M_m is not just a ball alone, for it carries an additional combinatorial structure in its boundary, which turns it into a polyhedron. This additional structure is given by

$$\underline{\underline{m}}_m := \{D_i \mid D_i \text{ is a component either of } \partial M_m \cap \partial M, \text{ or of } \\ \partial M_m \cap (\partial M_i - \partial M)^-,\ 0 \leqslant i \leqslant m-1\}.$$

Observe that $\underline{\underline{m}}_m$ is a tesselation of ∂M_m by discs. Similarly, we define $\underline{\underline{m}}'_m$.

Now, for any diffeomorphism, h, from $\mathfrak{H}(\mathfrak{M}, \mathfrak{M}')$, the restriction, $h|M_m$, maps $\underline{\underline{m}}_m$ onto $\underline{\underline{m}}'_m$. With other words, $h|M_m$ is a polyhedral isomorphism. – Vice versa, the extension (if exists) of any polyhedral isomorphism $M_m \to M'_m$ to some diffeomorphism $M \to M'$ is unique, up to isotopy (an inductive proof for this fact, using the combing process, can be found in [Joh 1, § 27]). In this way the diffeomorphisms from $\mathfrak{H}(\mathfrak{M}, \mathfrak{M}')$ are completely determined by their restrictions to M_m. In particular, it follows that $\mathfrak{H}(\mathfrak{M}, \mathfrak{M}')$ has to be a *finite* set since the set of all polyhedral isomorphisms $M_m \to M'_m$ certainly is finite (up to polyhedral isotopy).

It remains to *construct* $\mathfrak{H}(\mathfrak{M}, \mathfrak{M}')$. For this it suffices to know which of the polyhedral isomorphisms $M_m \to M'_m$ extend to diffeomorphisms $M \to M'$. We note that all even indexed manifolds from $\mathfrak{M}$, resp. $\mathfrak{M}'$ are simple. Thus, in particular, the finiteness result above applies to all of these manifolds. Using induction on the length of the great hierarchy, $\mathfrak{M}$, it therefore remains to decide when a diffeomorphism $M_{i+2} \to M'_{i+2}$ can be extended to a diffeomorphism $M_i \to M'_i$, $0 \leqslant i \leqslant m-2$. To work out such a decision procedure recall the special way in which M_i differs from M_{i+2}. It follows that the decision procedure is again easily obtained from the combing process [Joh 1, § 27] since the latter is actually a constructive process.

Haken 3-manifolds. Let V and V' be the characteristic submanifolds of the Haken 3-manifolds M and M', respectively. By what we have seen so far, we may suppose that V and V' are both non-empty and that $(M-V)^-$ and $(M'-V')^-$ are non-empty and simple. In the following we restrict ourselves to the worst case and let V be a (system of) Seifert fibre spaces in the interior of M. Then the same holds for V', too, since otherwise $\mathfrak{H}(\mathfrak{M}, \mathfrak{M}') = \emptyset$ and we are done. Now, $\mathfrak{H}(\mathfrak{M}, \mathfrak{M}')$ is non-empty iff there is some diffeomorphism $(M-V)^- \to (M'-V')^-$ which extends to a diffeomorphism $M \to M'$. But it follows from the previous theorem that the set of all diffeomorphisms $(M-V)^- \to (M'-V')^-$, up to isotopy, is finite and constructable. Thus we only have to check a finite number of given diffeomorphisms. Therefore the isomorphism-problem for Haken 3-manifolds finally reduces to the question: given a Seifert fibre space, X, when does a diffeomorphism $h\colon \partial X \to \partial X$ extend to a diffeomorphism of X.

A necessary condition is that h preserves the Seifert fibration [Wa 1]. Therefore the previous extension-problem is easily reduced to the case that the restriction of h to any component of ∂X is a multiple of some Dehn twist (along a fibre), which will be assumed in the following.

Now, if ∂X is connected, then h extends iff it is isotopic either to the identity, or to an even multiple of some Dehn twist, according as the orbit surface of X is orientable or not (this follows from [Joh 1, § 25]). Observe that this criterion is easy to decide.

In order to generalize the previous criterion to the case of disconnected boundary, let us first fix one component, T_0, of ∂X. Then, for each component of ∂X different from T_0, choose a vertical annulus in X, joining this component with T_0, and let d_i, $1 \leqslant i \leqslant q$, be the Dehn twist along this annulus. Then, given $h\colon \partial X \to \partial X$, it is easy to determine integers, $n_1, \ldots, n_q$, with $(\prod_{1 \leqslant i \leqslant q} d_i^{n_i}) \circ h|(\partial X - T_0)^- \simeq \mathrm{id}$. Then h extends iff $(\prod d_i^{n_i}) \circ h|T_0$ satisfies the criterion above.

This completes our outline of the classification of Haken 3-manifolds.

Remark on knots. The solution of the isomorphism-problem for Haken 3-manifolds especially solves the isomorphism-problem for knot spaces of classical knots (i.e. differentiable embeddings of S^1 into S^3). A slight enlargement of this solution also covers the isomorphism-problem for pairs (knot space, meridean curve). Such pairs determine knots completely, whereas this is not yet known for knot spaces alone. In particular, the isomorphism problem for knots is solvable, and hence knots are classified in the sense of Papakyriakopoulos. Thus the question arises whether there are even models for knot-types. For example, one might wish to give a list of knot positions in S^3 which shows as many symmetries as possible. The problem of whether such a position exists for all knot types will be discussed in [Joh 2].

§ 8. Homotopy equivalences. Having presented the classification of Haken 3-manifolds we now turn to the study of homotopy equivalences between them. The existence of characteristic submanifolds provides us with a splitting result similar to those discussed for homotopy equivalences between surfaces (see § 4). In order to describe this splitting result, let M and M' be Haken 3-manifolds (closed or not!) and denote by V and V' the characteristic submanifolds in M and M', respectively. Then we have the following

THEOREM. *Any homotopy equivalence $f\colon M \to M'$ can be deformed into g such that*

(1) $g(V) = V$ *and* $g(M-V) = (M'-V')$,

(2) $g|V\colon V \to V'$ *is a homotopy equivalence, and*

(3) $g|(M-V)^-\colon (M-V)^- \to (M'-V')^-$ *is a diffeomorphism.*

This theorem is one of the main results of [Joh 1]. It gives the possibility to split up a homotopy equivalence, $f\colon M \to M'$, into homotopy equivalences between I-bundles and Seifert fibre spaces (recall that V and V' consists of those), and diffeomorphisms between simple 3-manifolds. The next step is to study these pieces separately.

Homotopy equivalences between Seifert fibre spaces. As far as homotopy equivalences between I-bundles and Seifert fibrespaces are concerned we note that, besides some exceptional cases which we neglect, any such homotopy equivalence can be deformed so as to leave the fibre structure invariant [Joh 1, § 28] (see also Waldhausen's result concerning the case of diffeomorphisms [Wa 1]). With other words, there is a horizontal as well as a vertical contribution which together describe the whole homotopy equivalence in question. In the case of diffeomorphisms these contributions are essentially given by horizontal resp. vertical Dehn twists (along annuli and tori) [Joh 1, § 28]. The horizontal contribution also can be studied via the diffeomorphism of the orbit surface induced by the original one, and for this the results from chapter II are available. For exotic homotopy equivalences, however, the precise horizontal contribution still remains obscure in the presence of exceptional fibres.

Diffeomorphisms between simple 3-manifolds. As in the case of surface-diffeomorphisms, the existence of geometric structures is crucial in the study of diffeomorphisms between simple 3-manifolds. Fortunately, the recent work of Thurston, provides us with appropriate kinds of geometric structures. In particular, Thurston announced [Th 3]:

THEOREM (Thurston). *Every Haken 3-manifold (different from the I-bundle over the Klein bottle) carries a hyperbolic structure.*

In contrast to the surface-case, hyperbolic structures for Haken 3-manifolds are difficult to construct (the proof of the previous theorem is still not completely published yet; parts of it are contained in [Th 1, 2]). However, having established their existence the impact of hyperbolic structures on diffeomorphisms is much more drastic as in the surface-case. The reason for this is the rigidity of these structures for simple 3-manifolds [Mos, Pra]. As a consequence *any* diffeomorphism between simple 3-manifolds is isotopic to an isometry (and hence to a periodic diffeomorphism).

Altogether, this gives a rather precise, although not quite completed, picture of the structure of homotopy equivalences between Haken 3-manifolds, and we end this report by remarking that this picture can and has been used rather fruitfully in a further study of 3-manifold problems.

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