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EXISTENCE AND NONEXISTENCE OF CLASSICAL SOLUTIONS OF THE DIRICHLET PROBLEM FOR A CLASS OF FULLY NONLINEAR NONUNIFORMLY ELLIPTIC EQUATIONS

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1. Introduction and results

This paper concerns the existence and nonexistence of classical solutions for the equation

$$(1) \qquad F[u] \equiv F(u, D^2 u) = g(x) \quad \text{in } \Omega, \qquad u = \varphi \quad \text{on } \partial\Omega$$

in a given bounded domain Ω in $\boldsymbol{R}^n$ and arbitrarily assigned smooth boundary data. Here the real function $F(z, r)$ is defined on $\boldsymbol{R} \times \boldsymbol{R}^{n\times n}$ ($\boldsymbol{R}^{n\times n}$ denotes the $n(n+1)/2$-dimensional space of real symmetric $n\times n$ matrices) and satisfies the ellipticity condition

$$(2) \qquad 0 < F_{ij}(z, r)\xi^i \xi^j \leqslant \Lambda |\xi|^2$$

for $\xi \in \boldsymbol{R}^n \setminus 0$, $(z, r) \in \boldsymbol{R} \times \boldsymbol{R}^{n\times n}$, Λ a positive constant.

Without loss of generality we assume that $F(0, 0) = 0$. The short notation

$$F_{ij} = \frac{\partial F}{\partial r_{ij}}, \quad F_{ij,pq} = \frac{\partial^2 F}{\partial r_{ij}\,\partial r_{pq}}, \quad F_{z,ij} = \frac{\partial^2 F}{\partial z\,\partial r_{ij}}, \quad u_{ij} = u_{x_i x_j},$$

etc. is used and summation convention is understood.

Our basic assumption is that $F(z, r)$ is twice differentiable and $F(z, r)$ is a concave function of $(z, r) \in \boldsymbol{R} \times \boldsymbol{R}^{n\times n}$, i.e.

$$(3) \qquad F_{zz}(z, r)\eta^2 + 2F_{z,ij}(z, r)\eta\xi^{ij} + F_{ij,pq}(z, r)\xi^{ij}\xi^{pq} \leqslant 0$$

for $(\eta, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^{n^2}$ and $(z, r) \in \boldsymbol{R} \times \boldsymbol{R}^{n\times n}$.

Under the fundamental assumption (3), Trudinger [12] (see also [4]) proves the existence and uniqueness of a classical solution $u \in C^{4,\alpha}(\Omega) \cap C^{0,1}(\bar{\Omega})$ for the uniformly elliptic equation (1). Our purpose is to extend this result of Trudinger to a class of nonuniformly elliptic equations (1).

Let us recall that in the quasilinear case Serrin [11] introduced an important class of "regularly elliptic equations" which, as far as the Dirichlet problem is concerned, behave similarly to uniformly elliptic equations. For equations which are not regularly elliptic it is necessary to impose significant restrictions on the curvatures of the boundaries of the underlying domains in order for the Dirichlet problem to be generally solvable.

Thus, following Serrin [11], we will introduce and investigate the class of "fully nonlinear regularly elliptic equations" (1). For equations which do not belong to the above class we will prove some existence and nonexistence theorems depending on the geometric properties of the domain Ω.

In [9] we introduced the scalar function $E(x, z, p)$ for the more general equation

$$F[u] \equiv F(x, u, D^2 u) = 0 \quad \text{in } \Omega \tag{4}$$

by $E(x, z, p) = F(x, z, p \oplus p) - F(x, z, 0)$ for $(x, z, p) \in \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n$, where the matrix $p \oplus p$ is $\{p_i p_j\}_{i,j=1}^n$.

We say that the equation (4) is *regularly elliptic* provided that

$$\Psi(|p|) E(x, z, p) \geqslant |p| S_p F_{ij}(x, z, p \oplus p) + |F(x, z, 0)| \tag{5}$$

for $x \in \Omega$, $|z| \leqslant M$ and $|p| \geqslant L$, where $\Psi\colon (0, \infty) \to (0, \infty)$ is a continuous strictly concave function satisfying the condition

$$\int^{\infty} \frac{dt}{t\Psi^2(t)} = \infty \tag{6}$$

and $S_p F_{ij} = \operatorname{trace} F_{ij}$, M, L are positive constants.

EXAMPLE 1.

$$f_1(u_{x_1 x_1}) + f_2(u_{x_2 x_2}) + \ldots + f_n(u_{x_n x_n}) = f_0(x, u). \tag{7}$$

The functions $f_i(t)$ are defined, twice differentiable and strictly concave for $t \in \boldsymbol{R}$. Moreover, $f_i(0) = 0$, $f_i(t) = C_i \sqrt{t/\ln t}$, $C_i = \text{const} > 0$ when t is sufficiently large and $f_i(t)$ have the asymptotes $a_i t + b_i$, $0 < a_i < \infty$, for $t \to -\infty$.

In this case

$$\begin{aligned} E(p) = f_1(p_1^2) + \ldots + f_n(p_n^2) &\geqslant C^1 (|p_1|/\ln^{1/2}|p_1| + \ldots + |p_n|/\ln^{1/2}|p_n|) \\ &\geqslant C^1 |p|/\ln^{1/2}|p| \end{aligned}$$

and $S_p F_{ij}(p \oplus p) = f_1'(p_1^2) + \ldots + f_n'(p_n^2) \leqslant C''$, $|f_0(x, u)| \leqslant C''$ for $x \in \Omega$, $|z| \leqslant M$ and sufficiently large p. Hence (5) holds with $\Psi = C \ln^{1/2} t$ and $\int^{\infty} dt/(C^2 t \ln t) = \infty$. Consequently, (7) is a "fully nonlinear regularly elliptic equation".

We will prove the existence of a classical solution for the equation (1) under the more restrictive condition

$$\Psi(|p|)E(z, p) \geqslant |p| \tag{8}$$

for $|z| \leqslant M$, $|p| \geqslant L$, instead of (5), where $E(z, p) = F(z, p \oplus p) - F(z, 0)$ and Ψ is the same function as in (5).

For instance, for the equation (7) the conditions (5) and (8) are equivalent. More generally, they are equivalent for the equations

$$F_1(u, D_{x'}^2 u) + F_2(u, D_{x''}^2 u) = g(x), \qquad x = (x', x'').$$

Let us formulate the following existence theorem.

THEOREM 1. *Let the function $F \in C^{2,\alpha}$ $(0 < \alpha < 1)$ satisfy (2), (3), (8). Let g, $\varphi \in C^2(\bar{\Omega})$ and let Ω be a bounded domain in $\mathbf{R}^n$ with a C^3 smooth boundary. Suppose F is nonincreasing in z for each $r \in \mathbf{R}^{n \times n}$. Then there exists a unique solution $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ of the b.v.p.* (1).

Let us consider a wider class of nonuniformly elliptic equations (1), including equations which are not regularly elliptic. We now suppose that

$$E(z, p) \geqslant C|p|^\theta \tag{9}$$

for $|z| \leqslant M$, $|p| \geqslant L$ and some positive constants C, θ.

EXAMPLE 2. The equation (7) with the same functions $f_i(t)$ as in Example 1, but now we suppose that $f_i(t) = C_i t^\beta$, $C_i > 0$, $0 < \beta < 1/2$ for $i = 1, 2, \ldots, n$ and sufficiently large t.

In this case the equation (7) is not regularly elliptic, while $E(z, p) \geqslant C|p|^{1-2\beta}$ for large t.

For the wider class of equations (1), (9) we have the following existence theorem under the additional condition that the domain Ω is convex.

THEOREM 2. *Let the function $F \in C^{2,\alpha}$ $(0 < \alpha < 1)$ satisfy (2), (3), (9). Let g, $\varphi \in C^2(\bar{\Omega})$ and let Ω be a convex bounded domain in $\mathbf{R}^n$ with a C^3 smooth boundary. Suppose F is nonincreasing in z for each $r \in \mathbf{R}^{n \times n}$. Then there exists a unique solution $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ of the b.v.p.* (1).

To show that the result in Theorem 1 is in many ways the best possible, we need a more sharp nonexistence result than in Theorem 2 in [9] for the general equation (4).

More precisely, we suppose that

$$\Psi(|p|)E(x, z, p) \leqslant |p| S_p F_{ij}(x, z, p \oplus p) \tag{10}$$

for $x \in \Omega$, $|z| \geqslant M$ and $|p| \geqslant L$, where M, L are positive constants, $E(x, z, p) = F(x, z, p \oplus p) - F(x, z, 0)$ and Ψ is the same function as in (5).

Suppose the following conditions on the asymptotic behaviour of the coefficients $F_{ij}(x, z, p \oplus p)$, $F(x, z, 0)$ hold for large values of $|p|$:

$$\begin{aligned} &F_{ij}(x, z, p \oplus p)/S_p F_{ij}(x, z, p \oplus p) = f^{ij}(x, \delta) + O(1) \\ (11) \quad &F(x, z, 0)/[|p| S_p F_{ij}(x, z, p \oplus p)] = f(x, \delta) + O(1) \end{aligned} \quad \text{as } |p| \to \infty,$$
$$\delta = p/|p|.$$

Here $f^{ij}(x, \delta)$, $f(x, \delta)$ are continuous functions of their arguments. Using the matrix

$$\mathscr{F}(x, \delta) = \{f^{ij}(x, \delta)\}_{i,j=1}^{n}$$

as in [9] we introduce a generalized mean curvature by

$$\mathscr{K}(y, \nu) = \sum_{i=1}^{n-1} \lambda_i \mathscr{F}(y, \nu) \lambda_i k_i + \nu \mathscr{F}(y, \nu) \nu H \tag{12}$$

where $y \in \partial\Omega$, ν denotes the unit outer normal to $\partial\Omega$ at y, $k_1, k_2, \ldots, k_{n-1}$ and $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ are respectively the principal curvatures and principal directions of $\partial\Omega$ at y and H is the ordinary mean curvature at y.

Then we have the following nonexistence result depending on the generalized mean curvature.

THEOREM 3. *Let Ω be a bounded domain in $\boldsymbol{R}^n$ whose boundary is of class C^3 and let F be a real smooth function satisfying the conditions* (2), (3), (10), (11).

Moreover, let F be nonincreasing in z for $(x, r) \in \Omega \times \boldsymbol{R}^{n \times n}$ and $F(x, z, 0) \leqslant 0$ for $z \geqslant M$, $|p| \geqslant L$. If the geometric condition

$$\mathscr{K}(y, \nu) \geqslant -f(y, \nu) \tag{13}$$

fails at a single point y of the boundary surface, then there exist smooth boundary data for which no solution of the Dirichlet problem is possible.

For more details and examples for Theorem 3 see [9].

The paper is divided into two parts. In the second part we treat the a priori estimates required by the existence theory. Theorems 1, 2 are proved by means of reduction of equation (1) to a uniformly elliptic equation, using the local boundedness and Hölder continuity of the second derivatives of the solutions of the regularized problem.

As for Theorem 3, its proof is very close to the proof of Theorem 2 in [9] and is omitted in this paper.

2. A priori estimates

To prove Theorems 1, 2 we will use the following comparison principle (see Th. 17.1 in [4], p. 443) for general fully nonlinear equations

$$F[u] = F(x, u, Du, D^2 u) = 0 \quad \text{in } \Omega.$$

THEOREM 4. *Let Ω be a bounded domain in $\boldsymbol{R}^n$ and let $u, v \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ satisfy $F[u] \geqslant F[v]$ in Ω, $u \leqslant v$ on $\partial\Omega$ where*

(i) *the function F is continuously differentiable with respect to the z, p, r variables;*

(ii) *the operator F is elliptic on all functions of the form $\theta u+(1-\theta)v$, $0 \leqslant \theta \leqslant 1$;*

(iii) *the function F is nonincreasing in z for each $(x, p, r) \in \Omega \times \boldsymbol{R}^n \times \boldsymbol{R}^{n\times n}$.*

Then $u \leqslant v$ in Ω.

The proof of Theorem 4 is given in [4], p. 444.

Let us now consider the regularized problem

$$F^{\varepsilon}[u] \equiv F(u, D^2 u)+\varepsilon\big(F_{ij}(0, 0)u_{ij}+F_z(0, 0)u\big) = g(x) \tag{14}$$

in Ω, $u = \varphi$ on $\partial\Omega$.

The equation (14) is uniformly elliptic and there exists a solution $u^{\varepsilon} \in C^{0,1}(\bar{\Omega}) \cap C^{4,\alpha}(\Omega)$ (see [12], p. 41). We will prove that the solutions u^{ε} are equicontinuous with their first and second derivatives on compact subsets of Ω.

Let us begin with an estimate of the maximum absolute value of the solutions u^{ε}.

LEMMA 1. *Let $F \in C^2$ satisfy* (2), (3), (9) *and suppose F is nonincreasing in z for $r \in \boldsymbol{R}^{n\times n}$. Then the estimate*

$$\sup_{\bar{\Omega}} |u^{\varepsilon}| \leqslant K_0$$

holds, where the constant K_0 does not depend on ε.

Proof. Let $\Omega \subset \{x \in \boldsymbol{R}^n;\ |x-x_0| < R\}$, $x_0 \notin \bar{\Omega}$ and a positive constant N satisfy the inequality

$$N > \max|\varphi| + (\max|g(x)|/C)^{2/\theta}.$$

From the comparison principle, Theorem 4, for the functions u^{ε} and $v = N(|x-x_0|^2 - R^2 - 1)$ it follows that $u^{\varepsilon} \geqslant v$ on $\partial\Omega$ and

$$F^{\varepsilon}[v] = F^{\varepsilon}(v, D^2 v] \geqslant F^{\varepsilon}(0, D^2 v) = F^{\varepsilon}(0, 2N\delta^{ij}) \geqslant C(2N)^{\theta/2}$$
$$\geqslant g(x) = F^{\varepsilon}[u^{\varepsilon}]$$

in Ω and hence

$$u^{\varepsilon} \geqslant v \geqslant -N(R^2+1) \quad \text{in } \Omega.$$

Let us now define the function ϕ as a solution of the b.v.p.

$$F_{ij}(0, 0)\phi_{ij}+F_z(0, 0)\phi = g(x)/(1+\varepsilon) \quad \text{in } \Omega, \tag{15}$$
$$\phi = \varphi \quad \text{on } \partial\Omega.$$

The linear b.v.p. (15) has a unique solution $\phi^\varepsilon \in C^{2,\alpha}(\bar{\Omega})$ and $|\phi^\varepsilon|_{C^2(\bar{\Omega})} \leqslant C_1$, where C_1 does not depend on ε. From the concavity of F we immediately obtain the inequality

$$\begin{aligned} F^\varepsilon[\phi^\varepsilon] &\leqslant F(0, 0) + F^\varepsilon_z(0, 0)\phi^\varepsilon + F^\varepsilon_{ij}(0, 0)\phi^\varepsilon_{ij} \\ &= g(x) = F^\varepsilon[u^\varepsilon]. \end{aligned}$$

Since $\phi^\varepsilon \geqslant u^\varepsilon$ on $\partial\Omega$, from the comparison principle, Theorem 4, it follows that $\phi^\varepsilon \geqslant u^\varepsilon$ in Ω. Lemma 1 is proved.

For convenience, further we will omit the index ε.

LEMMA 2. *Let the hypotheses of Theorem 2 hold. Then the estimate*

$$\sup_\Omega |Du^\varepsilon| \leqslant K_1$$

holds, where K_1 does not depend on ε.

Proof. We first observe that the function ϕ defined in Lemma 1 is a global upper barrier for the operator F and arbitrary smooth domain Ω. To construct a global lower barrier for the operator F we need the following simple lemma.

LEMMA 3. *Let a continuous strictly concave function Ψ: $(0, \infty) \to (0, \infty)$ satisfy* (6). *Then for any positive constants β, q, δ_0, γ_0 there exists a constant $\delta(\beta, q, \delta_0, \gamma_0)$, $0 < \delta < \delta_0$, and a nonnegative function $h \in C^2[0, \delta]$ satisfying the conditions*

$$h' \geqslant \gamma_0, \quad h'' < 0, \quad h(0) = 0, \quad h(\delta) \geqslant \beta$$

and $\sqrt{-h''}/\Psi(\sqrt{-h''}) = qh'$.

Proof. From the concavity of Ψ it follows that the function $\omega(t) = \sqrt{t}/\Psi(\sqrt{t})$ is monotonically increasing and thus there exists a well-defined inverse function ω^{-1}. Let a constant $\gamma \geqslant \gamma_0$ satisfy the inequality

$$\int_\gamma^\infty \frac{dt}{\omega^{-1}(qt)} < \delta_0. \tag{16}$$

Since

$$\int_1^\infty \frac{dt}{\omega^{-1}(qt)} \leqslant \frac{2}{q}\int_1^\infty \frac{dt}{t^2\,\Psi(t)} < \infty$$

(16) follows when γ is a sufficiently large positive constant. We consider the positive monotonically increasing function

$$G(s) = \int_\gamma^s \frac{dt}{\omega^{-1}(qt)} \qquad \text{for } s \geqslant \gamma$$

and the inverse function

$$G^{-1}\colon \left(0, \int_\gamma^\infty \frac{dt}{\omega^{-1}(qt)}\right) \to (\gamma, \infty).$$

From (6) we can choose $\delta < \int_\gamma^\infty dt/(\omega^{-1}(qt))$, δ sufficiently close to $\int_\gamma^\infty dt/(\omega^{-1}(qt))$ for the estimate

$$\int_{\sqrt{\omega^{-1}(q\gamma)}}^{\sqrt{\omega^{-1}(qG^{-1}(\delta))}} \frac{dt}{t\Psi^2(t)} \geqslant \beta q^2 + 1/[2\Psi^2(\sqrt{\omega^{-1}(q\gamma)})]$$

to hold. Let us consider the function h given by $h = \int_0^t G^{-1}(\delta - s)\,ds$. It is evident that $h \geqslant 0$, $h(0) = 0$, $h' = G^{-1}(\delta - t) \geqslant \gamma > \gamma_0$,

$$h'' = -1/G'(G^{-1}(\delta - t)) = -\omega^{-1}(qh') < 0,$$

i.e. $h \in C^2[0, \delta]$ and $qh' = \omega(-h'') = \sqrt{-h''}/\Psi(\sqrt{-h''})$.

Moreover,

$$h(\delta) = \int_0^\delta G^{-1}(\delta - s)\,ds = -\int_{G^{-1}(\delta)}^{\gamma} \frac{t\,dt}{\omega^{-1}(qt)}$$

$$= 1/[2q^2\Psi^2(\sqrt{s})]\Big|_{\omega^{-1}(q\gamma)}^{\omega^{-1}(qG^{-1}(\delta))} + \frac{1}{q^2}\int_{\sqrt{\omega^{-1}(q\gamma)}}^{\sqrt{\omega^{-1}(qG^{-1}(\delta))}} \frac{dt}{t\Psi^2(t)} \geqslant \beta.$$

Lemma 3 is proved.

Let us go back to Lemma 2 and define the function $d(x)$ as the distance from x to the surface $\partial\Omega$. In the domain $\Omega_{d_0} = \{x \in \Omega;\ d(x) < d_0\}$ the function $d(x)$ is of class C^2 when d_0 is sufficiently small (see [11], p. 421, or [4], p. 381). Consider now the barrier function $\phi(x) - \frac{1}{4}h(d)$ in the domain Ω_δ, where h is defined in Lemma 3 with $\gamma_0 = \max(1, L)$, $\beta = 4\max|\phi| + 4N(R^2+1)$, $q = n^2\Lambda\max|K_i| + 4\max|g| + 2\max|F(\phi, 2D^2\phi)| + 1$ and $\delta_0 = d_0$.

Then $\phi - \frac{1}{4}h \leqslant u$ on $\partial\Omega_\delta$ and from the concavity of F we obtain the inequalities

$$\begin{aligned}
F[\phi - \tfrac{1}{4}h] &= F(\phi - \tfrac{1}{4}h,\ \phi_{ij} - \tfrac{1}{4}h'' d_i d_j - \tfrac{1}{4}h' d_{ij}) \\
&\geqslant \tfrac{1}{2}F(\phi, 2D^2\phi) + \tfrac{1}{4}F(\phi, -h'' d_i d_j) + \tfrac{1}{4}F(\phi, -h' d_{ij}) \\
&\geqslant \tfrac{1}{2}F(\phi, 2D^2\phi) + \frac{\sqrt{-h''}}{4\Psi(\sqrt{-h''})} - \tfrac{1}{4}F_{ij}(\phi, -h' d_{ij})\,h' d_{ij} \\
&= \tfrac{1}{2}F(\phi, D^2\phi) + \frac{\sqrt{-h''}}{4\Psi(\sqrt{-h''})} + \tfrac{1}{4}\sum_{i=1}^{n-1} \frac{\lambda_i\{F_{ke}\}\lambda_i k_i h'}{1 - k_i d} \\
&\geqslant g(x) = F[u].
\end{aligned}$$

Here $\{F_{Ke}\}$ is the matrix $\{F_{Ke}(\phi, -h' d_{ij})\}$, $k_1, \ldots, k_{n-1}$ and $\lambda_1, \ldots, \lambda_{n-1}$ are respectively the principal curvatures and the principal directions of $\partial\Omega$ at the point $y(x)$ on $\partial\Omega$ nearest to x. Also we use an important identity to calculate $F_{ij}d_{ij}$ (see [11], p. 422, Lemma 1).

From the comparison principle, Theorem 4, it follows that $\phi(x)-\frac{1}{4}h(d)$ is a global lower barrier in Ω_δ.

Let us now consider a barrier construction when Ω is a convex domain. We choose $h(t)=\gamma(2\delta t-t^2)\geqslant 0$ for $0<t<\delta$ where $\delta<d_0$ and

$$\gamma>[4\max|\phi|+N(R^2+1)]/\delta^2+[4\max|g|+2\max|F(\phi,\,2D^2\phi)|]^{2/\theta}.$$

Then $\phi-\frac{1}{4}h\leqslant u$ on $\partial\Omega_\delta$ and our previous calculations yield the following estimate:

$$F[\phi-\tfrac{1}{4}h]\geqslant\tfrac{1}{2}F(\phi,\,2D^2\phi)+\tfrac{1}{4}F(\phi,\,2\gamma\delta^{ij})+\tfrac{1}{4}\sum_{i=1}^{n-1}\frac{\lambda_i\{F_{Ke}\}\lambda_i k_i h'}{1-k_i d}$$

$$\geqslant\tfrac{1}{4}(2\gamma)^{\theta/2}-\tfrac{1}{2}\max|F(\phi,\,2D^2\phi)|\geqslant g(x)=F[u].$$

From the comparison principle it follows that $\phi-\frac{1}{4}h$ is a global lower barrier for u in the convex domain Ω.

By means of standard barrier technique we obtain the gradient boundary estimate

$$\sup_{\partial\Omega}|Du|\leqslant K_1'$$

where K_1' does not depend on ε.

To prove Lemma 2 let us apply the comparison principle for the functions $u\pm u_k$, $k=1,2,\ldots,n$ and $v_1=N_1(|x-x_0|^2-R^2-1)$.

From the concavity of F we obtain the inequality

$$F[u\pm u_k]\leqslant F(u,\,D^2u)\pm F_z(u,\,D^2u)u_k\pm F_{ij}(u,\,D^2u)(u_k)_{ij}=g+g_k$$

$$=C(2N_1)^{\theta/2}\leqslant F(0,\,2N_1\delta^{ij})=F(0,\,D^2v_1)\leqslant F[v_1]\quad\text{in }\Omega$$

and $u\pm u_k\geqslant v_1$ on $\partial\Omega$, when N_1 is a sufficiently large constant. Consequently $u\pm u_k\geqslant v_1\geqslant -N_1(R^2+1)$ in Ω for $k=1,2,\ldots,n$ and Lemma 2 is proved.

LEMMA 4. *Let the hypotheses of Theorem* 1 *or Theorem* 2 *hold. Then*

$$\sup_{\Omega}|d^{4/\theta}D^2u|\leqslant K_2$$

where $d(x)$ is the distance of x to $\partial\Omega$ and K_2 does not depend on ε.

Proof. To estimate D^2u at a point $y\in\Omega$ which we may take to be the origin, whose distance to $\partial\Omega$ is $2r$, we make use of the function $\zeta=r^2-|x|^2$ in $|x|<r$, $\zeta\equiv 0$ for $|x|\geqslant r$.

Let us consider in the domain $\Omega^+=\{x\in\Omega;\ u_{\tau\tau}(x)>0\}$ the function

$$w=\zeta^{2k}u_{\tau\tau}^2/m^{2-1/k}+N_2u_\tau^2-u$$

where τ is a unit vector in $\boldsymbol{R}^n$ and

$$m=\sup_{\Omega^+}\zeta^k u_{\tau\tau}=\zeta^k(p_0)u_{\tau\tau}(p_0)>0$$

for $k>2/\theta$ an integer.

From the concavity of F we obtain the estimates

$$\begin{aligned}
&\tfrac{1}{2}F(-2w,\ -2D^2w)+\tfrac{1}{2}F(-2u-2w,\ 4N_2\,Du_\tau\oplus Du_\tau)\\
&\quad\leqslant F(-u-2w,\ -D^2w+2N_2\,Du_\tau\oplus Du_\tau)\leqslant F_z(u,\ D^2u)(-2w-2u)\\
&\qquad+F(u,\ D^2u)-m^{1/k-2}F_{ij}(u,\ D^2u)\,[8k(2k-1)(r^2-|x|^2)^{2k-2}x_i\,x_j\,u_{\tau\tau}^2\\
&\qquad-4k(r^2-|x|^2)^{2k-1}u_{\tau\tau}^2\,\delta^{ij}-8kx_i\,u_{\tau\tau}\,u_{\tau\tau j}(r^2-|x|^2)^{2k-1}\\
&\qquad+2(r^2-|x|^2)^{2k}u_{\tau\tau i}\,u_{\tau\tau j}+2(r^2-|x|^2)^{2k}u_{\tau\tau}\,u_{\tau\tau ij}]\\
&\qquad-2N_2\,F_{ij}(u,\ D^2u)\,u_\tau\,u_{\tau ij}\leqslant C_3+C_4\,u_{\tau\tau}^{2/k}.
\end{aligned}$$

Consequently

$$\begin{aligned}
F[-2w]&\leqslant 2C_3+2C_4\,u_{\tau\tau}^{2/k}-F(-2u-2w,\ 4N_2\,Du_\tau\oplus Du_\tau)\\
&\leqslant 2C_3+2C_4\,u_{\tau\tau}^{2/k}-F(0,\ 4N_2\,Du_\tau\oplus Du_\tau)\\
&\leqslant 2C_3+2C_4\,u_{\tau\tau}^{2/k}-(2N_2^{1/2}\,|Du_\tau|)^\theta\leqslant F[N_3(|x|^2-R^2-1)]
\end{aligned}$$

when N_2, N_3 are sufficiently large.

Since $-2w\geqslant N_3(|x|^2-R^2-1)$ on $\partial\Omega^+$, from the comparison principle it follows that

$$w\leqslant N_3(R^2+1)\quad \text{in } \Omega^+.$$

Hence at p_0 we obtain the inequality $m\leqslant C_5$ and $r^{2k}u_{\tau\tau}(0)\leqslant C_5$, i.e.

$$d^{4/\theta}u_{\tau\tau}\leqslant C_5\quad \text{in } \Omega. \tag{17}$$

As in Evans [3] we will show the boundedness of $u_{\tau\tau}$ from below. Clearly from the concavity of F it follows that

$$g(x)\leqslant(1+\varepsilon)\big(F_{ij}(0,\ 0)\,u_{ij}+F_z(0,\ 0)\,u\big).$$

Let $\mathscr{P}$ be a suitable orthogonal constant matrix which defines a nonsingular linear transformation $y=x\mathscr{P}$ so that under this transformation

$$F_{ij}(0,\ 0)\,u_{x_ix_j}=u_{y_iy_i}=g(x)-F_z(0,\ 0)\,u\geqslant -C_6\quad \text{in } \Omega.$$

From (17) it follows that $d^{4/\theta}u_{y_ky_k}\geqslant -C_7$ in Ω, i.e. $|d^{4/\theta}u_{y_ky_k}|\leqslant C_8$ for $k=1,2,\ldots,n$. Lemma 4 is proved.

LEMMA 5. *Let the hypotheses of Theorem* 1 *or Theorem* 2 *hold. Then*

$$|u|_{2,\beta,\Omega'}\leqslant K(\Omega')$$

where $\Omega'\Subset\Omega$, $\beta=\beta(\Omega')$ *and* $K(\Omega')$ *does not depend on* ε.

Proof. From Lemma 4 it follows that the equation (1) is uniformly elliptic in $\overline{\Omega'}$ with an ellipticity constant independent of ε. The proof of Lemma 5 follows immediately from Theorem 8.1 in [12].

Proof of Theorems 1 *and* 2. The unique solvability of the Dirichlet problem (1) follows from the comparison principle.

The existence of solutions of the Dirichlet problem (1) follows from Lemmas 1–5. Indeed, the solutions u^ε of (14) are equicontinuous with their first and second derivatives on compact subsets of Ω. Hence, by the usual diagonalization process, there is a subsequence $\{u^{\varepsilon_m}\}$ converging in Ω to a solution u of (1) as $\varepsilon \to 0$.

The uniform gradient bound, Lemma 2, guarantees that the convergence is uniform in $\bar{\Omega}$ and hence $u = \varphi$ on $\partial\Omega$. This completes the proof of Theorems 1 and 2.

References

[1] S. Bernstein, *Conditions nécessaires et suffisantes pour la possibilité du problème de Dirichlet*, C. R. Acad. Sci. Paris 150 (1910), 514–515.

[2] –, *Sur les équations du calcul des variations*, Ann. Sci. École Norm. Sup. 29 (1912), 431–485.

[3] L. C. Evans, *Classical solutions of fully nonlinear, convex, second order elliptic equations*, Comm. Pure Appl. Math. 25 (1982), 333–363.

[4] P. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer, Berlin–Heidelberg–New York–Tokyo 1983.

[5] A. V. Ivanov, *Quasilinear degenerate and nonuniformly elliptic and parabolic equations of second order*, Trudy Mat. Inst. Steklov. 160 (1982) (in Russian).

[6] N. V. Krylov, *On degenerate nonlinear elliptic equations, I, II*, Mat. Sb. 120 (3) (1983), 311–330; 121 (2), (1983), 211–232 (in Russian).

[7] N. V. Krylov and M. V. Safonov, *Certain properties of solutions of parabolic equations with measurable coefficients*, Izv. Akad. Nauk SSSR 40 (1980), 161–175 (in Russian), English transl. Math. USSR-Izv. 16 (1981), 151–164.

[8] N. Kutev, *The problem of Dirichlet for a class of fully nonlinear nonuniformly elliptic equations*, in: Proc. Thirteenth Spring Conf. of the Union of Bulg. Math., 1984, 145–151.

[9] –, *Nonexistence of classical solutions of the Dirichlet problem for fully nonlinear elliptic equations*, to appear.

[10] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York 1968.

[11] T. Serrin, *The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables*, Philos. Trans. Roy. Soc. London Ser. A. 264 (1969), 413–496.

[12] N. S. Trudinger, *Fully nonlinear, uniformly elliptic equations under natural structure conditions*, Trans. Amer. Math. Soc. 287 (2) (1983), 751–769.