GEOMETRIC AND ALGEBRAIC TOPOLOGY
BANACH CENTER PUBLICATIONS, VOLUME 18
PWN – POLISH SCIENTIFIC PUBLISHERS
WARSZAWA 1986

LUSTERNIK-SCHNIRELMANN CATEGORY; A GEOMETRIC APPROACH *

L. MONTEJANO

Mexico

In 1934 a new topological invariant was defined by Lusternik and Schnirelmann [8] as a result of their research into the calculus of variations, in particular, as a result of their study of the geodesics of a surface. This invariant, called the *category* of a space, gives important information about the existence of critical points and has received a lot of attention over the years from very different points of view. See, for example, the survey article of I. M. James [7] concerning the algebraic topology viewpoint and the articles of Palais [11] and Takens [14] concerning the relations between the category and the existence of critical points. Our purpose is to show that there is a considerable amount of geometric topology involved in the study of this invariant and, of course, that the combination of these techniques with the classical ones give rise to beautiful and interesting results and problems.

Although most of our definitions and results can be stated in a more general setting, we will restrict ourselves by expository reasons, to the compact PL case.

§ 1. The category and the simple category

Let P be a compact polyhedron, the *category of* P, $\operatorname{cat}(P)$, is the smallest integer k such that P can be covered with k subpolyhedra each of which is null homotopic in P (or equivalently, each of which can be deformed in P into a single point). Thus, $\operatorname{cat}(P) = 1$ if and only if P is contractible, $\operatorname{cat}(P) \leqslant 2$ for P a suspension and the category of spheres is two.

Two simple but important propositions follow.

PROPOSITION 1.1. *Let P be a compact connected polyhedron. Then*

$$\operatorname{cat}(P) \leqslant \dim P+1.$$

Proof. Let T be a triangulation of P and let T'' be the second baricentric

* The paper is in final form and no version of it will be submitted for publication elsewhere.

subdivition of T. Suppose dim $P = n$. Let T_k be the subcomplex of T which is the disjoint union, over all k-simplices σ_i^k of T, of $\mathrm{St}(\hat{\sigma}_i^k, T'')$. Since $|T_k|$ is the disjoint union of contractible pieces, it is null homotopic in P. Therefore, $\{|T_0|, |T_1|, \ldots, |T_n|\}$ is a cover of $n+1$ subpolyhedra each of which is null homotopic in P. ∎

This inequality is the best possible, in particular, the category of the n-projective real space, $\boldsymbol{RP}^n$, is $n+1$. It is also easy to see, looking at the cellular decomposition of $\boldsymbol{RP}^n$, that $\mathrm{cat}(\boldsymbol{RP}^n) \leqslant n+1$. Let us suppose that $\boldsymbol{RP}^n = K_1 \cup \ldots \cup K_k$, where each K_i is a null homotopic subpolyhedron of $\boldsymbol{RP}^n$ and $k < n+1$, then $j_i^*\colon \tilde{H}^*(\boldsymbol{RP}^n, \boldsymbol{Z}_2) \to \tilde{H}^*(K_i, \boldsymbol{Z}_2)$ is trivial, where $j_i\colon K_i \to \boldsymbol{RP}^n$ is the inclusion, $1 \leqslant i \leqslant k$. Hence, by exactness, $\tilde{H}^*(\boldsymbol{RP}^n, K_i, \boldsymbol{Z}_2) \to \tilde{H}^*(\boldsymbol{RP}^n, \boldsymbol{Z}_2)$ is an epimorphism. Let $\gamma_1, \ldots, \gamma_k$ be any elements of $\tilde{H}^*(\boldsymbol{RP}^n, \boldsymbol{Z}_2)$. We can pull each γ_i back to $\tilde{H}^*(\boldsymbol{RP}^n, K_i, \boldsymbol{Z}_2)$ and hence pull the product $\gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_k$ back to $\tilde{H}^*(\boldsymbol{RP}^n, K_1 \cup \ldots \cup K_k, \boldsymbol{Z}_2) = 0$, thus proving that $\gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_k = 1$ which is a contradiction because $\tilde{H}^*(\boldsymbol{RP}^n, \boldsymbol{Z}_2)$ is a truncated polynomial ring of height n.

Using the same idea of the proof of Proposition 1.1, it is possible to prove inductively that if P is a p-connected compact polyhedron then $\mathrm{cat}(P) \leqslant 1 + \dim P/(p+1)$.

PROPOSITION 1.2. *If P dominates R then* $\mathrm{cat}(P) \geqslant \mathrm{cat}(R)$.

Proof. Let $g\colon R \to P$ and $f\colon P \to R$ be PL maps such that $fg \simeq \mathrm{Id}$. If P_1 is a null homotopic subpolyhedron of P then $g^{-1}(P_1)$ is a null homotopic subpolyhedron of R because the inclusion $g^{-1}(P_1) \hookrightarrow R$ is homotopic to $fg\colon g^{-1}(P_1) \to R$ which is homotopic to a constant map due to the fact that P_1 is null homotopic in P. Hence any null homotopic polyhedral cover of P can be pulled back to a null homotopic polyhedral cover of R. ∎

An important corollary of Proposition 1.2 is that

The category is a homotopy invariant.

Associated with the notion of category is the notion of geometric category. The *geometric category* of a compact polyhedron P, $\mathrm{gcat}(P)$, is the smallest integer k such that P can be covered with k subpolyhedra each of which is contractible (in itself).

An easy modification of the proof of Proposition 1.1 (add arcs to make each $|T_k|$ connected) shows that

PROPOSITION 1.3. *Let P be a compact connected polyhedron. Then*

$$\mathrm{gcat}(P) \leqslant \dim P + 1.$$

However, the geometric category is not a homotopy invariant and does not always agree with the category as the following example of Fox shows

EXAMPLE 1.4 (Fox, [5]). Let K be the polyhedron obtained from the sphere S^2 by an identification of three different points $a_1, a_2, a_3 \in S^2$. Then $\mathrm{cat}(K) = 2$ but $\mathrm{gcat}(K) = 3$.

Clearly, K can be covered with three contractible subpolyhedra. We will prove that it is impossible to cover K with two contractible subpolyhedra A and B, thus proving that $\mathrm{gcat}(K) = 3$. Let $q: S^2 \to K$ be the quotient map, $b = q(a_1) = q(a_2) = q(a_3)$ and suppose $b \in B$. It is easy to see, since $a_1, a_2, a_3 \in q^{-1}(B)$, that $q^{-1}(B)$ is the disjoint union of 3 contractible subpolyhedra B_1, B_2, B_3 such that $a_i \in B_i$, $i = 1, 2, 3$, and that $S^2 - q^{-1}(B)$ is connected. Also $S^2 - q^{-1}(A)$ is connected but since $q^{-1}(A) \cup q^{-1}(B) = S^2$ we have that $S^2 - q^{-1}(A) \subset q^{-1}(B)$ and therefore we may assume that $S^2 - q^{-1}(A) \subset B_1$. Then, the connected set $S^2 - B_1$ is contained in $q^{-1}(A)$ but hence a_2 and a_3 belong to the same component of $q^{-1}(A)$ which is a contradiction.

Now we will show that $\mathrm{cat}(K) = 2$. Let A be the subpolyhedron of K shown in figure 1a. The closure of $K - A$ is contractible and hence null homotopic in K, furthermore, although A is not contractible, it is easy to see from the following sequence of figures that A is null homotopic in K. Since K is not contractible, we have $\mathrm{cat}(K) = 2$.

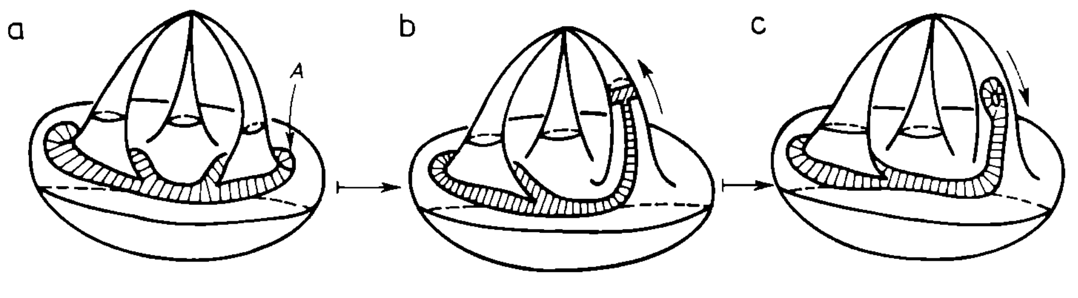

Fig. 1

Also, it is easy to see (filling up the three horns of K) that K has the homotopy type of $S^2 \vee S^1 \vee S^1$ which is a suspension and consequently has category and geometric category two.

Another interesting fact is that, although $\mathrm{gcat}(K) = 3$, $\mathrm{gcat}(K \times I) = 2 = \mathrm{cat}(K)$. Let $\pi: S^2 \times I \to K \times I$ be the obvious quotient map and let D_1, D_2 and D_3 be pairwise disjoint disks in S^2 centered at a_1, a_2 and a_3 respectively. Let $A = \pi((D_1 \times I) \cup (D_2 \times [0, 3/4]) \cup (D_3 \times [1/4, 1]))$ and B the closure of $(K \times I) - A$, then $\{A, B\}$ is a contractible polyhedral cover of $K \times I$.

For a compact polyhedron P, we have that $\mathrm{gcat}(P) \geqslant \mathrm{gcat}(P \times I) \geqslant \ldots \geqslant \mathrm{gcat}(P \times I^n) \geqslant \ldots \geqslant \mathrm{cat}(P)$. It is clear that this sequence must stabilize somewhere. Let us define the *simple category* of P, $\mathrm{scat}(P)$, as follows: $\mathrm{scat}(P) = \mathrm{Min}_{n \geqslant 0} \{\mathrm{gcat}(P \times I^n)\}$. We will prove that the simple category is a simple homotopy invariant that differs from the category in at most one unit.

THEOREM 1.5. *Let P be a compact polyhedron. Then*

$$\mathrm{cat}(P) \leqslant \mathrm{scat}(P) \leqslant \mathrm{cat}(P) + 1.$$

Proof. Since $\mathrm{cat}(P) = \mathrm{cat}(P \times I^n) \leqslant \mathrm{gcat}(P \times I^n)$ for every $n \geqslant 0$, we have that $\mathrm{cat}(P) \leqslant \mathrm{scat}(P)$. Let $\mathrm{cat}(P) = k$ and let $\{P_1, \ldots, P_k\}$ be a null homotopic polyhedral cover of P. Let n be a sufficiently big integer and for $1 \leqslant i \leqslant k$ let $T_i = P_i \times I^n \times [i, k+1] \subset P_i \times I^n \times [0, k+1]$ and $T_i' = P_i \times I^n \times [0, i] \subset P_i \times I^n \times [0, k+1]$. It is easy to see that $\bigcup_{i=1}^{k+1} (T_i' \cup T_i) = P \times I^n \times [0, k+1]$ and for every $i = 1, 2, \ldots, k$, $T_i' \cap T_{i+1} = \emptyset$. Since P_i is null homotopic in P, we may assume that n is so big that for every $i = 1, \ldots, k$, the cone of P_i, denoted by cP_i, is contained in $P \times I^n \times \{i\}$ in such a way that the base of cP_i coincides with $P_i \times \{0\} \times \{i\}$. Furthermore, we may embed another copy of the cone of P_i, cP_i^+, in $P \times I^n \times [i, i+1/3]$, by pushing $(cP_i - P_i \times \{0\} \times \{i\})$ off $P \times I^n \times [0, i]$ in such a way that $T_i' \cap cP_i^+ = P_i \times \{0\} \times \{i\}$ and $R_i^+ = T_i' \cup cP_i^+$ is a contractible subpolyhedron of $P \times I^n \times [0, i+1/3]$. Similarly, cP_i^- is obtained from cP_i by pushing $(cP_i - P_i \times \{0\} \times \{i\})$ off $P_i \times I^n \times [i, k+1]$ in such a way that $T_i \cap cP_i^- = P_i \times \{0\} \times \{i\}$ and $R_i^- = T_i \cup cP_i^-$ is a contractible subpolyhedron of $P \times I^n \times [i-1/3, k+1]$. Note that for every $i = 1, \ldots, k-1$, $R_i^+ \cap R_{i+1}^- = \emptyset$. For every $i = 1, \ldots, k-1$, let γ_i be an arc from R_i^+ to R_{i+1}^- in such a way that $R_i = R_i^+ \cup \gamma_i \cup R_{i+1}^-$ is a contractible subpolyhedron of $P \times I^n \times [0, k+1]$. Then $\{R_1^-, R_1, R_2, \ldots, R_{k-1}, R_k^+\}$ is a contractible polyhedral cover of $P \times I^n \times [0, k+1]$ and consequently $\mathrm{gcat}(P \times I^{n+1}) \leqslant k+1 = \mathrm{cat}(P) + 1$. ∎

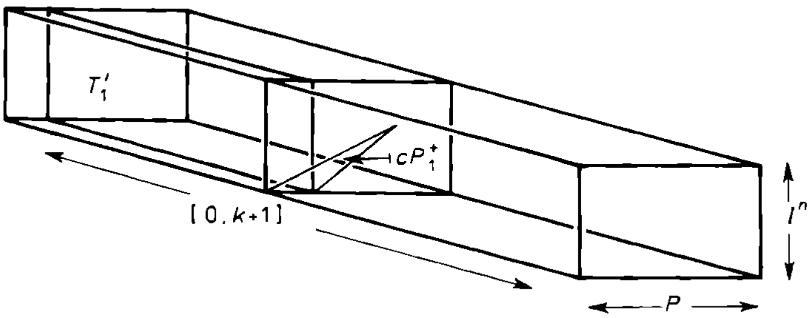

Fig. 2

Next we will prove that if K and L are compact polyhedra with the same simple homotopy type then $\mathrm{scat}(K) = \mathrm{Min}_{n\geqslant 0}\{\mathrm{gcat}(K \times I^n)\} = \mathrm{Min}_{n\geqslant 0}\{\mathrm{gcat}(L \times I^n)\} = \mathrm{scat}(L)$. First we need to review a little about

Q-manifolds. Let Q be the countable product of closed intervals, that is, $Q = I_1 \times \ldots \times I_n \times \ldots$, where each I_i is a copy of the closed interval $[0, 1]$. The following factorization of the Hilbert cube Q will be useful. Let $I^n = I_1 \times I_2 \times \ldots \times I_n$ and let $Q_n = I_n \times I_{n+1} \times \ldots$, in such a way that $Q = I^n \times Q_{n+1}$.

A Q-manifold is a separable metric space modelled in Q. It is well known that for every polyhedron K, $K \times Q$ is a Q-manifold and conversely, every Q-manifold is homeomorphic to $K \times Q$ for some polyhedron K. Furthermore, two Q-manifolds $K \times Q$ and $L \times Q$ are homeomorphic if and only if the polyhedra K and L have the same simple homotopy type. For more about Q-manifolds see [2].

Let P be a compact polyhedron and let $\mathscr{U}$ be an open subset of $P \times Q$ homeomorphic to a compact Q-manifold cross $\boldsymbol{R}$. An *splitting* of $\mathscr{U}$ consists of

(a) Two closed Q-submanifolds of $\mathscr{U}$, $\mathfrak{m}_1$ and $\mathfrak{m}_2$, such that $\mathfrak{m}_1 \cup \mathfrak{m}_2 = \mathscr{U}$ and,

(b) A compact bicollared subpolyhedron A of $P \times I^n$ such that $\mathfrak{m}_1 \cap \mathfrak{m}_2 = A \times Q_{n+1}$.

CHAPMAN SPLITTING THEOREM 1.6 ([2]). *Let P and $\mathscr{U}$ as above. Then there is an splitting of $\mathscr{U}$ for which the inclutions $A \times Q_{n+1} \hookrightarrow \mathfrak{m}_1$ and $A \times Q_{n+1} \hookrightarrow \mathfrak{m}_2$ are homotopy equivalences (and consequently there is a strong deformation retraction of $\mathfrak{m}_i$ onto $A \times Q_{n+1}$, $i = 1, 2$).*

Now we will use Theorem 1.6 to prove

THEOREM 1.7 ([10]). *The simple category is a simple homotopy invariant.*

Proof. Let K and L be compact polyhedra with the same simple homotopy type and suppose $\operatorname{gcat}(K) = k$. It will be enough to prove that, for n sufficiently big, $L \times I^n$ can be covered with k contractible subpolyhedra. Let $\{K_1, \ldots, K_k\}$ be a contractible polyhedral cover of K and let h: $L \times Q \to K \times Q$ be a homeomorphism. We will first show that for every $i = 1, \ldots, k$ there is a compact contractible subpolyhedron B_i of $L \times I^{n_i}$ such that $h^{-1}(K_i \times Q) \subset B_i \times Q_{n_i+1}$. Let N_i be a regular neighborhood of K_i in K and let U_i be the following open subset of $L \times Q$. $U_i = h^{-1}((\operatorname{Int} N_i - K_i) \times Q) \cong (\operatorname{Fr} N_i \times Q) \times \boldsymbol{R}$. Applying the Chapman Splitting Theorem, there is a bicollared subpolyhedron A_i of $L \times I^{n_i}$ such that $A_i \times Q_{n_i+1}$ splits homotopically U_i. Moreover, A_i is the boundary of a compact subpolyhedron B_i of $L \times I^{n_i}$ which has the following properties: $h^{-1}(K_i \times Q) \subset B_i \times Q_{n_i+1}$ and B_i is contractible because there is a strong deformation retraction of U_i, which is contractible, onto $B_i \times Q_{n_i+1}$. Let $n = \operatorname{Max}\{n_i | \, i \leqslant k\}$, then $\{B_i \times I_{n_i+1} \times \ldots \times I_n | i \leqslant k\}$ is a contractible polyhedral cover of $L \times I^n$. This completes the proof of Theorem 1.7. ∎

There are some interesting questions concerning the geometric category and the simple category

(i) Is $|\operatorname{gcat}(P) - \operatorname{gcat}(P \times I)| \leqslant 1$ for every compact polyhedron P?

(ii) Is there a compact polyhedron P with the property that $\operatorname{gcat}(P \times I) \neq \operatorname{scat}(P)$?

(iii) Is there an integer N, possibly depending on the dimension or/and the connectivity, such that for every compact polyhedron P, $\operatorname{scat}(P) = \operatorname{gcat}(P \times I^N)$?

(iv) When is $\operatorname{gcat}(P) = \operatorname{cat}(P)$, $\operatorname{gcat}(P) = \operatorname{scat}(P)$ or $\operatorname{scat}(P) = \operatorname{cat}(P)$? What if P is a closed PL manifold?

§ 2. The strong category

In 1967 Tudor Ganea [6] introduced the following homotopy invariant which is called the *strong category*. Let P be a compact polyhedron, then

$$\operatorname{Cat}(P) = \operatorname{Min}\{\operatorname{gcat}(K) \mid K \text{ is a compact polyhedron of the same homotopy type as } P\}.$$

The following properties of the strong category follow immediately

THEOREM 2.1. *Let P be a compact connected polyhedron. Then*

(i) $\operatorname{Cat}(P) \leqslant \dim P + 1$,

(ii) $\operatorname{cat}(P) \leqslant \operatorname{Cat}(P) \leqslant \operatorname{scat}(P) \leqslant \operatorname{cat}(P) + 1$, *and*

(iii) *If $\pi_1(P)$ is free or free abelian then*

$$\operatorname{Cat}(P) = \operatorname*{Min}_{n \geqslant 0}\{\operatorname{gcat}(P \times I^n)\} = \operatorname{scat}(P).$$

It would be interesting to know if $\operatorname{scat}(P) = \operatorname{Cat}(P)$ for every compact polyhedron P. In particular, $\operatorname{cat}(P) \leqslant 2$ if and only if P admits a coHopf structure [7], if this is the case, then $\pi_1(P)$ is free and therefore $\operatorname{Cat}(P) = \operatorname{scat}(P)$.

By (ii), the strong category is either the category or the category plus one. Both possibilities may occur as the following example of Berstein and Hilton [1] shows.

Let $f\colon S^m \to S^n$ be a map and let $S^n \cup_f e^{m+1}$ be the space obtained by attaching an $(m+1)$-cell to S^n by the map f. At this point, our main interest is to find a space for which the category and the strong category does not coincide, therefore we will study the category and the strong category of this kind of spaces.

Clearly, $\operatorname{Cat}(S^n \cup_f e^{m+1}) \leqslant 3$ and if f is homotopic to the suspension of a map $g\colon S^{m-1} \to S^{n-1}$ then $\operatorname{Cat}(S^n \cup_f e^{m+1}) \leqslant 2$. In fact, we will prove that the converse is true but first we need the following proposition.

PROPOSITION 2.2. *Let P be a compact connected polyhedron. Then $\operatorname{Cat}(P) \leqslant 2$ $(\operatorname{scat}(P) \leqslant 2)$ if and only if P has the (simple) homotopy type of a suspension.*

Proof. If P is (simple) homotopy equivalent to the suspension of a polyhedron K, ΣK, then it is clear that $(\operatorname{scat}(P) \leqslant 2)$ $\operatorname{Cat}(P) \leqslant 2$. Suppose now that P can be covered with two contractible subpolyhedra A and B. We will prove that P is simple homotopy equivalent to $\Sigma(A \cap B)$. Let $X = (A \times \{0\}) \cup ((A \cap B) \times I) \cup \cup (B \times \{1\}) \subset P \times I$ and let π: $X \to P$ be the projection. Let q: $X \to \Sigma(A \cap B)$ be the obvious quotient map. Since π and q are cell-like maps then, by [4], P and $\Sigma(A \cap B)$ have the same simple homotopy type. ∎

PROPOSITION 2.3. *Let* f: $S^m \to S^n$ *be a map,* $m > n \geqslant 3$. *Then* $\mathrm{Cat}(S^n \cup_f e^{m+1}) \leqslant 2$ *if and only if there is a map* g: $S^{m-1} \to S^{n-1}$ *such that* f *is homotopic to* Σg, *the suspension of* g.

Proof. Clearly $\mathrm{Cat}(S^n \cup_{\Sigma g} e^{m+1}) \leqslant 2$ because $S^n \cup_{\Sigma g} e^{m+1} \simeq \Sigma(S^{n-1} \cup_g e^m)$. Suppose now that $\mathrm{Cat}(S^n \cup_f e^{m+1}) \leqslant 2$. By Proposition 2.2, $S^n \cup_f e^{m+1} \simeq \Sigma Z$ for some polyhedron Z. Also $\tilde{H}_*(\Sigma Z) = \boldsymbol{Z}$ for $* = n$ and $* = m+1$ and zero otherwise. On the other hand Z is connected because $\pi_1(\Sigma Z) = 0$. Let $Y = Z/Z^1$ where Z^1 is the *1*-skeleton of Z and let q: $Z \to Y$ be the quotient map. Note that q_*: $\tilde{H}_*(Z) \to \tilde{H}_*(Y)$ is an isomorphism for $* \neq 2$ and $H_2(Y) = H_2(Z) \oplus F$, where F is a free group. Since Z is simply connected, then there is a homotopy equivalence λ: $Y \to (VS^2) \cup_\beta e^{n-1} \cup_\alpha e^m$ for some maps β and α. (see [1] for more details about λ). Let π: $(VS^2) \cup_\beta e^{n-1} \cup_\alpha e^m \to S^{n-1} \cup_g e^m$ be the map which collapses (VS^2) to a single point. Then $\pi\lambda q$: $Z \to S^{n-1} \cup_g e^m$ is a map which induces isomorphisms in homology and consequently we have $\Sigma Z \simeq S^n \cup_f e^{m+1} \simeq S^n \cup_{\Sigma g} e^{m+1} \simeq \Sigma(S^{n-1} \cup_g e^m)$. Let h: $S^n \cup_f e^{m+1} \to S^n \cup_{\Sigma g} e^{m+1}$ be a cellular homotopy equivalence. Since $m > n$, we may assume that h: $(S^n \cup_f e^{m+1}, S^n) \to (S^N \cup_{\Sigma g} e^{m+1}, S^n)$ is a homotopy equivalence of pairs. Let us consider the following diagram

$$\begin{tikzcd}
 & (S^n \cup_f e^{m+1}, S^n) \arrow[r, "q'"] \arrow[dd, "h"] & (S^{m+1}, *) \arrow[dd, "h'"] \\
(B^{m+1}, S^m) \arrow[ur, "f'"] \arrow[dr, "(\Sigma g)'"'] & & \\
 & (S^n \cup_{\Sigma g} e^{m+1}, S^n) \arrow[r, "q"] & (S^{m+1}, *)
\end{tikzcd}$$

where q and q' are the quotient maps and h' is the unique map such that $h'q' = qh$. Note that h' is also a homotopy equivalence. Furthermore, f' and $(\Sigma g)'$ are the attaching maps which correspond to f and Σg. Since $h'q'f' \simeq \pm q(\Sigma g)'$ and since $q_\#$: $\pi_{m+1}(S^n \cup_{\Sigma g} e^{m+1}, S^n) \to \pi_{m+1}(S^{m+1}, *)$ is an isomorphism, hf' and $(\Sigma g)'$ represent up to sign, the same element in $\pi_{m+1}(S^n \cup_{\Sigma g} e^{m+1}, S^n)$, but hence $f \simeq \pm \Sigma g$. That is, f is homotopic to a suspension. ∎

EXAMPLE 2.4 (Berstein–Hilton [1]). *Let* f: $S^6 \to S^3$ be the map which represents the element of order 3 in $\pi_6(S^3) \cong \boldsymbol{Z}_{12}$. Then $\mathrm{Cat}(S^3 \cup_f e^7) = 3$ but $\mathrm{cat}(S^3 \cup_f e^7) = 2$.

Let Σ: $\pi_5(S^2) \to \pi_6(S^3)$ be the suspension homomorphism. Since $\pi_5(S^2) = \boldsymbol{Z}_2$, $\pi_6(S^3) = \boldsymbol{Z}_{12}$ and f: $S^6 \to S^3$ represents the element of order 3, this element is not in the image of Σ and consequently, by Proposition 2.3,

$\operatorname{Cat}(S^3 \cup_{\bar{f}} e^7) = 3$. Next we will prove that $\operatorname{cat}(S^3 \cup_{\bar{f}} e^7) = 2$. Berstein–Hilton original proof uses the fact that the category of a space is less or equal than two if and only if the space admits a coHopf structure. Obstructions to the existence of coHopf structures had been investigated, particularly in connection with spaces of the form $S^m \cup_g e^{m+1}$. This analysis involves the study of the homotopy groups of a wedge of spheres. We will present here another proof.

Given a map g: $S^3 \times S^2 \to S^2$, we have associated a map $\bar{g}$: $S^6 \to S^3$ as follows. Let us think in S^3 as the suspension of S^2, that is, $S^3 = S^2 \times [-1, 1]/\sim$, where $(x, 1) \sim (x', 1)$ and $(x, -1) \sim (x', -1)$ for every $x, x' \in S^2$. Let q: $S^2 \times [-1, 1] \to S^3$ be the quotient map and let us denote by $v_0 = q(S^2 \times \{-1\})$ the south pole of S^3. Also let us think in S^6 as $\partial B^7 = \partial(B^4 \times B^3) = B^4 \times S^2 \cup_{S^3 \times S^2} S^3 \times B^3$. Define $\bar{g}|B^4 \times S^2$ by

$$\bar{g}(x, y) = \begin{cases} q(g(x/\|x\|, y), 1 - \|x\|) & \text{if} \quad x \neq 0, \text{ and} \\ q(S^2 \times \{1\}) & \text{if} \quad x = 0. \end{cases}$$

Furthermore, $\bar{g}|S^3 \times B^3$ is given by

$$\bar{g}(x, y) = \begin{cases} q(g(x, y/\|y\|), \|y\| - 1) & \text{if} \quad y \neq 0 \quad \text{and} \\ q(S^2 \times \{-1\}) & \text{if} \quad y = 0. \end{cases}$$

Note that $\bar{g}|S^3 \times S^2 = g$ and that $\bar{g}^{-1}(v_0) = S^3 \times \{0\} \subset S^6$.

Let us first identify the generator of $\pi_6(S^3) = \mathbf{Z}_{12}$. Let h: $S^3 \times S^2 \to S^2$ be the following composition of maps: $S^3 \times S^2 \xrightarrow{\pi \times 1} SO(3) \times S^2 \xrightarrow{ev} S^2$, where π is the covering map and ev is the evaluation map. Note that for every $p \in S^2$, $h|S^3 \times \{p\}$: $S^3 \to S^2$ is homotopic to the Hopf map. The map $\bar{h}$: $S^6 \to S^3$ associated with h as above, represents the generator of $\pi_6(S^3)$ [16]. Consequently, a representant of the element of order 3 in $\pi_6(S^3) = \mathbf{Z}_{12}$ is the map $\bar{f}$ associated with the following composition f: $S^3 \times S^2 \xrightarrow{4 \times 1} S^3 \times S^2 \xrightarrow{\pi \times 1} SO(3) \times S^2 \xrightarrow{ev} S^2$. Note that for every $p \in S^2$, $f|S^3 \times \{p\}$: $S^3 \to S^2$ is homotopic to 4 times the Hopf map, that is, represents 4 times the generator of $\pi_3(S^2) = \mathbf{Z}$. Also note that $\bar{f}^{-1}(v_0)$ is a 3-sphere contained in S^6.

Let D^4 be the 4-disk in e^7 obtained by coning $\bar{f}^{-1}(v_0) \cong S^3$ from the origin of e^7 and let $\Sigma^4 \subset S^3 \cup_{\bar{f}} e^7$ be the following 4-sphere. $\Sigma^4 = \{v_0\} \cup_{\bar{f}} D^4 \subset S^3 \cup_{\bar{f}} e^7$. It is easy to see that there is a strong deformation retraction of $S^3 \cup_{\bar{f}} e^7 - \Sigma^4$ onto $S^3 - \{v_0\}$ because $D^4 \subset e^7$ was obtained by coning $\bar{f}^{-1}(v_0)$ from the origin of e^7.

Therefore $S^3 \cup_{\bar{f}} e^7 - \Sigma^4$ is contractible. Next we will prove that Σ^4 is null homotopic in $S^3 \cup_{\bar{f}} e^7$. Let p be a fixed point of S^2 and let $D_1^4 = B^4 \times \{p\} \cup_{S^3 \times \{p\}} S^3 \times \{tp \in B^3 / 0 \leq t \leq 1\} \subset S^6$. Clearly, D_1^4 is a 4-disk such that $\partial D_1^4 = \bar{f}^{-1}(v_0) = \partial D^4$ and there is a deformation of D^4 in e^7 onto D_1^4

keeping fixed $\partial D_1^4 = \partial D^4$. Furthermore, $\bar{f}|D_1^4$: $(D_1^4, \partial D_1^4) \to (S^3, v_0)$ represents in $\pi_4(S^3)$ the suspension of 4 times the generator of $\pi_3(S^2)$ due to the fact that $f|S^3 \times \{p\}$: $S^3 \to S^2$ is homotopic to 4 times the Hopf map and the form in which we constructed $\bar{f}$ from f. It follows that Σ^4 is null homotopic in $S^3 \cup_{\bar{f}} e^7$ if the suspension homomorphism Σ: $\pi_3(S^2) \to \pi_4(S^3)$ sends 4 times the generator of $\pi_3(S^2)$ to zero, but this is so because $\pi_4(S^3) = \mathbf{Z}_2$.

Let A be a derived neighborhood of Σ^4 in $S^3 \cup_{\bar{f}} e^7$. Then $\{A, \mathrm{cl}(S^3 \cup_{\bar{f}} e^7 - A)\}$ is a null homotopic polyhedral cover of $S^3 \cup_{\bar{f}} e^7$ and consequently $2 = \mathrm{cat}(S^3 \cup_{\bar{f}} e^7) \neq \mathrm{Cat}(S^3 \cup_{\bar{f}} e^7) = 3$. Note also that since $\pi_1(S^3 \cup_{\bar{f}} e^7) = 0$ then $\mathrm{scat}(S^3 \cup_{\bar{f}} e^7) = 3$.

It would be interesting to know if there is a compact polyhedron P for which $3 \leqslant \mathrm{cat}(P) \neq \mathrm{Cat}(P)$.

The coincidence between the category and the strong category was studied by T. Ganea in 1967 [16]. He proved that for a p-connected polyhedron P ($p \geqslant 1$) with the property that $\dim P \leqslant (\mathrm{cat}(P)+1)(p+1)-3$, the category and the strong category coincide. Recently, M. Clapp and D. Puppe [3] were able to prove that if $\dim P \leqslant (2\,\mathrm{cat}(P)-1)(p+1)-3$ then $\mathrm{cat}(P) = \mathrm{Cat}(P)$. Note that for spaces of category two both bounds are the same and that the Berstein–Hilton example shows that these bounds are the best possible. It would be important fo find examples showing that the second bound is the best possible when the category is greater than two.

§ 3. Product spaces and the category

For cartesian products of compact connected polyhedra, the following formula concerning the category is known

$$\mathrm{cat}(P_1 \times P_2) \leqslant \mathrm{cat}(P_1) + \mathrm{cat}(P_2) - 1 \tag{3.1}$$

The same formula holds for the strong category, that is

$$\mathrm{Cat}(P_1 \times P_2) \leqslant \mathrm{Cat}(P_1) + \mathrm{Cat}(P_2) - 1 \tag{3.2}$$

In fact, if P_1 is not contractible, Takens [15] proved the following mixed formula

$$\mathrm{Cat}(P_1 \times P_2) \leqslant \mathrm{Cat}(P_1) + \mathrm{cat}(P_2) - 1 \tag{3.3}$$

Equality does not always hold. Let $S^2 \cup_p e^3$ be the space obtained by attaching a 3-cell to S^2 by a map of degree p. It follows that $S^2 \cup_p e^3$ is a suspension for every p and that $\mathrm{cat}(S^2 \cup_p e^3) = \mathrm{Cat}(S^2 \cup_p e^3) = 2$ if $p \neq 1$. On the other hand, if p and q are muttually prime integers $(S^2 \cup_p e^3)$

$\times(S^2 \cup_q e^3)$ contains $(S^2 \cup_p e^3) \vee (S^2 \cup_q e^3)$ as a deformation retract and so $\operatorname{cat}((S^2 \cup_p e^3) \times (S^2 \cup_q e^3)) = \operatorname{Cat}((S^2 \cup_p e^3) \times (S^2 \cup_q e^3)) = 2$.

There is a long-standing conjecture to the effect that for every compact connected polyhedron P and any positive integer $r \geqslant 1$.

$$\operatorname{cat}(P \times S^r) = \operatorname{cat}(P) + 1 \tag{3.4}$$

From 1.2 and 3.1, it follows that $\operatorname{cat}(P) \leqslant \operatorname{cat}(P \times S^r) \leqslant \operatorname{cat}(P) + 1$, and hence the difficult part of the conjecture consists in proving that for every compact connected polyhedron P and every positive integer r

$$\operatorname{cat}(P) \neq \operatorname{cat}(P \times S^r)$$

It is interesting to note that there is a polyhedron P for which $\operatorname{Cat}(P \times S^1) = \operatorname{Cat}(P)$. In fact, let $P = S^3 \cup_f e^7$ be the Berstein–Hilton example. Then $\operatorname{cat}(P) = 2$, $\operatorname{Cat}(P) = 3$ and by 3.3, $\operatorname{Cat}(P \times S^1) \leqslant 3$, but by proposition 2.2, since $P \times S^1$ is not homotopy equivalent to a suspension, $\operatorname{Cat}(P \times S^1) = \operatorname{Cat}(P) = 3$. Furthermore, by Theorem 2.1, there is a polyhedron K for which $\operatorname{gcat}(K \times S^1) = \operatorname{gcat}(K)$.

The conjecture 3.3 was solved by Singhof [13] when P is a closed PL manifold and the category of P is not too small with respect the dimension of P. Actually he derived the proof from the following theorem.

THEOREM 3.5 (Singhof). *Let M^n be a closed p-connected* PL *n-manifold with* $\operatorname{cat}(M) = N$, $n \geqslant 5$. *If* $N \geqslant (n+p+4)/2(p+1)$, *then there are N n-balls which cover M.*

COROLLARY 3.6. *Let M^n be a closed* PL *n-manifold*, $n \geqslant 4$. *If* $\operatorname{cat}(M) \geqslant (n+5)/2$, *then* $\operatorname{cat}(M \times S^1) = \operatorname{cat}(M) + 1$.

Proof. Suppose $N = \operatorname{cat}(M \times S^1) = \operatorname{cat}(M) \geqslant (n+5)/2$. By Theorem 3.5, there exists a cover $\{B_1, \ldots, B_N\}$ of $M \times S^1$, where each B_i is an $(n+1)$-ball. By means of a homeomorphism, we can assume B_1 is so small that $B_1 \cap (M \times \{a\}) = \emptyset$ for some $a \in S^1$. Then $\{B_2 \cap (M \times \{a\}), \ldots, B_N \cap (M \times \{a\})\}$ is a null homotopic polyhedral cover of $M \times \{a\}$, which is impossible. ∎

Singhof's original proof of Theorem 3.5 will not be given here, instead, we will use the following theorem, whose proof exploits the linear structure between the k-skeleton of a polyhedron and its dual skeleton [9].

THEOREM 3.7. *Let P be a compact p-connected n-dimensional polyhedron and let $\{P_1, \ldots, P_N\}$ be a polyhedral cover of P. Then there exists a polyhedral cover $\{R_1, \ldots, R_N\}$ of P such that*

a) *R_i is null homotopic in P if P_i is so,* $1 \leqslant i \leqslant N$, *and*

b) *R_i is a derived neighborhood of N_i, where*

$$\dim N_i \leqslant \operatorname{Max}\{n-(N+1)(p+1), p\}, \qquad 1 \leqslant i \leqslant N.$$

Proof. We start the proof by proving the following fact: Let X, Y be subpolyhedra of P, $\{P_1, \ldots, P_N\}$ be a polyhedral cover of X in P and dim $Y < N(p+1)$. Then there exists a polyhedral cover $\{R_1, \ldots, R_N\}$ of $X \cup Y$ in P such that R_i is a derived neighborhood of $P_i \cup N_i$, where dim $N_i \leqslant p$, $1 \leqslant i \leqslant N$. The proof is by induction on N. If $N = 1$ then there is nothing to prove. We will suppose it is true for $N-1$, and prove it for N. Let T be a triangulation of M such that K, L, $T_1, \ldots, T_N$ are subcomplexes of T which triangulate X, Y, $P_1, \ldots, P_N$, respectively. Let L' be the $((N-1)(p+1)-1)$-skeleton of L and let L'' be its dual skeleton. Note that dim $L'' \leqslant p$. Let $R = X \cap (\bigcup_1^{N-1} P_i)$. By induction, since dim $L' < (N-1)(p+1)$, there is a polyhedral cover $\{J_1, \ldots, J_{N-1}\}$ of $R \cup |L'|$ in P such that J_i is a derived neighborhood of $P_i \cup N_i$ where dim $N_i \leqslant p$, $1 \leqslant i \leqslant N-1$. Let R_i be a derived neighborhood of J_i, $1 \leqslant i \leqslant N-1$, and let H be a subpolyhedron of $|T_N \cup L|$ such that H collapses to $|T_N \cup L''|$ and $X \cup Y \subset H \cup (\bigcup_1^{N-1} R_i)$. Finally, let R_N be a derived neighborhood of H. Hence, $\{R_1, \ldots, R_N\}$ is our desired polyhedral cover. This completes the inductive step.

We now return to the proof of Theorem 3.7. Let T be a triangulation of P such that there are subcomplexes $T_1, \ldots, T_N$ which triangulate $P_1, \ldots, P_N$ respectively. Let T_N' be the $(n-(N-1)(p+1))$-skeleton of T_N and let T_N'' be its dual skeleton. Note that dim $T_N'' < (N-1)(p+1)$. By the first part of the proof, there is a polyhedral cover $\{R_1', \ldots, R_{N-1}'\}$ of $(\bigcup_1^{N-1} P_i) \cup |T_N''|$ in P such that R_i' is a derived neighborhood of $P \cup N_i$, where dim $N_i < p$, $1 \leqslant i \leqslant N-1$. Let $R_1, \ldots, R_{N-1}$ be derived neighborhoods of $R_1', \ldots, R_{N-1}'$ respectively and let R_N be a derived neighborhood of $|T_N'|$ such that $\bigcup_1^N R_i = P$. Since P is p-connected, R_i is null homotopic in P if P_i is so, $1 \leqslant i \leqslant N$. Furthermore, R_N is a derived neighborhood of $|T_N'|$ where dim $T_N' \leqslant n-(N-1)(p+1)$. By repeating this process, and using Lemma 1.63 of [12] in order to preserve property b) in the process, we obtain our desired polyhedral cover. ∎

ZEEMAN'S ENGULFING THEOREM 3.8. *Let M be a closed p-connected* PL *n-manifold and let X be a compact subpolyhedron of dimension q, $q \leqslant n-3$ and $2q \leqslant n+p-2$. Then X is null homotopic in M if and only if there exists an n-ball B such that $X \subset B \subset M$.*

Proof of Theorem 3.5. Let $\{P_1, \ldots, P_N\}$ be a polyhedral cover of M such that each P_i is null homotopic in M. By Theorem 3.6, we may assume that P_i is a derived neighborhood of N_i, where dim $N_i \leqslant$ Max $\{n-(N-1)(q+1), q\}$ and where $q =$ Min $\{p, n-3\}$, $1 \leqslant i \leqslant N-1$. Since dim $N_i \leqslant n-3$ and $2\dim N_i \leqslant n+p-2$, by Zeeman's Engulfing Theorem there are n-balls $B_1, \ldots, B_N$ such that $N_i \subset B_i$. Since P_i collapses to N_i, $1 \leqslant i \leqslant N$, we may assume without loss of generality that $P_i \subset B_i$, $1 \leqslant i \leqslant N$. This concludes the proof of the theorem. ∎

Let M be a closed PL n-manifold and let $B(M)$ be the smallest integer k such that M can be covered with k PL n-balls. Theorem 3.5 can be restated as follows: If M is a closed p-connected PL n-manifold, $n \geqslant 5$, with the property that $\dim M \leqslant (2\operatorname{cat}(M)-1)(p+1)-3$ then $\operatorname{cat}(M) = B(M)$. Note that this bound is exactly the same bound founded by M. Clapp and D. Puppe [3] when studing the coincidence of the category and the strong category of a polyhedron. Also note that, since homotopy spheres have category two and $B(M) = 2$ if and only if M^n is an n-sphere, Theorem 3.5 implies the Generalized Poincaré Conjecture.

As we have seen, to realize the category in terms of well-known pieces (PL balls) can be very useful. We would like to finish this paper by stating some characterizations of the category in terms of covers whose elements are well known [10].

(a) Let P be a compact connected polyhedron, then $\operatorname{cat}(P) =$ smallest integer k such that $P \times Q$ can be covered with $k+1$ open subsets each of which is homeomorphic to $Q \times [0, 1) \cong (Q - *)$.

Let N^n be a PL n-manifold with boundary. A boundary ball B of N^n is a PL n-ball contained in N^n such that $B \cap \operatorname{Bd} N^n$ is a PL $(n-1)$-ball.

(b) Let M^m be a compact connected PL m-manifold, then $\operatorname{cat}(M^m)$ $=$ smallest integer k such that $M^m \times I^n$ (n big enough) can be covered with $k+1$ boundary balls.

(c) Let M^m be a closed connected PL m-manifold, then $\operatorname{cat}(M^m)$ $=$ smallest integer k such that $M^m \times S^n$ (n big enough) can be covered with $k+1$ PL balls.

References

[1] I. Berstein and P. J. Hilton, *Category and generalized Hopf invariants*, Illinois J. Math. 4 (1960), 437–451.

[2] T. A. Chapman, *Lectures on Hilbert cube manifolds*, C. B. M. S. Regional Conference Series in Math., No. 28, 1976.

[3] M. Clapp and D. Puppe, *Lusternik–Schnirelmann's type invariants and the topology of critical sets*, to appear in Trans. Amer. Math. Soc.

[4] M. Cohen, *Simplicial structures and transverse cellularity*, Ann. of Math. 85 (1967), 218–245.

[5] R. H. Fox, *On the Lusternik–Schnirelmann category*, Ann. of Math. 42 (1941), 333–370.

[6] T. Ganea, *Lusternik–Schnirelmann category and strong category*, Illinois J. Math. 11 (1967), 417–427.

[7] I. M. James, *On category, in the sense of Lusternik–Schnirelmann*, Topology 17 (1978), 331–349.

[8] L. Lusternik and L. Schnirelmann, *Méthodes Topologiques dans les Problèmes Variationnels*, Herman, Paris 1934.

[9] L. Montejano, *A quick proof of Singhof's* $\operatorname{cat}(M \times S^1) = \operatorname{cat}(M) + 1$ *theorem*, Manuscripta Math. 42 (1983), 49–52.

[10] —, *Lusternik-Schnirelmann category and Hilbert cube manifolds*, to appear in Topology Appl.

[11] R. S. Palais, *Lusternik-Schnirelmann theory on Banach manifolds*, Topology 5 (1966), 115–132.

[12] T. B. Rushing, *Topological Embeddings*, Academic Press, New York 1973.

[13] W. Singhof, *Minimal coverings of manifolds with balls*, Manuscripta Math. 29 (1979), 385–415.

[14] F. Takens, *The minimal number of critical points of a function on a compact manifold and the Lusternik-Schnirelmann category*, Invent. Math. 6 (1968), 197–244.

[15] —, *The Lusternik-Schnirelmann categories of a product space*, Compositio math. 22 (1970), 175–180.

[16] H. Toda, *Composition Methods in Homotopy Groups of Spheres*, Ann. of Math. Stud., No. 49, Princeton 1962.

Presented to the Topology Semester
April 3 – June 29, 1984