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CONTINUITY OF QUASIMINIMA UNDER THE PRESENCE OF IRREGULAR OBSTACLES

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The notion of quasiminimum for a variational integral was introduced by Giaquinta and Giusti in [6] and [7] in connection with direct methods for establishing the regularity of minima for nondifferentiable functionals of integral type. It was shown in [7] that solutions to a large class of elliptic differential equations and variational inequalities are quasiminima for rather simple variational integrals, and thus their Hölder continuity can be established with direct methods. We first recall the definition from [7]:

DEFINITION. Let Ω be a domain in the Euclidean space $\boldsymbol{R}^n$ and let F be a Carathéodory function on $\Omega \times \boldsymbol{R}^N \times \boldsymbol{R}^{Nn}$ satisfying

$$|\xi|^p - a|u|^\alpha - g(x) \leqslant F(x, u, \xi) \leqslant a(|\xi|^p + |u|^\alpha) + g(x)$$

where g is a given nonnegative function and a, α, p are nonnegative constants with $p > 1$ and $1 \leqslant \alpha < np/(n-p)$ if $p < n$, $1 \leqslant \alpha < +\infty$ if $p \geqslant n$. Let $Q \geqslant 1$.

A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \boldsymbol{R}^N)$ is called a (local) *Q-minimum* for the functional

$$\mathscr{F}(u; \Omega) = \int_\Omega F(x, u(x), \nabla u(x))\,dx$$

if for every open set A with compact closure in Ω and for every $v \in W^{1,p}_{\mathrm{loc}}(\Omega; \boldsymbol{R}^N)$ with $v(x) = u(x)$ outside of A we have $\mathscr{F}(u; A) \leqslant Q\mathscr{F}(v; A)$.

Clearly, due to the absolute continuity of the integral and the outer regularity of Lebesgue measure, this is equivalent to the statement that for every $\varphi \in W^{1,p}_0(\Omega; \boldsymbol{R}^N)$ with compact support in Ω we have

$$\mathscr{F}(u; \{x;\ \varphi(x) \neq 0\}) \leqslant Q\mathscr{F}(u+\varphi; \{x;\ \varphi(x) \neq 0\}).$$

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In the present work we will study scalar-valued quasiminima ($N = 1$) under the presence of obstacles, i.e. we will impose a restriction $\psi_1(x) \leqslant u(x)$, $v(x) \leqslant \psi_2(x)$ on u and on the comparison functions v, where ψ_1 and ψ_2 are prescribed obstacle functions. If these are regular, then, as was pointed out in [7], the problem can be reduced to a problem without obstacles for the modified functional

$$\mathscr{F}'(u; \Omega) = \mathscr{F}(u; \Omega) + \mathscr{F}(\psi_1; \Omega) + \mathscr{F}(\psi_2; \Omega)$$

and thus the Hölderness of u can, if in addition $|\nabla\psi_1|$ and $|\nabla\psi_2|$ are summable to some power larger than p, be established directly. However, we are concerned with irregular obstacles, which may even be defined only on lower-dimensional manifolds. Our aim is to establish criteria and a priori estimates for the pointwise continuity of a quasiminimum. These criteria will be in terms of the $(1, p)$-capacities of one-sided level sets of the obstacles (Theorem 2.1), and we also include the question of boundary regularity in our investigations. Concerning the latter, we obtain a criterion for the regularity of a boundary point which is somewhat weaker than the ones deduced by Maz'ya [12] and by Gariepy and Ziemer [5], which generalize the classical Wiener criterion ([14], [15]). However, since we do not have any Euler equation for our variational problem, their methods do not seem to carry over.

The present work originates in some investigations on nonlinear variational inequalities ([8]) of the form

$$\sum_{\alpha=1}^{n} \int_{\Omega} a^\alpha(x, u(x), \nabla u(x))\left(\frac{\partial u}{\partial x_\alpha} - \frac{\partial v}{\partial x_\alpha}\right) dx \leqslant \int_{\Omega} b(x, u(x), \nabla u(x))(u(x) - v(x))\, dx$$

assumed to hold for all $v \in W^{1,p}(\Omega)$ with $\psi_1(x) \leqslant v(x) \leqslant \psi_2(x)$ and $v - u \in W_0^{1,p}(\Omega)$. All the main results of [8] are contained in our present results. In this connection we wish to mention the work of Frehse and Mosco [4], announced in [3]. They are concerned with quasilinear variational inequalities with a one-sided obstacle, and they obtain very precise continuity criteria, also related to the Wiener criterion. Similar results for vector-valued solutions of elliptic variational inequalities of diagonal type were obtained by Karlsson ([9], [10]). Finally we want to mention the Harnack inequality for quasiminima, proved by Di Benedetto and Trudinger [2].

The methods employed in the present work go back to De Giorgi [1] but are also highly influenced by [2].

Basic notation

We study real-valued functions in the Sobolev space $W^{1,p}(\Omega)$, i.e. functions in $L^p(\Omega)$ having first-order distributional derivatives in $L^p(\Omega)$. Here Ω is a domain in the Euclidean space $\boldsymbol{R}^n$. We limit ourselves to the case $1 < p < n$, but remark that Lemmas 2.1, 2.2 remain valid for $p = 1$. When $x_0 \in \boldsymbol{R}^n$ and $\varrho > 0$, we denote by $B_\varrho(x_0)$ the n-ball (not necessarily contained in Ω) with center at x_0 and radius ϱ. $\Omega_\varrho(x_0)$ is the intersection $B_\varrho(x_0) \cap \Omega$.

For a measurable function u defined in Ω we use the notation

$$A^+_{k,\varrho}(x_0) = \{x \in \Omega_\varrho(x_0);\ u(x) > k\}, \qquad A^-_{k,\varrho}(x_0) = \{x \in \Omega_\varrho(x_0);\ u(x) < k\},$$

$$M_\varrho(x_0) = \operatorname*{ess\,sup}_{\Omega_\varrho(x_0)} u(x), \quad m_\varrho(x_0) = \operatorname*{ess\,inf}_{\Omega_\varrho(x_0)} u(x) \quad \text{and} \quad \omega_\varrho(x_0) = M_\varrho(x_0) - m_\varrho(x_0).$$

The argument "x_0" in $B_\varrho(x_0)$, $A^+_{k,\varrho}(x_0)$, etc. is often omitted when there is no fear of confusion.

If $u \in W^{1,p}(\Omega)$ and if S is a relatively open subset of the boundary $\partial\Omega$, the number $\sup_S u(x)$ is defined as the infimum of the set of numbers k such that the function $\eta(x)(u(x)-k)_+$ belongs to $W_0^{1,p}(\Omega)$ for every Lipschitz continuous function η on $\bar{\Omega}$ vanishing on $\partial\Omega \setminus S$. Similarly for $\inf_S u(x)$. Thus, if $S = \emptyset$, we have $\sup_S u(x) = -\infty$ and $\inf_S u(x) = +\infty$. Here $(u(x)-k)_+$ denotes the positive part of $u(x)-k$, that is, $\max(0, u(x)-k)$. We also use the notation $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$.

For a measurable set E, $|E|$ is its Lebesgue measure, and the symbol $⨍$ stands for arithmetic mean value:

$$⨍_E u(x)\,dx = \frac{1}{|E|} \int_E u(x)\,dx.$$

1. Auxiliary lemmas

Our first lemma is a variant of Lemma 5.4 in [11], Chapter 2:

LEMMA 1.1. *Let $1 \leqslant p < n$, $u \in W^{1,p}(\Omega)$, $x_0 \in \Omega$ and $0 < \varrho \leqslant 1$. Suppose that for arbitrary numbers k greater than some $\hat{k}$ and for arbitrary $x_1 \in \Omega_\varrho(x_0)$ and ϱ_1, ϱ_2 with $0 < \varrho_1 < \varrho_2 \leqslant \varrho$, the function u satisfies the inequalities*

$$\int_{A_{k,\varrho_1}(x_1)} |\nabla u(x)|^p\,dx$$
$$\leqslant [(\varrho_2-\varrho_1)^{-p} \int_{A_{k,\varrho_2}(x_1)} (u(x)-k)^p\,dx + |k|^p |A_{k,\varrho_2}(x_1)| + \lambda^p \varrho_2^{n-p+p\sigma}]$$

where γ and λ are nonnegative constants, and $0 < \sigma \leqslant 1$.

Then there exists a number $c_1 > 0$, depending only on n, p, γ and σ, such that

$$M_\varrho(x_0) \leqslant k + c_1 \Big[|B_{2\varrho}|^{-1} \int_{A_{k,2\varrho}(x_0)} (u(x)-k)^{p\,dx}\Big]^{1/p} + c_1(\lambda\varrho^\sigma + |k|\,\varrho)$$

for every $k \geqslant \hat{k}$.

Proof. Fix a representative (pointwise defined function) for u and let

$x_1 \in \Omega_\varrho(x_0)$ be such that

$$u(x_1) = \lim_{r\to 0} \fint_{\Omega_r(x_1)} u(x)\,dx$$

(which holds for almost all x_1).

Put, for $\nu = 0, 1, 2, \ldots$, $\varrho_\nu = 2^{-\nu/\sigma}\varrho$ and $k_\nu = k+(1-2^{-\nu})m$, where m is a positive number to be determined later. Put

$$J_\nu = \int_{A_{k_\nu,\varrho_\nu}(x_1)} (u(x)-k_\nu)^p\,dx = \int_{\Omega_{\varrho_\nu}(x_1)} (u(x)-k_\nu)_+^p\,dx.$$

The Hölder and Sobolev inequalities yield

$$J_{\nu+1} \leqslant |A_{k_{\nu+1},\varrho_{\nu+1}}|^{p/n}\Big[\int_{\Omega_{\varrho_{\nu+1}}(x_1)} (u(x)-k_{\nu+1})_+^{np/(n-p)}\,dx\Big]^{1-p/n}$$

$$\leqslant c(n,p)|A_{k_{\nu+1},\varrho_{\nu+1}}|^{p/n}\Big[\int_{A_{k_{\nu+1},\varrho_{\nu+1}}} |\nabla u(x)|^p\,dx$$

$$+\varrho_{\nu+1}^{-p}\int_{A_{k_{\nu+1},\varrho_{\nu+1}}} (u(x)-k_{\nu+1})^p\,dx\Big].$$

Thus (since $\varrho_{\nu+1}^{-p} \geqslant (\varrho_\nu-\varrho_{\nu+1})^{-p}$)

$$J_{\nu+1} \leqslant c(n,p,\gamma)|A_{k_{\nu+1},\varrho_\nu}|^{p/n}[\varrho_{\nu+1}^{-p}J_\nu+|k_{\nu+1}|^p|A_{k_{\nu+1},\varrho_\nu}|+\lambda^p\varrho_\nu^{n-p+p\sigma}]$$

$$\leqslant c(n,p,\gamma)|A_{k_{\nu+1},\varrho_\nu}|^{p/n}[\varrho_{\nu+1}^{-p}J_\nu+(k_{\nu+1}-k)^p|A_{k_{\nu+1},\varrho_\nu}|$$
$$+\lambda^p\varrho_\nu^{n-p+p\sigma}+|k|^p\varrho_\nu^n].$$

Now $|A_{k_{\nu+1},\varrho_\nu}| \leqslant (k_{\nu+1}-k_\nu)^{-p}J_\nu = 2^{p(\nu+1)}m^{-p}J_\nu$, and

$$(k_{\nu+1}-k)^p|A_{k_{\nu+1},\varrho_\nu}| \leqslant 2^{p(\nu+1)}J_\nu \leqslant \varrho_{\nu+1}^{-p}J_\nu,$$

so

$$J_{\nu+1} \leqslant c(n,p,\gamma)\cdot 2^{\nu p^2/n}\cdot m^{-p^2/n}\cdot J_\nu^{p/n}\cdot(\varrho_{\nu+1}^{-p}J_\nu+\lambda^p\varrho_\nu^{n-p+p\sigma}+|k|^p\varrho_\nu^n).$$

Put $M_\nu = |B_{\varrho_\nu}(x_1)|^{-1}J_\nu$:

$$M_{\nu+1} \leqslant c(n,p,\gamma,\sigma)\cdot 2^{\nu p^2/n}\cdot m^{-p^2/n}\cdot M_\nu^{p/n}\cdot(M_\nu+\lambda^p\varrho_\nu^{p\sigma}+|k|^p\varrho_\nu^p).$$

Put $\alpha_\nu = 2^{\nu p}\cdot M_\nu$:

$$\alpha_{\nu+1} \leqslant c_0 m^{-p^2/n}\alpha_\nu^{p/n}(\alpha_\nu+\lambda^p\varrho^{p\sigma}+|k|^p\varrho^p),$$

where c_0 depends on n, p, γ and σ.

Now choose θ, depending on m, so that $c_0 m^{-p^2/n}\theta^{p/n} = \frac{1}{3}$, i.e. $\theta = (3c_0)^{-n/p}m^p$.

Choose m so large that

(i) $\lambda^p\varrho^{p\sigma} \leqslant \theta$, i.e. $m \geqslant (3c_0)^{n/p^2}\lambda\varrho^\sigma$,

(ii) $|k|^p \varrho^p \leqslant \theta$, i.e. $m \geqslant (3c_0)^{n/p^2} |k| \varrho$, and

(iii) $\alpha_0 \leqslant \theta$, i.e. $m \geqslant (3c_0)^{n/p^2} \left[|B_\varrho(x_1)|^{-1} \int_{A_{k,\varrho}(x_1)} (u(x)-k)^p \, dx\right]^{1/p}$.

Then as long as $\alpha_\nu \leqslant \theta$ we also have $\alpha_{\nu+1} \leqslant \frac{1}{3}(\alpha_\nu + \lambda^p \varrho^{p\sigma} + |k|^p \varrho^p) \leqslant \frac{1}{3}(\theta+\theta+\theta) = \theta$, that is, $\alpha_\nu \leqslant \theta$ for all ν, which implies that $\lim_{\nu\to\infty} M_\nu = 0$. Thus if

$$m \geqslant (3c_0)^{n/p^2} \max\left(\left[|B_\varrho|^{-1} \int_{\Omega_{2\varrho}(x_0)} (u(x)-k)_+^p \, dx\right]^{1/p}, \lambda\varrho^\sigma, |k|\varrho\right),$$

we have for almost all $x_1 \in \Omega_\varrho(x_0)$

$$u(x_1) = \lim_{\nu\to\infty}\left[|B_{\varrho_\nu}|^{-n} \int_{\Omega_{\varrho_\nu}(x_1)} (u(x)-k_\nu)\,dx + k_\nu \frac{|\Omega_{\varrho_\nu}(x_1)|}{|B_{\varrho_\nu}|}\right]$$
$$\leqslant \lim_{\nu\to\infty} k_\nu = k+m,$$

which proves the lemma with $c_1 = 2^{n/p}(3c_0)^{n/p^2}$.

The following lemma will be used to estimate the modulus of continuity of a quasiminimum:

LEMMA 1.2. *Let f and p be nonnegative functions on $[0, 4]$, p nondecreasing, and suppose that $f(\varrho) \leqslant M$ for $\varrho \leqslant 4$ and that for some θ with $0 < \theta < 1$*

$$f(\varrho) \leqslant \theta f(4\varrho) + p(\varrho) \quad \textit{for all } \varrho \in [0, 1].$$

Then, for all $\varrho \in [0, 4]$, the following inequality holds:

$$f(\varrho) \leqslant M\varrho^\alpha + \frac{\alpha \cdot 4^\alpha}{4^\alpha - 1}\varrho^\alpha \int_\varrho^4 r^{-\alpha-1} p(r)\,dr, \quad \textit{where } \alpha = \frac{\log(1/\theta)}{\log 4}.$$

Lemma 1.2 is proved by induction. The stated inequality holds trivially for $\varrho \in [1, 4]$, and it is easily seen that it holds for $\varrho/4$ whenever it holds for ϱ. We note the following consequences of the lemma, leaving the details to the reader:

$$\lim_{\varrho\to 0} p(\varrho) \leqslant \varepsilon \quad \text{implies} \quad \lim_{\varrho\to 0} f(\varrho) \leqslant \frac{4^\alpha \cdot \varepsilon}{4^\alpha - 1},$$

$$p(\varrho) = O(\varrho^\beta) \text{ when } \varrho \to 0 \quad \text{implies} \quad f(\varrho) = O(\varrho^{\alpha\wedge\beta}) \text{ if } \beta \neq \alpha,$$

$$p(\varrho) = O(\varrho^\alpha) \text{ when } \varrho \to 0 \quad \text{implies} \quad f(\varrho) = O\left(\varrho^\alpha \log\frac{1}{\varrho}\right).$$

We will now state the basic facts about Riesz potentials and capacities that will be needed. The reader is referred to e.g. [13] for more complete information on these topics.

The *Riesz kernel* R_1 is defined by $R_1(x) = \frac{1}{2}\pi^{-(n+1)/2}\Gamma((n-1)/2)|x|^{1-n}$. If

μ is a positive Borel measure in $\boldsymbol{R}^n$, we define its *Riesz potential* ${}^{\mu}U_1$ by

$$ {}^{\mu}U_1(x) = \int_{\boldsymbol{R}^n} R_1(x-y)\,d\mu(y). $$

Similarly, the *Riesz potential* U_1^f of a function $f \in L^p(\boldsymbol{R}^n)$ with $f(x) \geqslant 0$ is

$$ U_1^f(x) = \int_{\boldsymbol{R}^n} R_1(x-y) f(y)\,dy. $$

For $p > 1$, the *outer* $(1, p)$*-capacity* of a set $E \subset \boldsymbol{R}^n$ is defined by

$$ C_{1,p}(E) = \inf\{\int_{\boldsymbol{R}^n} f(x)^p\,dx;\ f \in L^p(\boldsymbol{R}^n),\ f \geqslant 0,\ U_1^f(x) \geqslant 1 \text{ on } E\}, $$

and the *inner* $(1, p)$*-capacity* by

$$ c_{1,p}(E) = \sup\{[\int_{\boldsymbol{R}^n} {}^{\mu}U_1(x)^{p'}\,dx]^{-1/p'};\ \mu \text{ probability measure with support in } E\}. $$

Here p' is the exponent conjugate to p: $1/p' + 1/p = 1$.

It is known that $c_{1,p}(E)^p \leqslant C_{1,p}(E)$ with equality at least for all Suslin sets, especially for all Borel sets.

EXAMPLES. 1. If $1 < p < n$ and E is Lebesgue measurable, then $C_{1,p}(E) \geqslant c(n, p)|E|^{1-p/n}$ with a constant $c(n, p) > 0$.

2. If $1 < p < n$ and E is a k-dimensional ball of radius r, then $C_{1,p}(E) = c(n, k, p) r^{n-p}$, where the constant $c(n, k, p)$ is strictly positive if $n-p < k < n$.

We will use the expression "quasi-everywhere" (q.e.) or "for quasi all x" when the exceptional set has outer $(1, p)$-capacity zero.

Now it is well known (see e.g. [12]) that if $u \in W^{1,p}(\Omega)$, then the limit

$$ \tilde{u}(x) = \lim_{\varrho \to 0} \fint_{\Omega_\varrho(x)} u(y)\,dy $$

exists for quasi all $x \in \Omega$. From now on we will identify u with its canonical representative $\tilde{u}$. Hence it will make sense to consider inequalities for u imposed on sets of positive capacity even if the Lebesgue measure is zero.

We will also make use of the following lemma:

LEMMA 1.3. *Let B be a ball of radius R, and let $v \in W^{1,p}(B)$. Let also μ be a probability measure supported in $\{x \in B;\ v(x) = 0\}$ and let $k > 0$. Then if $A_k = \{x \in B;\ |v(x)| \geqslant k\}$, we have*

$$ k|A_k| \leqslant cR^n \int_B {}^{\mu}U_1(z)|\nabla v(z)|\,dz $$

where the constant c depends only on n.

Proof. Suppose first that v is smooth. If $x, y \in B$ and $v(y) = 0$, then

$$|v(x)| \leqslant \int_0^{|x-y|} \left| \nabla v\left(y + t\frac{x-y}{|x-y|}\right)\right| dt.$$

We introduce polar coordinates around x and extend ∇v to be zero outside of B (then it is of course not a gradient any more):

$$\int_B |v(x)|\,dx \leqslant \int_0^{2R} r^{n-1} \int_{S^{n-1}} \int_0^r |\nabla v(x+t\xi)|\,dx\,d\sigma_\xi\,dr$$

$$\leqslant \frac{(2R)^n}{n} \int_B |x-z|^{1-n} |\nabla v(z)|\,dz.$$

Integrating with respect to $d\mu(x)$ we arrive at

$$k|A_k| \leqslant \int_B |v(x)|\,dx \leqslant cR^n \int_B {}^{\mu}U_1(z)\,|\nabla v(z)|\,dz,$$

which is the desired result. A simple approximation argument shows that it is valid also for general $v \in W^{1,p}(B)$.

2. Quasiminima in convex subsets of $W^{1,p}$

We are now ready to describe precisely the assumptions under which we shall work.

General Assumptions. 1. F is a Carathéodory function on $\Omega \times \boldsymbol{R} \times \boldsymbol{R}^n$, i.e. $F(\cdot, u, \xi)$ is measurable in Ω for all $(u, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^n$, $F(x, \cdot, \cdot)$ is continuous in $\boldsymbol{R} \times \boldsymbol{R}^n$ for almost all $x \in \Omega$.

2. For some nonnegative constant a we have

$$|\xi|^p - a|u|^p - f(x)^p \leqslant F(x, u, \xi) \leqslant a(|\xi|^p + |u|^p) + f(x)^p$$

for all $(x, u, \xi) \in \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n$, where f is a nonnegative function belonging to the Morrey space $L^{p,n-p+p\sigma}(\Omega)$ for some $\sigma > 0$, i.e. there is a $\lambda \geqslant 0$ such that

$$\int_E f(x)^p\,dx \leqslant \lambda^p (\operatorname{diam} E)^{n-p+p\sigma} \quad \text{for every measurable subset } E \text{ of } \Omega.$$

3. ψ_1 and ψ_2 are functions defined q.e. in Ω with

$$-\infty \leqslant \psi_1(x) < +\infty, \qquad -\infty < \psi_2(x) \leqslant +\infty, \qquad \psi_1(x) \leqslant \psi_2(x) \quad \text{q.e.}$$

4. u_0 is a fixed element of $W^{1,p}(\Omega)$, and we put

$$\mathcal{K} = \{v \in W^{1,p}(\Omega);\ \psi_1(x) \leqslant v(x) \leqslant \psi_2(x) \text{ q.e.},\ v - u_0 \in W_0^{1,p}(\Omega)\}.$$

Before stating our main results we have to introduce some further notation and terminology.

We say that $u \in \mathscr{K}$ is a *Q-minimum* for the functional

$$\mathscr{F}(u; A) = \int_A F(x, u(x), \nabla u(x)) dx$$

in $\mathscr{K}$ if, for every $v \in \mathscr{K}$, we have

$$\mathscr{F}(u; \{x;\ u(x) \neq v(x)\}) \leqslant Q\mathscr{F}(v; \{x;\ u(x) \neq v(x)\}).$$

u is a *sub-Q-minimum* for $\mathscr{F}$ *in* $\mathscr{K}$ if for every $v \in \mathscr{K}$ with $v(x) \leqslant u(x)$ q.e. we have $\mathscr{F}(u; \{x;\ u(x) \neq v(x)\}) \leqslant Q\mathscr{F}(v; \{x;\ u(x) \neq v(x)\})$.

u is a *super-Q-minimum* for $\mathscr{F}$ *in* $\mathscr{K}$ if for every $v \in K$ with $v(x) \geqslant u(x)$ q.e. we have $\mathscr{F}(u; \{x;\ u(x) \neq v(x)\}) \leqslant Q\mathscr{F}(v; \{x;\ u(x) \neq v(x)\})$.

Clearly a Q-minimum is both a sub- and a super-Q-minimum.

Let

$$\text{q.e.}\sup_{\Omega_0(x_0)} \psi_1(x) := \inf\{t;\ C_{1,p}\{x \in \Omega_\varrho(x_0);\ \psi_1(x) > t\} = 0\}$$

and define q.e.$\inf \psi_2(x)$ correspondingly.

Throughout the rest of the paper we will use the notation

$$\hat{k}_\varrho(x_0) = \max\big(\sup_{\partial\Omega \cap B_\varrho(x_0)} u(x),\ \text{q.e.}\sup_{\Omega_\varrho(x_0)} \psi_1(x)\big),$$

$$\hat{l}_\varrho(x_0) = \min\big(\inf_{\partial\Omega \cap B_\varrho(x_0)} u(x),\ \text{q.e.}\inf_{\Omega_\varrho(x_0)} \psi_2(x)\big),$$

$$E_1(x_0, \varrho, \varepsilon) = \{x \in \Omega_\varrho(x_0);\ \psi_1(x) \geqslant \hat{k}_\varrho(x_0) - \varepsilon\} \cup (B_\varrho(x_0) \setminus \Omega),$$

$$E_2(x_0, \varrho, \varepsilon) = \{x \in \Omega_\varrho(x_0);\ \psi_2(x) \leqslant \hat{l}_\varrho(x_0) + \varepsilon\} \cup (B_\varrho(x_0) \setminus \Omega).$$

We are now ready to state

THEOREM 2.1. *Let u be a Q-minimum in $\mathscr{K}$ for the functional $\mathscr{F}$ satisfying the General Assumptions and let $x_0 \in \bar{\Omega}$.*

Then if

$$\lim_{\varrho \to 0} (\hat{k}_\varrho(x_0) - \hat{l}_\varrho(x_0)) \leqslant 0 \ \textit{and}\ \liminf_{\varrho \to 0} \varrho^{p-n} C_{1,p}(E_i(x_0, \varrho, \varepsilon)) > 0$$

for $i = 1, 2$ and for all $\varepsilon > 0$, then u is continuous at x_0.

Remark. Compared to the introductory definition of quasiminima, our growth conditions are slightly less general. However, under the more general conditions of the introduction one easily proves, following [7], that our quasiminima are locally bounded where ψ_1 is bounded from above and ψ_2 bounded from below. Then the difference in growth conditions disappears.

Before going into the proof of Theorem 2.1, we will deduce some lemmas for sub-Q-minima. The first ones follow easily from Lemma 1.1 via an elementary lemma of Giaquinta and Giusti. Lemma 2.4 is developed from a lemma of De Giorgi [1].

LEMMA 2.1. *Let u be a sub-Q-minimum for $\mathscr{F}$ in $\mathscr{K}$. Then there exists a* *positive constant c_1, depending only on n, p, a, Q and σ, such that if $x_0 \in \bar{\Omega}$ and $0 < \varrho \leq 1$ the following estimate is valid*:

$$M_\varrho(x_0) \leq k + c_1 \left[|B_{2\varrho}|^{-1} \int_{A_{k,2\varrho}(x_0)} (u(x)-k)^p\, dx\right]^{1/p} + c_1 (\lambda\varrho^\sigma + |k|\,\varrho)$$

for every $k \geq \hat{k}_{2\varrho}(x_0)$.

Proof. Let $x_1 \in \Omega_\varrho(x_0)$ and let r and R be numbers with $0 < r < R$. Let η be the (unique) Lipschitz function which is 1 on $B_r(x_1)$, 0 in $R^n \backslash B_R(x_1)$ and satisfies $|\nabla\eta(x)| \leq (R-r)^{-1}$.

Let $k \geq k_{2\varrho}(x_0)$ and put $v(x) = u(x) - \eta(x)(u(x)-k)_+$. Then clearly $v \in \mathcal{K}$ and $v(x) \leq u(x)$. From the General Assumptions we easily get

$$\int_{A_{k,R}(x_1)} |\nabla u(x)|^p\, dx \leq c \int_{A_{k,R}(x_1)} \Big[(1-\eta(x))^p |\nabla u(x)|^p + \big(|\nabla\eta(x)|^p + \eta(x)^p\big)\big(u(x)-k\big)^p + |u(x)|^p + f(x)^p\Big]\, dx$$

whence

$$\int_{A_{k,r}} |\nabla u(x)|^p\, dx \leq c \int_{A_{k,R}\backslash A_{k,r}} |\nabla u(x)|^p\, dx + c(R-r)^{-p} \int_{A_{k,R}} (u(x)-k)^p\, dx + c\,|k|^p |A_{k,R}| + c\lambda^p R^{n-p+p\sigma}.$$

We now add the term $c \int_{A_{k,r}} |\nabla u(x)|^p\, dx$ to both sides and divide by $c+1$ to get

$$\int_{A_{k,r}} |\nabla u(x)|^p\, dx \leq \theta \int_{A_{k,R}} |\nabla u(x)|^p\, dx + (R-r)^{-p} \int_{A_{k,R}} (u(x)-k)^p\, dx + |k|^p |A_{k,R}| + \lambda^p R^{n-p+p\sigma},$$

with $\theta = c/(c+1) < 1$. Lemma 1.1 of [6] yields

$$\int_{A_{k,r}} |\nabla u(x)|^p\, dx \leq \gamma\left[(R-r)^{-p} \int_{A_{k,R}} (u(x)-k)^p\, dx + |k|^p |A_{k,R}| + \lambda^p R^{n-p+p\sigma}\right]$$

with a constant γ that depends only on n, p, a and Q. The conclusion now follows from Lemma 1.1.

The following two lemmas are immediate consequences of Lemma 2.1:

LEMMA 2.2. *Let u be a sub-Q-minimum for $\mathcal{F}$ in $\mathcal{K}$, and let Ω' be a subdomain of Ω. Then if* q.e. $\sup \psi_1(x) < +\infty$ *and F is a subset of Ω' with positive distance to $\Omega\backslash\Omega'$, then u is essentially bounded in F, and* $\operatorname{ess\,sup}_F u(x)$ *is majorized by a number that depends only on* q.e. $\sup\psi_1(x)$, $\operatorname{dist}(F, \Omega\backslash\Omega')$, *n, p, a, Q, σ and* $\left(\int_\Omega |u(x)|^p\, dx\right)^{1/p}$.

LEMMA 2.3. *If u is a sub-Q-minimum for $\mathcal{F}$ in $\mathcal{K}$, $x_0 \in \Omega$ and $0 < \varrho \leq 1$,* *then*

$$M_\varrho(x_0) \leqslant k + c_1 \left(\frac{|A^+_{k,2\varrho}(x_0)|}{|B_{2\varrho}|}\right)^{1/p} (M_{2\varrho}(x_0)-k) + c_1(\lambda\varrho^\sigma + |k|\,\varrho)$$

if $k \geqslant \hat{k}_{2\varrho}(x_0)$.

We now seek an estimate for how large k must be chosen to guarantee that the ratio $|A^+_{k,2\varrho}|/|B_{2\varrho}|$ is small enough:

LEMMA 2.4. *Let u be a sub-Q-minimum for $\mathscr{F}$ in $\mathscr{K}$, $x_0 \in \bar{\Omega}$, $0 < \varrho \leqslant 1$. Let δ and θ be positive numbers, and suppose that for some number k_0 with $\hat{k}_{2\varrho}(x_0) \leqslant k_0 \leqslant M_{2\varrho}(x_0)$*

$$C_{1,p}\left[(B_\varrho(x_0)\setminus\Omega) \cup \{x \in \Omega_\varrho(x_0);\ u(x) \leqslant k_0\}\right] \geqslant \delta^p \varrho^{n-p}.$$

Then there is a number $\eta \in (0,1)$, depending only on n, p, a, Q, σ, δ and θ, such that if $k = M_{2\varrho}(x_0) - \eta(M_{2\varrho}(x_0)-k_0)$, then

$$\frac{|A^+_{k,\varrho}(x_0)|}{|B_\varrho(x_0)|} \leqslant \theta\left(1 + \frac{(|M_{2\varrho}|+|k_0|)\,\varrho + \lambda\varrho^\sigma}{M_{2\varrho}-k}\right).$$

Proof. Let μ be a probability measure supported in $B_\varrho(x_0)\setminus A^+_{k_0,\varrho}(x_0)$ with

$$\int_{\mathbf{R}^n} {}^\mu U_1(x)^{p'}\,dx \leqslant 2\delta^{-p'}\varrho^{-(n-p)/(p-1)}.$$

For $v = 0, 1, 2, \ldots, s$, where s is a positive integer to be determined later, we put

$$k_v = M_{2\varrho}(x_0) - 2^{-v}(M_{2\varrho}(x_0)-k_0),$$

so that $k_{v+1}-k_v = \frac{1}{2}(M_{2\varrho}(x_0)-k_v)$. Put $v_v(x) = (u(x)-k_v)_+$ in $\Omega_{2\varrho}(x_0)$ and $v_v(x) = 0$ in $B_{2\varrho}(x_0)\setminus\Omega$. We apply Lemma 1.3 to v_v and the set $A^+_{k_{v+1},\varrho}(x_0)$ (and we put, for short, $T_v = A^+_{k_v,\varrho}\setminus A^+_{k_{v+1},\varrho}$):

$$\begin{aligned}(k_{v+1}-k_v)\,|A^+_{k_{v+1},\varrho}| &\leqslant c\varrho \int_{T_v} |\nabla u(z)|\,dz + c\varrho^n \int_{T_v} |\nabla u(z)|\,{}^\mu U_1(z)\,dz \\ &\leqslant c\Big[\int_{A^+_{k_v,\varrho}} |\nabla u(z)|^p\,dz\Big]^{1/p}\Big(\varrho\,|T_v|^{1-1/p} \\ &\quad + \varrho^n\Big[\int_{T_v} {}^\mu U_1(z)^{p'}\,dz\Big]^{1-1/p}\Big).\end{aligned}$$

Now, from the proof of Lemma 2.1, we see that

$$\begin{aligned}\int_{A^+_{k_v,\varrho}} |\nabla u(z)|^p\,dz &\leqslant c\left[\varrho^{n-p}(M_{2\varrho}-k_v)^p + |k_v|^p\,|A^+_{k_v,2p}| + \lambda^p\varrho^{n-p+p\sigma}\right] \\ &\leqslant c\varrho^{n-p}\,(k_{v+1}-k_v)^p\left(1+\frac{(|k_0|+|M_{2\varrho}|)\,\varrho+\lambda\varrho^\sigma}{M_{2\varrho}-k_s}\right)^p.\end{aligned}$$

We put, for short, $R = (|k_0| + |M_{2\varrho}|)\varrho + \lambda\varrho^{\sigma}$, and get

$$|A^+_{k_{\nu+1},\varrho}| \leqslant c\varrho^{n/p}\left(1+\frac{R}{M_{2\varrho}-k_s}\right)\Big(|T_\nu|^{1-1/p} + \varrho^{n-1}\big[\int_{T_\nu} {}^{\mu}U_1(z)^{p'}\,dz\big]^{1-1/p}\Big),$$

i.e.

$$|A^+_{k_s,\varrho}|^{p/(p-1)} \leqslant |A^+_{k_{\nu+1},\varrho}|^{p/(p-1)}$$
$$\leqslant c\varrho^{n/(p-1)}\left(1+\frac{R}{M_{2\varrho}-k_s}\right)^{p/(p-1)}\Big[|T_\nu| + \varrho^{p(n-1)/(p-1)}\int_{T_\nu} {}^{\mu}U_1(z)^{p'}\,dz\Big].$$

Summing over ν from 0 to $s-1$ we obtain

$$s|A^+_{k_s,\varrho}|^{p/(p-1)} \leqslant c\varrho^{n/(p-1)}\left(1+\frac{R}{M_{2\varrho}-k_s}\right)^{p/(p-1)}\left(\varrho^n + \frac{\varrho^n}{\delta^{p'}}\right)$$
$$\leqslant c\delta^{-p'}\varrho^{np/(p-1)}\left(1+\frac{R}{M_{2\varrho}-k_s}\right)^{p/(p-1)},$$

i.e.

$$\frac{|A^+_{k_s,\varrho}(x_0)|}{|B_\varrho|} \leqslant \frac{C}{\delta\cdot s^{1-1/p}}\left(1+\frac{R}{M_{2\varrho}-k_s}\right).$$

Now we can choose s so large that $C/(\delta\cdot s^{1-1/p}) \leqslant \theta$, which proves the lemma with $\eta = 2^{-s}$.

Proof of Theorem 2.1. Fix a point $x_0 \in \bar{\Omega}$ and a positive number ε. From the assumptions of the theorem, there are positive numbers δ and ϱ_0 such that if $\varrho \leqslant \varrho_0$, then $\hat{k}_{4\varrho}(x_0) - \hat{l}_{4\varrho}(x_0) \leqslant \varepsilon$, and

$$C_{1,p}(E_i(x_0, 4\varrho, \varepsilon)) \geqslant \delta^p\varrho^{n-p} \quad \text{for } i = 1, 2.$$

Let $\varrho \leqslant \varrho_0$, and put

$$\mu = \tfrac{1}{2}(M_{4\varrho}(x_0) + m_{4\varrho}(x_0)),$$
$$k_0 = \min(\mu \vee \hat{k}_{4\varrho}(x_0);\ l_{4\varrho}(x_0) + \varepsilon),$$
$$l_0 = \max(\mu \wedge \hat{l}_{4\varrho}(x_0);\ k_{4\varrho}(x_0) - \varepsilon).$$

Obviously k_0 and l_0 are finite numbers, $m_{4\varrho} \leqslant l_0 \leqslant k_0 \leqslant M_{4\varrho}$, and $k_0 \geqslant \hat{k}_{4\varrho}(x_0)$, $l_0 \leqslant \hat{l}_{4\varrho}(x_0)$.

Since $l_0 \leqslant k_0$, at least one of the two sets $B_{2\varrho}(x_0)\backslash A^+_{k_0,2\varrho}(x_0)$ and $B_{2\varrho}(x_0)\backslash A^-_{l_0,2\varrho}(x_0)$ has Lebesgue measure not less than $\frac{1}{2}|B_{2\varrho}|$, i.e. its $(1, p)$-capacity is not less than $c\varrho^{n-p}$.

We may assume that $C_{1,p}(B_{2\varrho}(x_0)\backslash A^+_{k_0,2\varrho}(x_0)) \geqslant c(n, p)\varrho^{n-p}$, otherwise we consider $-u$, which is a Q-minimum to the functional $\int F(x, -u, -\nabla u)\,dx$, with $-\psi_2$ as inferior and $-\psi_1$ as superior obstacle.

Choose θ so small that $c_1\,\theta^{1/p} \leqslant \frac{1}{2}$, where c_1 is the constant from Lemma 2.3. According to Lemmas 2.3, 2.4 there is an $\eta \in (0, 1)$ such that

$$M_\varrho(x_0) \leqslant \left(1-\frac{\eta}{2}\right)M_{4\varrho}(x_0)+\frac{\eta}{2}k_0+c\left[(|M_{4\varrho}(x_0)|+|k_0|)\varrho+\lambda\varrho^\sigma\right].$$

Now, if $k_0 \leqslant \mu+\varepsilon$, we add the trivial inequality $-m_\varrho(x_0) \leqslant -m_{4\varrho}(x_0)$ and get

$$(*)\qquad \omega_\varrho(x_0) \leqslant \left(1-\frac{\eta}{4}\right)\omega_{4\varrho}(x_0)+c\left[(|M_{4\varrho}(x_0)|+|m_{4\varrho}(x_0)|)\varrho+\lambda\varrho^\sigma\right]+\frac{\eta}{2}\varepsilon.$$

Suppose, however, that $k_0 > \mu+\varepsilon$. Then $\mu \vee \hat{k}_{4\varrho}(x_0) > \mu+\varepsilon$ and thus

$$\hat{k}_{4\varrho}(x_0) > \mu+\varepsilon, \qquad \hat{l}_{4\varrho}(x_0) \geqslant \mu, \qquad l_0 = \hat{k}_{4\varrho}(x_0)-\varepsilon > \mu.$$

Then $u(x) \geqslant l_0$ on $E_1(x_0, 4\varrho, \varepsilon)$, and we can apply Lemma 2.4 to $-u$ and $-l_0$. Hence there is a positive number η, which now depends on δ, such that, with $l = m_{4\varrho}(x_0)+\eta\,(l_0-m_{4\varrho}(x_0))$,

$$\frac{|A^-_{l,2\varrho}(x_0)|}{|B_{2\varrho}|} \leqslant \theta\left(1+\frac{(|m_{4\varrho}(x_0)|+|l_0|)\varrho+\lambda\varrho^\sigma}{l-m_{4\varrho}}\right),$$

and we still get the estimate $(*)$.

Lemma 1.2 now yields

$$\lim_{\varrho\to 0}\omega_\varrho(x_0) \leqslant C\varepsilon,$$

with a constant C independent of ε, and since ε is arbitrary, the conclusion of the theorem follows.

With the same proof, only by changing ε to $p(4\varrho)$, we can derive an a priori estimate:

THEOREM 2.2. *Let u be a Q-minimum for $\mathscr{F}$ in $\mathscr{K}$, let $x_0 \in \Omega$ and let p be a nondecreasing function with* $\lim_{\varrho\to 0+} p(\varrho) = 0$. *Suppose that for $\varrho \leqslant \varrho_0$,*

$$\hat{k}_\varrho(x_0)-\hat{l}_\varrho(x_0) \leqslant p(\varrho), \qquad C_{1,p}\big(E_i(x_0, \varrho, p(\varrho))\big) > \delta\varrho^{n-p} \quad \textit{for } i = 1, 2.$$

Then u is continuous at x_0, and the modulus of continuity for u at x_0, $\omega_\varrho(x_0)$, can, for $\varrho \in (0, \varrho_0)$, be estimated in terms of $\hat{k}_{2\varrho}(x_0)$, $\hat{l}_{2\varrho}(x_0)$, n, p, a, σ, Q, δ, λ and $\int_{\Omega_{2\varrho}(x_0)}|u(x)|^p\,dx$.

Remark. After preparing this manuscript, the author was informed that Ziemer [16] obtained the same criterion for continuity of quasiminima as the ones in this paper and afterwards even a somewhat weaker one (personal communication).

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