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REPRESENTATIONS OF GROUPS ON EIGENSPACES OF INVARIANT DIFFERENTIAL OPERATORS

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I. Introduction

Historically, Lie groups originated from Sophus Lie's investigations of differential equations around 1875. A problem that he studied and gave fundamental contributions to was:

Given a system of differential equations, to exploit knowledge about its invariance group towards its integration. See p. 176–177 of [Li2].

The present paper deals with a special aspect of Lie's problem about the interplay between differential equations and their invariance groups: We shall study how the invariance group of a differential operator D acts on the eigenspaces $\{u \mid Du = \lambda u\}$ for $\lambda \in \boldsymbol{C}$.

To facilitate the formulation of our problem we introduce the following (standard) terminology:

DEFINITION. Let $\mathscr{S}$ be a set of endomorphisms of a topological vector space E.

(α) $\mathscr{S}$ is *topologically irreducible* if the only closed $\mathscr{S}$-invariant subspaces of E are $\{0\}$ and E.

(β) A continuous linear operator $A\colon E \to E$ is an *intertwining operator for* $\mathscr{S}$ if $SA = AS$ for all $S \in \mathscr{S}$.

(γ) $\mathscr{S}$ is *scalarly irreducible* if the only intertwining operators for $\mathscr{S}$ are scalar multiples of the identity operator on E.

The topological vector spaces that we shall encounter below will be spaces of C^∞-functions. We shall always equip them with their standard topology: uniform convergence of the functions and each of their derivatives on compact sets.

PROBLEM. Let D be a linear partial differential operator on $\boldsymbol{R}^n$, and let π be a representation of a Lie group G on $C^\infty(\boldsymbol{R}^n)$ such that $\pi(g)D = D\pi(g)$ for all $g \in G$.

Since each eigenspace $\mathscr{E}_\lambda := \{f \in C^\infty(\mathbf{R}^n) \mid Df = \lambda f\}$, $\lambda \in \mathbf{C}$, is invariant under π, we get by restriction a representation π_λ of G on $\mathscr{E}_\lambda$. Is $\pi_\lambda(G)$ scalarly irreducible?

As we shall see, the answer to the question of the problem is yes for some of the classical partial differential operators of mathematical physics and the natural representations of their invariance groups. Those include the Laplace operator on $\mathbf{R}^n$, i.e.

$$\Delta = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_n^2,$$

the generalized wave operator on $\mathbf{R}^n$, i.e.

$$\Box = \Box_p = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_p^2 - \partial^2/\partial x_{p+1}^2 - \ldots - \partial^2/\partial x_n^2,$$

the heat equation operator on $\mathbf{R}^{1+n}$, i.e. $\partial/\partial t - \Delta$, and the Schrödinger operator on $\mathbf{R}^{1+n}$, i.e. $i^{-1}\,\partial/\partial t - \Delta$.

Our main result, Theorem 1, gives a general condition for a representation on an eigenspace of a constant coefficient partial differential operator to be scalarly irreducible. As corollaries we get new irreducibility results for the operators $\Box$, $\partial/\partial t - \Delta$ and $i^{-1}\,\partial/\partial t - \Delta$. It might be mentioned that topological irreducibility implies scalar irreducibility in this setting (Theorem 4). Information about other aspects of eigenspace representations can be found in the survey articles [He3] and [St1].

I would finally like to thank the University of Warszawa for its kind invitation and support of my visit.

II. Notation etc.

$$\langle \zeta, x\rangle = \zeta_1 x_1 + \ldots + \zeta_n x_n \quad \text{for } \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbf{C}^n \text{ and } x = (x_1, \ldots, x_n) \in \mathbf{R}^n.$$

$e_\zeta \in C^\infty(\mathbf{R}^n)$ is defined by

$$e_\zeta(x) := \exp\langle \zeta, x\rangle \quad \text{for } \zeta \in \mathbf{C}^n \text{ and } x \in \mathbf{R}^n.$$

$V^{\mathbf{C}}$ = the complexification of the vector space V.

$M_n(\mathbf{C})$, resp. $M_n(\mathbf{R})$ = all $n \times n$ matrices with complex, resp. real, entries.

X^t = the transpose of X.

$\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_n)^t$.

$\mathbf{C}[\zeta_1, \ldots, \zeta_n]$ = the ring of all polynomial functions from $\mathbf{C}^n$ to $\mathbf{C}$.

Terminology. If Φ is a subgroup of the group of diffeomorphisms of $\mathbf{R}^n$, we define the *natural representation* R of Φ on $C^\infty(\mathbf{R}^n)$ by

$$R(\varphi)f := f \circ \varphi^{-1} \quad \text{for } \varphi \in \Phi \text{ and } f \in C^\infty(\mathbf{R}^n).$$

III. A sufficient condition for scalar irreducibility

In this section we state a general sufficient condition for scalar irreducibility of a set of operators on an eigenspace for a constant coefficient partial differential operator (Theorem 1). The set of operators that we have in mind will be the operators in an infinitesimal representation of a Lie group. Our applications of Theorem 1 to specific partial differential operators and invariance groups are of this nature (see Section V).

THEOREM 1. *Let Ω be an open, convex, nonempty subset of $\boldsymbol{R}^n$. Let π be a map of a finite-dimensional complex vector space V into the continuous linear operators on $C^\infty(\Omega)$ of the form*

$$[\pi(X)f](x) = \langle \mu(X)f(x)+\sigma(X)\nabla f(x), x\rangle \quad \text{for } f\in C^\infty(\Omega),\ X\in V,\ x\in\Omega,$$

where μ: $V \to \boldsymbol{C}^n$ and σ: $V \to M_n(\boldsymbol{C})$ are complex linear.

Let $p\in \boldsymbol{C}[\zeta_1, \ldots, \zeta_n]$ be an irreducible polynomial of degree > 0, and let $\mathscr{E}_0$ be the eigenspace

$$\mathscr{E}_0 := \{f\in C^\infty(\Omega) \mid p(\nabla)f = 0\}.$$

If π, p and V are related as follows:

(i) *$\mathscr{E}_0$ is invariant under $\pi(V)$,*

(ii) $\dim\{X\in V \mid \mu(X)+\sigma(X)\zeta = 0\} = \dim V-(n-1)$ *for all $\zeta\in p^{-1}(0)$ with $\nabla p(\zeta)\neq 0$,*

then the restriction to $\mathscr{E}_0$ of the set of operators $\{\partial/\partial x_1, \ldots, \partial/\partial x_n, \pi(V)\}$ is scalarly irreducible.

Remark 2. Any irreducible polynomial $p\in \boldsymbol{C}[\zeta_1, \ldots, \zeta_n]$ has the following 3 properties:

(α) p has no multiple factors.

(β) The set $\{\zeta\in p^{-1}(0) \mid \nabla p(\zeta)\neq 0\}$ is dense in $p^{-1}(0)$ (see Remark on p. 12 of [Mi]).

(γ) $p^{-1}(0)$ is connected (see Remark on p. 105 of [Mi]).

It transpires from the proof of Theorem 1 that we do not need the irreducibility of p, but only the 3 properties (α), (β) and (γ).

Remark 3. A result similar to Theorem 1 can be found as Theorem 6 of [St2]. However, Theorem 1 is stronger on the following 3 counts: It can be applied to Lie algebra representations, not just group representations. It can deal with multiplier representations. The underlying domain need not be all of $\boldsymbol{R}^n$.

Proof of Theorem 1. The main ideas and the procedure are the same as in the proof of Theorem 6 of [St2].

The reference to Björk's book should be replaced by a reference to the remark on p. 39 of [Hö]. ∎

The next theorem establishes a connection between topological and scalar irreducibility in the setting of Theorem 1:

THEOREM 4. *Let Ω be an open, nonempty, connected subset of $\mathbf{R}^n$. Let $p \in \mathbf{C}[\zeta_1, \ldots, \zeta_n]$ be a polynomial of degree $\geqslant 1$, let $\mathscr{E}_0$ be the eigenspace $\mathscr{E}_0 := \{f \in C^\infty(\Omega) \mid p(\nabla) f = 0\}$, and let $\mathscr{S}$ be a set of linear operators on $\mathscr{E}_0$.*

If the set of operators $\{\partial/\partial x_1, \ldots, \partial/\partial x_n, \mathscr{S}\}$ acts topologically irreducibly on $\mathscr{E}_0$, then it also acts scalarly irreducibly.

Proof. Let $A: \mathscr{E}_0 \to \mathscr{E}_0$ be an intertwining operator for $\mathscr{S}$.

Take any $\zeta \in p^{-1}(0)$; then $e_\zeta \in \mathscr{E}_0$. Since A commutes with $\partial/\partial x_j$ for $j = 1, \ldots, n$ we get

$$\partial/\partial x_j (e_{-\zeta} A e_\zeta) = 0 \quad \text{for } j = 1, 2, \ldots, n.$$

It follows from the connectedness of Ω that there exists a constant $c \in \mathbf{C}$ such that $Ae_\zeta = ce_\zeta$.

We see that the eigenspace $\{f \in \mathscr{E}_0 \mid Af = cf\}$ is an invariant, nonzero, closed subspace of $\mathscr{E}_0$, so by the topological irreducibility

$$\{f \in \mathscr{E}_0 \mid Af = cf\} = \mathscr{E}_0, \quad \text{i.e.} \quad A = cI. \ \blacksquare$$

IV. The general set-up of our examples

The specific examples that we shall encounter in the next section are special cases of the following set-up:

Let R be a representation of a Lie group H on $C^\infty(\mathbf{R}^n)$ of the particular form

$$[R(h)f](x) = e^{\langle \varphi(h), x \rangle} f(\varrho(h)^{-1} x) \quad \text{for } h \in H, f \in C^\infty(\mathbf{R}^n), x \in \mathbf{R}^n,$$

where $\varphi \in C^\infty(H, \mathbf{C}^n)$ and ϱ is a continuous (and hence differentiable) representation of H on $\mathbf{R}^n$.

The derived representation $=$ the infinitesimal representation dR of $\mathfrak{h}$ ($=$ the Lie algebra of H) on $C^\infty(\mathbf{R}^n)$ can be computed to be

$$[dR(X)f](x) = \langle \mu(X), x \rangle f(x) + \langle \sigma(X) \nabla f(x), x \rangle$$
$$\text{for } X \in \mathfrak{h}, f \in C^\infty(\mathbf{R}^n) \text{ and } x \in \mathbf{R}^n,$$

where μ and σ are given by the expressions

$$\mu(X) = \frac{d}{ds}\bigg|_{s=0} \varphi(\exp sX) \in \mathbf{C}^n, \quad \text{and}$$

$$\sigma(X) = -d\varrho(X)^t = -\frac{d}{ds}\bigg|_{s=0} (\varrho(\exp sX))^t \in M_n(\mathbf{C}).$$

Finally, we are given a polynomial $p \in \mathbf{C}[\zeta_1, \ldots, \zeta_n]$ of degree $\geqslant 1$ with the property that

$$p(\nabla)R(h) = R(h)p(\nabla) \quad \text{for all } h \in H.$$

Theorem 1 above has the following consequence:

COROLLARY 5. *Let the set-up and the assumptions be as above in this section, and let Ω be an open, convex, nonempty subset of $\boldsymbol{R}^n$. If*

(i) *p is irreducible or at least satisfies the 3 conditions of Remark* 1, *and*

(ii) $\dim_{\boldsymbol{C}}\{X \in \mathfrak{h}^{\boldsymbol{C}} \mid \mu(X)+\sigma(X)\zeta = 0\} = \dim \mathfrak{h}-(n-1)$
for all $\zeta \in p^{-1}(0)$ *with* $\nabla p(\zeta) \neq 0$,

then the set $\{\partial/\partial x_1, \ldots, \partial/\partial x_n, dR(\mathfrak{h})\}$ acts scalarly irreducibly on the eigenspace $\{f \in C^\infty(\Omega) \mid p(\nabla)f = 0\}$; in particular, $\{\partial/\partial x_1, \ldots, \partial/\partial x_n, R(H)\}$ acts scalarly irreducibly on $\{f \in C^\infty(\boldsymbol{R}^n) \mid p(\nabla)f = 0\}$.

If R is the natural representation of $H \subseteq \mathrm{GL}(n, \boldsymbol{R})$ on $C^\infty(\boldsymbol{R}^n)$ we find that

$$\mu = 0 \quad \text{and} \quad \sigma(X) = -X^t \text{ for all } X \in \mathfrak{h}^{\boldsymbol{C}}.$$

V. Applications

A. The Laplace operator

The first study of irreducibility of eigenspaces for the Laplace operator is due to S. Helgason who in [He1] proved that the eigenspace $\mathscr{E}_\lambda = \{f \in C^\infty(\boldsymbol{R}^n) \mid \Delta f = \lambda f\}$ is topologically irreducible under the natural action of the group of rigid motions if and only if $\lambda \neq 0$. He examined the special case $\lambda = 0$ further in [He2] where he showed that $\mathscr{E}_0$ is scalarly irreducible under the action of (the Lie algebra of) the conformal group (which is bigger than the group of rigid motions).

We will here point out that $\mathscr{E}_\lambda$ for each $\lambda \in \boldsymbol{C}$ is scalarly irreducible under the natural action of the group of rigid motions:

THEOREM 6. *The eigenspace $\{f \in C^\infty(\boldsymbol{R}^n) \mid \Delta f = \lambda f\}$ is scalarly irreducible under the natural action of $\boldsymbol{R}^n \times_s \mathrm{SO}(n)$ for each integer $n \geq 2$ and each $\lambda \in \boldsymbol{C}$.*

Proof. The proof is a particular case of the proof of the next theorem. ∎

Remark. The result above about Δ can be generalized to the setting of the action of the Cartan motion group of a symmetric space of the noncompact type. See Theorem 6.6 of [He5] and Theorem 8 of [St2] for details. Δ would be associated to the symmetric space $\mathrm{SO}(n, 1)/\mathrm{SO}(n)$.

B. Generalized wave operators

Let us, for integers $n \geq 2$ and $p \in]0, n[$, consider the generalized wave operator

$$\Box = \Box_{n,p} = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_p^2 - \partial^2/\partial x_{p+1}^2 - \ldots - \partial^2/\partial x_n^2.$$

It commutes on $C^\infty(\boldsymbol{R}^n)$ with the natural action of the subgroup $G = \boldsymbol{R}^n \times_s \mathrm{SO}_0(p, n-p)$ of the group of affine motions of $\boldsymbol{R}^n$ (the Poincaré group when $n = 4$ and $p = 1$).

The question about topological irreducibility of the eigenspaces of $\Box$ under G has apparently never been settled, but for scalar irreducibility we have

THEOREM 7. *Let $n \geq 2$, let Ω be an open, convex, nonempty subset of $\boldsymbol{R}^n$ and let $\lambda \in \boldsymbol{C}$.*

The eigenspace $\{f \in C^\infty(\Omega) \mid \Box f = \lambda f\}$ is scalarly irreducible under the infinitesimal action of $\boldsymbol{R}^n \times_s \mathrm{SO}_0(p, n-p)$.

Proof. We shall apply Corollary 5 with

$$p(\zeta_1, \ldots, \zeta_n) = \zeta_1^2 + \ldots + \zeta_p^2 - \zeta_{p+1}^2 - \ldots - \zeta_n^2 - \lambda,$$
$$\mathfrak{h} = \mathfrak{so}(p, n-p),$$
$$\mu = 0, \text{ and } \sigma(X) = -X^t \text{ for } X \in \mathfrak{so}(p, n-p)^{\boldsymbol{C}}.$$

We leave the verification (easy) of condition (i) of Corollary 5 to the reader, and concentrate on (ii) that reduces to

$$\dim\{X \in \mathfrak{h}^{\boldsymbol{C}} \mid X^t \zeta = 0\} = \tfrac{1}{2}(n-1)(n-2).$$

Let $q = n-p$. Denoting $k \times l$ matrices by X_{kl} it is known (see e.g. [He4, p. 446]) that

$$\mathfrak{h}^{\boldsymbol{C}} = \left\{ \begin{pmatrix} X_{pp} & X_{pq} \\ X_{pq}^t & X_{qq} \end{pmatrix} \in M_n(\boldsymbol{C}) \,\middle|\, X_{pp}^t = -X_{pp},\ X_{qq} = -X_{qq}^t \right\}.$$

Introducing the vector space isomorphism $\Phi = \mathfrak{h}^{\boldsymbol{C}} \to \mathfrak{o}(n)^{\boldsymbol{C}}$ given by

$$\Phi\begin{pmatrix} X_{pp} & X_{pq} \\ X_{pq}^t & X_{qq} \end{pmatrix} := \begin{pmatrix} X_{pp} & X_{pq} \\ -X_{pq}^t & -X_{qq} \end{pmatrix}$$

and the vector $\zeta' := (\zeta_1, \ldots, \zeta_p, -\zeta_{p+1}, \ldots, -\zeta_n)^t \in \boldsymbol{C}^n$, we get

$$\begin{aligned} \dim\{Y \in \mathfrak{h}^{\boldsymbol{C}} \mid Y^t \zeta = 0\} &= \dim\{X \in \mathfrak{o}(n)^{\boldsymbol{C}} \mid \Phi^{-1}(X)^t \zeta = 0\} \\ &= \dim\left\{ \begin{pmatrix} X_{pp} & X_{pq} \\ -X_{pq}^t & X_{qq} \end{pmatrix} \in \mathfrak{o}(n)^{\boldsymbol{C}} \,\middle|\, \begin{pmatrix} X_{pp} & X_{pq} \\ X_{pq}^t & -X_{qq} \end{pmatrix}^t \zeta = 0 \right\} \\ &= \dim\{X \in \mathfrak{o}(n)^{\boldsymbol{C}} \mid X^t \zeta' = 0\}. \end{aligned}$$

Since $\mathfrak{o}(n)^{\boldsymbol{C}} = \{X \in M_n(\boldsymbol{C}) \mid X^t = -X\}$ we see that condition (ii) is true if

$$\dim\{X \in M_n(\boldsymbol{C}) \mid X^t = -X \text{ and } X\zeta = 0\} = \tfrac{1}{2}(n-1)(n-2) \quad \text{for all } \zeta \in \boldsymbol{C}^n \setminus \{0\}.$$

So let $\zeta \in \boldsymbol{C}^n \setminus \{0\}$.

Since $\zeta \neq 0$ one of its components is different from 0, say the first one; normalizing ζ we may assume that the first component is 1. Writing $\zeta = \begin{pmatrix} 1 \\ \zeta' \end{pmatrix}$ where $\zeta' \in C^{n-1}$ and

$$X = \begin{bmatrix} 0 & x_2 \ \dots \ x_n \\ -x_2 & \\ \vdots & X' \\ -x_n & \end{bmatrix},$$

where $X' \in \mathfrak{gl}(n-1, C)$ and $X'' = -X'$, we get $X\zeta = 0 \Leftrightarrow X'\zeta' = (x_2, \dots, x_n)^t$, and so

$$\begin{aligned} &\dim \{X \in \mathfrak{gl}(n, C) \mid X^t = -X \text{ and } X\zeta = 0\} \\ &= \dim \{X' \in \mathfrak{gl}(n-1, C) \mid X'' = -X'\} = \dim(\mathfrak{o}(n-1)^C) = \dim_R \mathfrak{o}(n-1) \\ &= (n-1)(n-2)/2 \end{aligned}$$

as desired.

C. The differential operator $2\alpha\partial/\partial t + \Delta$

In this section we shall, for any $\alpha \in C$, study eigenspace representations of the partial differential operator

$$D_\alpha := 2\alpha \frac{\partial}{\partial t} + \Delta \quad \text{on } R^{1+n}, \quad \text{where } n \geqslant 1.$$

As special cases we get the heat equation operator $(\alpha = -\frac{1}{2})$ and the nonrelativistic Schrödinger operator for a free particle $(\alpha = i)$.

Our invariance group G will be the universal covering group of the $(2n+2)$-dimensional real nilpotent Lie algebra

$$\mathfrak{g} = \operatorname{span}\{H, K_1, \dots, K_n, P_1, \dots, P_n, Z\}$$

which is characterized by the commutator relations

$$[K_i, P_j] = \delta_{ij} Z \quad \text{and} \quad [K_i, H] = P_i \quad \text{for } i, j = 1, 2, \dots, n.$$

For any $\alpha \in C$ the invariance group G acts on $C^\infty(R^{1+n})$ via the representation π_α which is given by the following identities for all $f \in C^\infty(R^{1+n})$, $(t, x) \in R^{1+n}$, $t_0, z \in R$, $x_0, v \in R^n$:

$$\begin{aligned} [\pi_\alpha(\exp t_0 H) f](t, x) &= f(t - t_0, x), \\ [\pi_\alpha(\exp x_0 \cdot P) f](t, x) &= f(t, x - x_0), \\ [\pi_\alpha(\exp zZ) f](t, x) &= \exp(\alpha z) f(t, x), \\ [\pi_\alpha(\exp v \cdot K) f](t, x) &= \exp(\alpha(\langle v, x\rangle - \tfrac{1}{2}\langle v, v\rangle t)) f(t, x - tv), \end{aligned}$$

where we have used the abbreviations

$$x_0 \cdot P = x_{01} P_1 + \ldots + x_{0n} P_n \quad \text{for} \quad x_0 = (x_{01}, \ldots, x_{0n}) \in \boldsymbol{R}^n \quad \text{and}$$

$$v \cdot K = v_1 K_1 + \ldots + v_n K_n \quad \text{for} \quad v = (v_1, \ldots, v_n) \in \boldsymbol{R}^n.$$

It is easy to check that D_α is invariant under the action of π_α.

Remark. This is of course well known (see e.g. section 6g of [Ba]). It might be added that when $n = 3$ then our invariance group G is a subgroup of a central extension $\tilde{\mathscr{G}}$ of the Galilei group, viz. $\tilde{\mathscr{G}}$ without the space rotations. Although we below could work with $\tilde{\mathscr{G}}$ instead of G, we wish to point out that our irreducibility result does not need the action of the full group $\tilde{\mathscr{G}}$, but only the action of G.

THEOREM. *If* $\alpha \in \boldsymbol{C} \setminus \{0\}$ *and* $\lambda \in \boldsymbol{C}$ *then the restriction of* π_α *to the eigenspace* $\{f \in C^\infty(\boldsymbol{R}^{1+n}) \mid D_\alpha f = \lambda f\}$ *is scalarly irreducible.*

Proof. We shall use Corollary 5 with

$$p(\tau, \xi) = 2\alpha\tau + \xi^2 - \lambda,$$

$$\mathfrak{h} = \operatorname{span}\{K_1, \ldots, K_n\}$$

$$R = \text{the restriction of } \pi_\alpha \text{ to the subgroup } H = \exp \mathfrak{h}.$$

Since p is an irreducible polynomial it remains to check condition (ii) of Corollary 5: We find

$$\mu(v \cdot K) = \alpha(0, v) \quad \text{and} \quad \sigma(v \cdot K) = -\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \quad \forall v \in \boldsymbol{R}^n.$$

Condition (ii) reduces to

$$\dim\left\{v \in \boldsymbol{C}^n \,\middle|\, \alpha\begin{pmatrix} 0 \\ v^t \end{pmatrix} - \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \tau \\ \xi^t \end{pmatrix} = 0\right\} = \dim \mathfrak{h} - ((n+1) - 1) = 0,$$

which is true when $\alpha \neq 0$. ∎

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