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A DISCRETE PHENOMENON IN PROPAGATION OF C^∞ SINGULARITIES

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1. Introduction and statement of the results

The study of the differential operator $G_\lambda = D_x^2 + x^2 D_y^2 - \lambda D_y$, $\lambda \in \boldsymbol{C}$, $(x, y) \in \boldsymbol{R}^2$, has played an important role in the development of the theory of pseudodifferential operators with multiple characteristics. Grushin ([3], [4]) proved that G_λ is hypoelliptic when $\lambda \notin 2\boldsymbol{Z}+1$ whereas it is not hypoelliptic when $\lambda \in 2\boldsymbol{Z}+1$. This result was extended in [3], [4], [8], [1], and many other papers (see for example [7] and the references given there). Let now Ω be an open subset of $\boldsymbol{R}^{n+2}$, $n \geqslant 1$, with coordinates (s, x, y), $s \in \boldsymbol{R}^n$, and denote the dual coordinates by (σ, ξ, η). G_λ defines a differential operator P_λ in Ω by the formula $P_\lambda u(s, x, y) = (D_x^2 + x^2 D_y^2 - \lambda D_y) u(s, x, y)$. Of course P_λ cannot be hypoelliptic. When $x \neq 0$, the characteristic set of P_λ is a smooth involutive manifold and the results of [9] show that, when $x \neq 0$, P_λ propagates microlocal C^∞ singularities along two-dimensional leaves on which s, σ, ξ, η are constants. In this note we give some results about what happens when $x = 0$. Denoting wavefront sets by WF, we prove the following results.

THEOREM 1. *If $\lambda \in \boldsymbol{C} \backslash (2\boldsymbol{Z}+1)$, P_λ propagates C^∞ singularities in the direction of $\partial/\partial y$. More precisely, assume that $u \in \mathscr{D}'(\Omega)$ and that $\varrho \in \mathrm{WF}(u)$, with $x(\varrho) = 0$. If $\mathrm{WF}(P_\lambda u)$ does not meet an open conic neighborhood Γ of ϱ and if $\lambda \notin 2\boldsymbol{Z}+1$, then the connected component of the set $\{\varrho + (0, 0, h, 0, 0, 0) \in \boldsymbol{R}^{2n+4}, h \in \boldsymbol{R}\} \cap \Gamma$ through ϱ is contained in $\mathrm{WF}(u)$.*

THEOREM 2. *Assume that $\lambda \in 2\boldsymbol{Z}+1$. Let $F^\pm$ be closed subcones of $T^*\Omega \backslash 0$ such that $\eta \gtrless 0 = x = \xi$ on $F^\pm$. If $\lambda \in 2\boldsymbol{Z}^+ + 1$, one can find $u \in \mathscr{D}'(\Omega)$ such that $\mathrm{WF}(u) = F^+$ and $P_\lambda u \in C^\infty(\Omega)$. If $-\lambda \in 2\boldsymbol{Z}^+ + 1$, one can find $u \in \mathscr{D}'(\Omega)$ such that $\mathrm{WF}(u) = F^-$ and $P_\lambda u \in C^\infty(\Omega)$.*

Remark. Theorems 1 and 2 show that, for P_λ, the lower order term plays an important role in the propagation of C^∞ singularities when $x = 0$. However, it plays no role in the propagation of analytic singularities, as the following very special case of a result of Grigis, Schapira, Sjöstrand [2] shows.

THEOREM ([2]). *Denote analytic wavefront sets by* WF_a. *Let* λ *be any complex number. Assume that* $u \in \mathscr{D}'(\Omega)$ *and that* $\varrho \in \mathrm{WF}_a(u)$. *If* $\mathrm{WF}_a(P_\lambda u)$ *does not meet an open conic neighborhood* Γ *of* ϱ, *then the connected component of the set* $\{\varrho+(0, h_1, h_2, 0, 0, 0) \in \boldsymbol{R}^{2n+4}, (h_1, h_2) \in \boldsymbol{R}^2\} \cap \Gamma$ *through* ϱ *is contained in* $\mathrm{WF}_a(u)$.

The plan of this paper is the following: In Section 2, we prove Theorem 1 by integration of estimates of Carleman type, following the method of [9]. Section 3 is devoted to the proof of Theorem 2.

2. Proof of Theorem 1

Theorem 1 will be deduced from the following estimate of Carleman type, the proof of which is based on results of [6] (see also [10]):

PROPOSITION 1. *If* $\lambda \notin 2\boldsymbol{Z}+1$, *there exists* $m \in \boldsymbol{Z}^+$ *such that the following holds: if* K *is a compact subset of* $\boldsymbol{R}^2$, *one can find a constant* $C_{K,\lambda}$ *such that the estimate*

$$\|e^{\gamma y} v\|_{L^2(\boldsymbol{R}^2)} \leqslant C_{K,\lambda} \|e^{\gamma y} D_y^m G_\lambda v\|_{L^2(\boldsymbol{R}^2)} \tag{1}$$

holds for all $\gamma \in \boldsymbol{R}$ *and all* $v \in C_0^\infty(K)$.

Proof of Proposition 1. Put $\tilde{G}_\lambda(\gamma, \eta) = D_x^2 + x^2(\eta+i\gamma)^2 - \lambda(\eta+i\gamma)$, $(\gamma, \eta) \in \boldsymbol{R}^2$. We will show that the estimate

$$\eta^{2(N+1)}\|w\| \leqslant C_\lambda(\eta^2+\gamma^2)^{N+1/2}\|\tilde{G}_\lambda(\gamma, \eta) w\| \tag{2}$$

holds for all $w \in \mathscr{S}(\boldsymbol{R}_x)$ and all $(\gamma, \eta) \in \boldsymbol{R}^2$ if $\lambda \notin 2\boldsymbol{Z}+1$ and $|\mathrm{Re}\,\lambda| \leqslant 4N$. Here N is any positive integer and $\| \ \|$ denotes the $L^2(\boldsymbol{R}_x)$ norm. This of course will imply (1) if one takes $w(x)$ equal to the partial Fourier transform of $e^{\gamma y} v(x, y)$ with respect to y. Since $\tilde{G}_\lambda(\gamma, \eta) = \tilde{G}_{-\lambda}(-\gamma, -\eta)$, (2) is a consequence of

$$\eta^{2(N+1)}\|w\| \leqslant C_\lambda(\eta^2+\gamma^2)^{N+1/2}\|\tilde{G}_\lambda(\gamma, \eta) w\| \tag{3}$$
$$\text{for all } w \in \mathscr{S}(\boldsymbol{R}_x),\ \eta > 0,\ \gamma \in \boldsymbol{R},\ \lambda \notin 2\boldsymbol{Z}^+ + 1,\ \mathrm{Re}\,\lambda \leqslant 4N.$$

To prove (3), put $\tilde{Q}_\lambda(\gamma, \eta) = e^{-i\gamma x^2/2}\tilde{G}_\lambda(\gamma, \eta) e^{i\gamma x^2/2}$. Then

$$\tilde{Q}_\lambda(\gamma, \eta) = \frac{-\gamma+i\eta}{i\eta}[X(X^* + \omega X) + (1-\lambda)\eta],$$

where $X = D_x + i\eta x$, $*$ means adjoint with respect to the $L^2(\boldsymbol{R}_x)$ scalar product and $\omega = \gamma/(-\gamma+i\eta)$. Therefore the results of Section 3 of [6] imply that (3) holds with $\tilde{G}_\lambda(\gamma, \eta)$ replaced by $\tilde{Q}_\lambda(\gamma, \eta)$. Hence (3) holds. This completes the proof of Proposition 1.

To obtain Theorem 1 from Proposition 1 we shall use the technique of Sjöstrand [9], which we recall briefly. Write $t' = s$, $t'' = (x, y)$, $t = (t', t'')$, $\tau' = \sigma$, $\tau'' = (\xi, \eta)$, $\tau = (\tau', \tau'')$. Take $0 \not\equiv \psi \in C_0^\infty(\boldsymbol{R}^n)$ and choose a symbol $\chi(t', \bar{t}', \tau') \in S^{n/4}_{1,1/2}((\boldsymbol{R}^n \times \boldsymbol{R}^n) \times \boldsymbol{R}^n)$ with support in a set $|t' - \bar{t}'| \leqslant$ const and equal to $\psi((t' - \bar{t}')|\tau'|^{1/2})|\tau'|^{n/4}$ if $|\tau'| \geqslant 1$. If $z \in \mathcal{S}'(\boldsymbol{R}^{n+2})$ and $\mathrm{WF}(z)$ does not meet the set $\tau' = 0$, one defines

$$T_\chi z(t, \tau') = \int e^{i\langle t' - \bar{t}', \tau'\rangle} \chi(t', \bar{t}', \tau') z(\bar{t}', t'') d\bar{t}',$$

which belongs to $C^\infty(\boldsymbol{R}^{n+2} \times \boldsymbol{R}^n)$. The following result is contained in [9].

Lemma 1 ([9]). *If $z \in \mathcal{D}'(\boldsymbol{R}^{n+2})$ satisfies $\mathrm{WF}(z) \subset \{(t, \tau) \in T^* \boldsymbol{R}^{n+2} \backslash 0,\ \tau'' = 0\}$, the following assertions are equivalent*:

(1) *$z \in H^\mu$ microlocally near $(t_0, \tau_0', 0)$.*

(2) *There exists a conic neighborhood $V \subset \boldsymbol{R}^{n+2} \times (\boldsymbol{R}^n \backslash \{0\})$ of (t_0, τ_0') such that $(1 + |\tau'|)^\mu T_\chi z(t, \tau') \in L^2(V)$.*

For later use we introduce, as in [9], the following notation: if $f \in C^\infty(\boldsymbol{R}^{n+2} \times \boldsymbol{R}^n)$ and $(t_0, \tau_0') \in \boldsymbol{R}^{n+2} \times (\boldsymbol{R}^n \backslash \{0\})$, we put

$$F_f(t_0, \tau_0') = \sup\{\mu \in \boldsymbol{R},\ (1 + |\tau'|)^\mu f(t, \tau') \text{ is square integrable in some conic neighborhood of } (t_0, \tau_0')\}.$$

We can now prove Theorem 1. First it is no restriction to assume that $\Omega = \boldsymbol{R}^{n+2}$ and we shall do so. If $\eta \neq 0$, P_λ is microlocally hypoelliptic since $\lambda \notin 2\boldsymbol{Z} + 1$ (see e.g. [8]). Hence $\tau'' = 0$ on $\mathrm{WF}(u) \cap \Gamma$. Therefore Theorem 1 will be proved if we can show the following:

(4) *Assume that $w \in \mathcal{S}'(\boldsymbol{R}^{n+2})$ is such that $\tau'' = 0$ on $\mathrm{WF}(w) \cap \Gamma$, and satisfies $\mathrm{WF}(P_\lambda w) \cap \Gamma = \emptyset$, $\lambda \notin 2\boldsymbol{Z} + 1$. If $(s_0, 0, y_0, \sigma^0, 0, 0) \in \Gamma \backslash \mathrm{WF}(w)$, then the connected component of $\{(s_0, 0, y_0 + h, \sigma^0, 0, 0),\ h \in \boldsymbol{R}\} \cap \Gamma$ through $(s_0, 0, y_0, 0, 0, \sigma^0)$ does not meet $\mathrm{WF}(w)$.*

When proving (4), it is of course no restriction to assume that $s_0 = y_0 = 0$. We may also assume that $h < 0$, since a change of variables $(s, x, y) \mapsto (s, x, -y)$ will then take care of the case $h > 0$. Take $R_1, R_2, R > 0$ such that the set

$$\Gamma_1 = \{(s, x, y, \sigma^0, 0, 0),\ |s| \leqslant R_1,\ |x| \leqslant R_2,\ -R \leqslant y \leqslant 0\}$$

is contained in Γ. Choose R_1, R_2 so small that $(s, x, 0, \sigma^0, 0, 0) \notin \mathrm{WF}(w)$ if $(s, x, 0, \sigma^0, 0, 0) \in \Gamma_1$. Then, by the results of [9], $x = 0$ on $\mathrm{WF}(w) \cap \Gamma_1$. Choose $r > 0$ such that $y < -r$ on $\mathrm{WF}(w) \cap \Gamma_1$, and put $a = (R + r)/2$. We will show the following:

(5) *$\mathrm{WF}(w) \cap \Gamma_1$ dos not meet the slab $-a < y < -r$.*

An iteration of the proof of (5) will give (4) for $h < 0$, completing the proof of Theorem 1. To prove (5), choose a properly supported classical pseudodifferential operator φ of order 0 such that $\mathrm{WF}(\varphi) \subset \Gamma$ and such that $\mathrm{WF}(I-\varphi)$ does not meet a conic neighborhood of Γ_1. Also take $\theta(t'') \in C_0^\infty(\{t'' \in \boldsymbol{R}^2,\ -R < y < -r, |x| < R_2\})$ with $\theta(t'')=1$ if $-a<y<-r-\varepsilon$ and $|x|<R_2/2$. Put $L_\lambda = D_y^m P_\lambda$ with m as in (1), and $g(t,\tau') = T_\chi(\varphi w)(t,\tau')$. Let us apply (1) to $\theta(t'')g(t,\tau')$ with $\gamma = \nu \log\langle\tau'\rangle$, where $\nu \in \boldsymbol{Z}^+$. Since $[L_\lambda, T_\chi]=0$, it follows that, if $M \in \boldsymbol{R}$,

$$\text{(6)} \qquad \mathrm{const}\,\|\langle\tau'\rangle^{\nu(y+a)-M}\theta(t'')g(t,\tau')\|^2_{L^2(\boldsymbol{R}^2_{t''})} \leqslant A(t',\tau')+B(t',\tau')+C(t',\tau'),$$

where

$$A(t',\tau') = \|\langle\tau'\rangle^{\nu(y+a)-M}\theta T_\chi L_\lambda \varphi w(t,\tau')\|^2_{L^2(\boldsymbol{R}^2_{t''})},$$

$$B(t',\tau') = \|\langle\tau'\rangle^{\nu(y+a)-M}[L_\lambda,\theta]g(t,\tau')\|^2_{L^2(\mathscr{B})},$$

$$C(t',\tau') = \|\langle\tau'\rangle^{\nu(y+a)-M}[L_\lambda,\theta]g(t,\tau')\|^2_{L^2(\{t''\in\boldsymbol{R}^2, y<-a\})},$$

where we have put $\mathscr{B} = \{t''\in\boldsymbol{R}^2,\ -r-\varepsilon<y<-r\} \cup \{t''\in\boldsymbol{R}^2,\ |x|>R_2/2\}$.

Let Δ be a small conic neighborhood of σ^0 in $T^*\boldsymbol{R}^n\backslash 0$, and put

$$E = \{(s,\sigma)\in\boldsymbol{R}^n\times(\boldsymbol{R}^n\backslash\{0\}),\ |s|<R_1,\ \sigma\in\Delta\}.$$

Since $\mathrm{WF}(L_\lambda\varphi w)\cap\Gamma_1 = \emptyset$, Lemma 1 above implies that $\int_E A(t',\tau')\,dt'\,d\tau' < \infty$ if Δ is narrow enough. But

$$[L_\lambda,\theta] = \sum_{|\alpha|\leqslant m+1} \theta_\alpha(t'')D^\alpha_{t''}$$

commutes with T_χ. Since $y<-r-\varepsilon$ and $x=0$ on $\mathrm{WF}(w)\cap\Gamma_1$, it follows that $\int_E B(t',\tau')\,dt'\,d\tau' < \infty$ if Δ is narrow enough. If we now choose M so large that

$$F_{T_\chi D^\alpha\varphi w}(t,\tau') > -M \qquad \text{if } (t,\tau',0)\in\Gamma_1 \text{ and } |\alpha|\leqslant m+1,$$

we obtain $\int_E C(t',\tau')\,dt'\,d\tau'<\infty$ if Δ is narrow enough. Using (6) for every $\nu\in\boldsymbol{Z}^+$, we conclude, using Lemma 1, that (5) holds. The proof of Theorem 1 is complete.

3. Proof of Theorem 2

Put $A^\pm = D_x \pm ixD_y$. Theorem 3.3.7 of [5] implies that there exists $u\in\mathscr{D}'(\Omega)$ with $\mathrm{WF}(u)=F^+$ and $A^-u\in C^\infty(\Omega)$. Hence Theorem 2 follows immediately for P_1, since $P_1 = A^+A^-$. On the other hand $P_{\lambda+2}A^+ = A^+P_\lambda$ and A^+ is microlocally hypoelliptic if $\eta>0$, so Theorem 2 follows for P_λ if $\lambda\in 2\boldsymbol{Z}^+ + 1$. Using the change of variables $(s,x,y)\mapsto(s,x,-y)$, one gets Theorem 2 when $-\lambda\in 2\boldsymbol{Z}^+ + 1$. The proof is complete.

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