PARTIAL DIFFERENTIAL EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 19
PWN-POLISH SCIENTIFIC PUBLISHERS
WARSAW 1987

SENSITIVITY ANALYSIS OF THE SIGNORINI VARIATIONAL INEQUALITY

JAN SOKOŁOWSKI

Systems Research Institute, Polish Academy of Sciences
Warsaw, Poland

Introduction

In this paper we present the form of the conical differential of a solution to the Signorini variational inequality with respect to elastic coefficients, body forces and surface loads.

In Section 1 we recall known results [3, 5] concerning the conical differentiability of the projection in a Hilbert space onto a closed convex subset. Several examples are presented. In Section 2, under some assumptions, the conical differentiability of solutions for a class of variational inequalities with respect to perturbations of the right-hand side as well as of the bilinear form is obtained. In Section 3 abstract results of Section 2 are applied to the Signorini variational inequality.

1. Differentiability of the projection

Let K be a closed convex subset of a separable Hilbert space H. For a given element $f \in H$ we denote by $P_K(f) \in K$ the projection of the element f onto the set K, i.e., the element $y = P_K(f)$ is the unique element which satisfies the following variational inequality:

$$\begin{cases} y \in K, \\ (y-f, z-y)_H \geqslant 0, \qquad \forall z \in K. \end{cases} \tag{1.1}$$

It can be verified that the mapping $P_K(\cdot)\colon H \to K \subset H$ is Lipschitz continuous, i.e.,

$$\|P_K(f_1)-P_K(f_2)\|_H \leqslant \|f_1-f_2\|_H, \qquad \forall f_1, f_2 \in H. \tag{1.2}$$

Therefore the projection operator $P_K(\cdot)$ is differentiable on a dense subset of the space H — this follows by the generalization of the Rademacher theorem [5].

DEFINITION 1. Let V_1, V_2 be two Banach spaces, let

$$F\colon V_1 \to V_2 \tag{1.3}$$

be a continuous nonlinear mapping. The mapping (1.3) is said to be *conically differentiable* at $v \in V_1$ if there exists a continuous and positively homogeneous mapping

$$Q\colon V_1 \to V_2 \tag{1.4}$$

such that for $t > 0$, t small enough,

$$\forall w \in V_1\colon \quad F(v+tw) = F(v) + tQ(w) + o(t; w) \tag{1.5}$$

where $\|o(t; w)\|_{V_2}/t \to 0$ as $t \to 0$, uniformly with respect to $w \in V_1$ on compact subsets of the space V_1.

We present some examples of conically differentiable mappings.

EXAMPLE 1. Let $\Omega \subset \boldsymbol{R}^n$ be a given domain. We take

$$H = L^2(\Omega), \tag{1.6}$$

$$K = \{z \in L^2(\Omega) \mid z(x) \geqslant 0 \text{ a.e. in } \Omega\}. \tag{1.7}$$

Let us recall that

$$(y, z)_{L^2(\Omega)} = \int_\Omega y(x) z(x) dx, \qquad \forall y, z \in L^2(\Omega). \tag{1.8}$$

In this case

$$P_K(f) = \max\{0, f\} = f^+, \qquad \forall f \in L^2(\Omega) \tag{1.9}$$

where

$$f^+(x) = \max\{0, f(x)\} \qquad \text{for a.a. } x \in \Omega. \tag{1.10}$$

It can be verifed that for $t > 0$, t small enough,

$$\forall w \in L^2(\Omega)\colon \; P_K(f+tw) = P_K(f) + tQ(w) + o(t; w) \tag{1.11}$$

where

$$Q(w)(x) = \begin{cases} 0, & x \in \Omega^-, \\ w(x), & x \in \Omega^+, \\ w^+(x), & x \in \Omega^0, \end{cases} \tag{1.12}$$

and

$$\Omega^- = \{x \in \Omega \mid f(x) \leqslant 0\}, \tag{1.13}$$

$$\Omega^+ = \{x \in \Omega \mid f(x) \geqslant 0\}, \tag{1.14}$$

$$\Omega^0 = \{x \in \Omega \mid f(x) = 0\}. \tag{1.15}$$

Let us note that if for a given element $f \in L^2(\Omega)$ we have meas $\Omega^0 > 0$ then the mapping $L^2(\Omega) \ni w \mapsto Q(w) \in L^2(\Omega)$ is not linear, therefore the projection operator $P_K(\cdot)$ is not differentiable.

EXAMPLE 2. Consider

$$V_1 = L^2(0, 1), \tag{1.16}$$

$$V_2 = \{z \in H^1(0, 1) | z(0) = 0\}, \tag{1.17}$$

where

$$H^1(0, 1) = \{z \in L^2(0, 1) | dz/dx \in L^2(0, 1)\}. \tag{1.18}$$

The mapping $F\colon V_1 \to V_2$ is defined as follows: for a given element $f \in L^2(0, 1)$, the element $F(f) \in V_2$ is a unique solution of the following variational inequality:

$$\begin{cases} F = F(f) \in K = \{z \in V_2 | z(1) \geqslant 0\}, \\ \int_0^1 \frac{dF}{dx}\left(\frac{dz}{dx} - \frac{dF}{dx}\right) dx \geqslant \int_0^1 f(z-F)\,dx, \qquad \forall z \in K. \end{cases} \tag{1.19}$$

The solution to (1.19) is given by

$$\begin{aligned} F(f)(x) = {} & -\int_0^x \int_0^\xi \Big(f(s) - 3s\int_0^1 \eta f(\eta)\,d\eta\Big)\,ds\,d\xi \\ & + 3x \max\Big\{0, \int_0^1 \eta f(\eta)\,d\eta\Big\}. \end{aligned} \tag{1.20}$$

From (1.20) it follows that the mapping

$$L^2(0, 1) \ni f \mapsto F(f) \in H^1(0, 1) \tag{1.21}$$

is not differentiable at $f^* \in L^2(0, 1)$ if and only if

$$\int_0^1 \eta f^*(\eta)\,d\eta = 0. \tag{1.22}$$

In this case we have for $t > 0$

$$\forall w \in L^2(0, 1)\colon\ F(f^* + tw) = F(f^*) + tQ(w) \tag{1.23}$$

where

$$\begin{aligned} Q(w)(x) = {} & -\int_0^x \int_0^\xi \Big(w(s) - 3s\int_0^1 \eta w(\eta)\,d\eta\Big)\,ds\,d\xi \\ & + 3x \max\Big\{0, \int_0^1 \eta w(\eta)\,d\eta\Big\}. \end{aligned}$$

We need the following notation. For a given element $y \in K$ we denote by

$$N_K(y) = \{v \in H | (v, \varphi - y)_H \leqslant 0,\ \forall \varphi \in K\}, \tag{1.24}$$

(1.25) $$C_K(y) = \{v \in H \mid \exists \tau > 0 \text{ such that } y + \tau v \in K\}$$

the normal cone and the tangent cone, respectively. Furthermore, for a given element $f \in H$ we write

(1.26) $$S_K(f) = \{v \in \overline{C_K(P_K(f))} \mid (f - P_K(f), v)_H = 0\}$$

where $\overline{C_K(y)}$ is the closure in H of the tangent cone $C_K(y)$. It can be verified that the set $S_K(f)$ is a closed convex cone.

Let us assume that there is given a continuous mapping $f(\cdot)$: $[0, \delta) \to H$ which is right-differentiable at 0, i.e., there exists an element $f'(0) \in H$ such that

(1.27) $$\lim_{\tau \downarrow 0} \|(f(\tau) - f(0))/\tau - f'(0)\|_H = 0.$$

Write $y(\tau) = P_K(f(\tau))$, $\gamma(\tau) = (y(\tau) - y(0))/\tau$ and observe that in view of (1.2), $\|\gamma(\tau)\|_H \leqslant C$ for all $\tau \in (0, \delta)$ for some $\delta > 0$, where C is a constant.

PROPOSITION 1. *Every weak limit point γ of $\gamma(\tau)$ for $\tau \to 0$ satisfies*

(1.28) $$\gamma \in S_K(f(0)).$$

The proof of Proposition 1 is given e.g. in [3].

DEFINITION 2. The set K is called *polyhedral* if the following condition is satisfied for all $f \in H$:

(1.29) $$S_K(f) = \overline{\{v \in C_K(P_K(f)) \mid (f - P_K(f), v)_H = 0\}}.$$

THEOREM 1. *Let us assume that the set K is polyhedral. Then for $\tau > 0$, τ small enough,*

(1.30) $$P_K(f(\tau)) = P_K(f(0)) + \tau P_{S_K(f(0))}(f'(0)) + o(\tau)$$

where $\|o(\tau)\|_H/\tau \to 0$ *as* $\tau \to 0$.

The proof of Theorem 1 is given in [3, 5]. We present an example which will be useful for us in Section 3.

EXAMPLE 3. Let $\Omega \subset \boldsymbol{R}^n$ be a domain with smooth boundary $\Gamma = \partial\Omega$. Let Γ_c be a part of the boundary Γ. Let us recall that the Sobolev space $H^{1/2}(\Gamma_c)$ is defined [4] in the following way:

(1.31) $$H^{1/2}(\Gamma_c) = \{h \in L^2(\Gamma_c) \mid \exists \Psi \in H^1(\Omega),\ \Psi|_{\Gamma_c} = h\}.$$

Let $H \subset H^{1/2}(\Gamma_c)$ be a closed linear subspace. We assume that there is given a symmetric bilinear form $b(\cdot, \cdot)$: $H \times H \to \boldsymbol{R}$ such that

(1.32) $$\alpha_1 \|h\|^2_{H^{1/2}(\Gamma_c)} \leqslant b(h, h) \leqslant \beta_1 \|h\|^2_{H^{1/2}(\Gamma_c)},$$
$$0 < \alpha_1 \leqslant \beta_1 < +\infty, \quad \forall h \in H.$$

We write

(1.33) $$h^+ = \max\{0, h\},$$

(1.34) $$h^- = \max\{0, h\}, \quad \forall h \in L^2(\Gamma_c),$$

and we assume that the following conditions are satisfied:

(1.35) (i) $h^+, h^- \in H, \quad \forall h \in H,$

(1.36) (ii) $b(h^+, h^-) \leqslant 0, \quad \forall h \in H,$

(1.37) (iii) $H \cap C_0(\Gamma_c)$ is dense in $C_0(\Gamma_c)$,

where $C_0(\Gamma_c)$ denotes the space of continuous functions with compact support on Γ_c. The pair $\{H, b(\cdot, \cdot)\}$ is a so-called Dirichlet space [5].

We denote by $K \subset H$ the set

(1.38) $$K = \{h \in H \mid h(x) \leqslant 0 \text{ a.e. on } \Gamma_c\}.$$

Let us define the mapping $P_K(\cdot)$: $H \to K \subset H$ in the following way: for any element $f \in H$, the element $p = P_K(f)$ is a unique solution of the following variational inequality:

(1.39) $$\begin{cases} p \in K, \\ b(p-f, h-p) \geqslant 0, \quad \forall h \in K. \end{cases}$$

In this case we write $(p, h)_H = b(p, h)$, $\forall p, h \in H$. Using the results obtained by F. Mignot [5] it can be verified that the set (1.38) is polyhedral, therefore the projection $P_K(\cdot)$ is conically differentiable. The mapping $Q(\cdot)$ takes the form $Q(w) = P_{S_K(f)}(w)$, $\forall w \in H$, i.e.,

(1.40) $$\begin{cases} Q = Q(w) \in S_K(f), \\ b(Q-w, h-Q) \geqslant 0, \quad \forall h \in S_K(f), \end{cases}$$

where

(1.41) $$S_K(f) = \{h \in H \mid h(x) \leqslant 0 \text{ a.e. on } Z(p), b(p-f, h) = 0\},$$
$$p = P_K(f),$$

(1.42) $$Z(p) = \{x \in \Gamma_c \mid p(x) = 0\}.$$

2. Differential stability of solutions to an abstract variational inequality

Let W be a Hilbert space; denote by W' the dual space. Given a continuous linear mapping $R \in L(W; H)$, we denote by U the closed and convex subset of the space W of the form

(2.1) $$U = \{\varphi \in W \mid R\varphi \in K \subset H\}.$$

Let there be given a symmetric bilinear form $a(\cdot, \cdot)$: $W \times W \to \boldsymbol{R}$ which is continuous and coercive, i.e.,

$$a(\varphi, \varphi) \geqslant \alpha \|\varphi\|_W^2, \quad \alpha > 0, \ \forall \varphi \in W, \tag{2.2}$$

$$|a(\varphi, \psi)| \leqslant M \|\varphi\|_W \|\psi\|_W, \quad \forall \varphi, \psi \in W. \tag{2.3}$$

In this section we consider the differentiability properties of the mapping

$$W' \ni f \mapsto \Pi(f) \in U \subset W, \tag{2.4}$$

where for a given element $f \in W'$, the element $\Pi(f) \in U$ is a unique solution of the following variational inequality:

$$\begin{cases} \Pi(f) \in U, \\ a(\Pi(f), \varphi - \Pi(f)) \geqslant \langle f, \varphi - \Pi(f) \rangle, \quad \forall \varphi \in U. \end{cases} \tag{2.5}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between W' and W.

EXAMPLE 4. Let $\Omega \subset \boldsymbol{R}^n$ be a bounded domain with smooth boundary $\Gamma = \partial\Omega$. We take

$$W = H^1(\Omega), \quad H = H^{1/2}(\Gamma),$$

$$R\varphi = \varphi|_\Gamma, \quad \forall \varphi \in H^1(\Omega), \tag{2.6}$$

$$K = \{h \in H^{1/2}(\Gamma) \mid h(x) \geqslant 0 \text{ a.e. on } \Gamma\}; \tag{2.7}$$

then

$$U = \{\varphi \in H^1(\Omega) \mid \varphi|_\Gamma \geqslant 0 \text{ a.e. on } \Gamma\}. \tag{2.8}$$

EXAMPLE 5. Let $\Gamma_2 \subset \partial\Omega$ be a given part of the boundary of the domain $\Omega \subset \boldsymbol{R}^2$. Let

$$W = H^1(\Omega; \boldsymbol{R}^2), \quad U = \{\varphi \in W \mid R\varphi \in K\}, \tag{2.9}$$

$$H = H^{1/2}(\Gamma_2), \quad K = \{h \in H^{1/2}(\Gamma_2) \mid h(x) \leqslant 0 \text{ on } \Gamma_2\}, \tag{2.10}$$

where

$$(R\varphi)(x) = \varphi_n(x) = \sum_{i=1}^{2} \varphi_i(x) n_i(x), \quad x \in \Gamma_2, \qquad \forall \boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in H^1(\Omega; \boldsymbol{R}^2), \tag{2.11}$$

and $\boldsymbol{n} = (n_1, n_2)$ denotes the unit outward normal on $\partial\Omega$.

In this section we prove that the conical differentiability of the mapping (2.4) is equivalent to the conical differentiability of the projection mapping P_K: $H \to K \subset H$.

Let us consider the variational inequality (2.5). We assume that the operator R maps W onto H and that $0 \in K \subset H$, therefore

$$\ker R \cap U = \ker R. \tag{2.12}$$

Write

$$W_1 = \ker R, \quad W_2 = W_1^{\perp}; \tag{2.13}$$

thus

$$W = W_1 \oplus W_2 \tag{2.14}$$

and there exists an inverse operator $R^{-1} \in L(H; W_2)$. We define a scalar product $((\cdot, \cdot))_H$ in the following way:

$$((h_1, h_2))_H = a(R^{-1} h_1, R^{-1} h_2), \quad \forall h_1, h_2 \in H. \tag{2.15}$$

We introduce the projection operator

$$P_K\colon H \to K \subset H \tag{2.16}$$

in the following way: for a given element $\xi \in H$, the element $p = P_K(\xi)$ is a unique solution of the variational inequality

$$\begin{cases} p = P_K(\xi) \in K, \\ ((p-\xi, h-p))_H \geqslant 0, \quad \forall h \in K. \end{cases} \tag{2.17}$$

For a given element $f \in W'$, we denote by $\Phi(f) \in H$ the unique solution of the variational equation

$$((\Phi(f), h))_H = \langle f, R^{-1} h \rangle, \quad \forall h \in H. \tag{2.18}$$

Let us note that the linear mapping

$$W' \ni f \mapsto \Phi(f) \in H \tag{2.19}$$

is continuous.

Now we are in a position to decompose the variational inequality (2.5) in the following way: the solution $y = \Pi(f)$ to (2.5) takes the form

$$\Pi(f) = y_1 + y_2, \quad y_i \in W_i, \ i = 1, 2, \tag{2.20}$$

where $y_1 \in W_1$ is a unique solution of the variational equation

$$\begin{cases} y_1 \in W_1, \\ a(y_1, \eta) = \langle f, \eta \rangle, \quad \forall \eta \in W_1. \end{cases} \tag{2.21}$$

The element $y_2 \in W_2$ is given by

$$y_2 = R^{-1} P_K(\Phi(f)). \tag{2.22}$$

Using (2.20)–(2.22) we obtain the following

LEMMA 1. *The mapping* (2.4) *is conically differentiable if and only if the projection operator* (2.16) *is conically differentiable.*

Finally let us consider the variational inequality

$$\begin{cases} y_\varepsilon \in U, \\ a_\varepsilon(y_\varepsilon, \varphi - y_\varepsilon) \geqslant \langle f_\varepsilon, \varphi - y_\varepsilon \rangle, \quad \forall \varphi \in U, \end{cases} \tag{2.23}$$

where $\varepsilon \in [0, \delta)$ is a parameter, $\delta > 0$. Here $a_\varepsilon(\cdot, \cdot)$: $W \times W \to \boldsymbol{R}$ denotes a family of bilinear forms such that conditions (2.2), (2.3) are satisfied uniformly with respect to the parameter $\varepsilon \in [0, \delta)$. We denote by $A_\varepsilon \in L(W; W')$ the linear operator given by

$$\langle A_\varepsilon z, \varphi \rangle = a_\varepsilon(z, \varphi), \quad \forall z, \varphi \in W. \tag{2.24}$$

Furthermore, we denote by $((\cdot, \cdot))_H$ the scalar product of the form $((h, \eta))_H = a_0(R^{-1} h, R^{-1} \eta)$, $\forall h, \eta \in H$.

We assume that the mapping (2.16) is conically differentiable, i.e., for $\tau > 0$, τ small enough,

$$\forall h \in H\colon\ P_K(\xi + \tau h) = P_K(\xi) + \tau Q(h) + o(\tau) \tag{2.25}$$

where $\|o(\tau)\|_H/\tau \to 0$ as $\tau \to 0$.

THEOREM 2. *Assume that*

(i) *there exists an operator* $A' \in L(W; W')$ *such that*

$$\lim_{\varepsilon \downarrow 0} \|(A_\varepsilon - A_0)/\varepsilon - A'\|_{L(W;W')} = 0, \tag{2.26}$$

(ii) *there exists an element* $f' \in W'$ *such that*

$$\lim_{\varepsilon \downarrow 0} \|(f_\varepsilon - f_0)/\varepsilon - f'\|_{W'} = 0. \tag{2.27}$$

Furthermore, assume that (2.25) *holds. Then for* $\varepsilon > 0$, ε *small enough, the solution to* (2.23) *satisfies*

$$y_\varepsilon = y_0 + \varepsilon y' + o(\varepsilon) \quad \text{in } W \tag{2.28}$$

where $\|o(\varepsilon)\|_W/\varepsilon \to 0$ *as* $\varepsilon \to 0$. *The element* $y' \in W$ *is given by*

$$y' = \Pi_1(f' - A' y_0) + R^{-1} Q(\Phi(f' - A' y_0)) \tag{2.29}$$

where for any $\Theta \in W'$ *the elements* $\Pi_1(\Theta)$, $\Phi(\Theta)$ *are unique solutions of the variational equations*

$$\begin{cases} \Pi_1 = \Pi_1(\Theta) \in W_1, \\ a_0(\Pi_1, \eta) = \langle \Theta, \eta \rangle, \quad \forall \eta \in W_1, \end{cases} \tag{2.30}$$

$$\begin{cases} \Phi = \Phi(\Theta) \in W_2, \\ a_0(R^{-1}\Phi, R^{-1} h) = \langle \Theta, R^{-1} h \rangle, \quad \forall h \in H. \end{cases} \tag{2.31}$$

The proof of Theorem 2 is given in [6].

3. Sensitivity analysis of the Signorini variational inequality

This section is devoted to the sensitivity analysis of solutions to a system of equations of elliptic type with respect to perturbations of the right-hand side and functional coefficients of the elliptic operator. The system under consideration describes the deformations of plane elastic solids. We start with the description of the mathematical model. We will use the summation convention over repeated indices i, j, k, $l = 1, 2$.

Let us consider the deformations of a plane elastic body of reference configuration $\bar{\Omega} \subset \boldsymbol{R}^2$. Assume that the body is subjected to body forces $\boldsymbol{f} = (f_1, f_2)$ and that surface tractions $\boldsymbol{P} = (P_1, P_2)$ are applied to a portion Γ_1 of the boundary $\Gamma = \partial\Omega$ of the body. We assume that the body is fixed along a portion Γ_0 of its boundary, and that frictionless contact conditions are prescribed on a portion Γ_2 of the boundary $\partial\Omega$.

Let $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{\sigma} = \{\sigma_{ij}\}$, $i, j = 1, 2$, denote arbitrary displacement and stress fields in the body. We consider Hookean elastic materials, i.e.,

$$\sigma_{ij}(x) = c_{ijkl}(x)\, u_{k,l}(x), \qquad x \in \Omega, \tag{3.1}$$

where $\{c_{ijkl}(x)\}$, i, j, k, $l = 1, 2$, denote the components of Hooke's tensor $\boldsymbol{C}$ at $x \in \Omega$; $u_{k,l} = \partial u_k/\partial x_l$ and we use the summation convention over repeated indices i, j, k, $l = 1, 2$. We assume that

$$\begin{aligned} &c_{ijkl}(x) = c_{jikl}(x) = c_{klij}(x), \qquad \forall x \in \Omega, \\ &c_{ijkl}(\cdot) \in L^\infty(\Omega), \qquad \forall i, j, k, l = 1, 2 \end{aligned} \tag{3.2}$$

and that there exists a positive constant $\alpha_0 > 0$ such that

$$c_{ijkl}(x)\, e_{ij}\, e_{kl} \geqslant \alpha_0\, e_{ij}\, e_{ij}, \qquad \forall x \in \Omega, \quad \text{for all symmetric matrices } [e_{ij}]_{2\times 2}. \tag{3.3}$$

A stress field $\boldsymbol{\sigma} = \boldsymbol{\sigma}(x)$ is in equilibrium at a point x in the interior of Ω if

$$-\sigma_{ij,j}(x) = f_i(x), \qquad x \in \Omega,\ i = 1, 2, \tag{3.4}$$

where $\sigma_{ij,j} = \sum_{j=1}^{2} \partial\sigma_{ij}/\partial x_j$, $i = 1, 2$.

A displacement field $\boldsymbol{u} = \boldsymbol{u}(x)$ satisfies the kinematic boundary conditions on Γ_0 if

$$u_i(x) = 0, \qquad x \in \Gamma_0, \qquad i = 1, 2. \tag{3.5}$$

If $\boldsymbol{P}$ is the traction applied on Γ_1, the stress produced there must satisfy

$$\sigma_{ij}(x)\, n_j(x) = P_i(x), \qquad x \in \Gamma_1. \tag{3.6}$$

If the body is unilaterally suported by a frictionless rigid foundation and the portion Γ_2 of the boundary $\partial\Omega$ is a candidate for the contact region, i.e., contact occurs at a portion $Z \subset \Gamma_2$ which is not known *a priori*, the unilateral boundary conditions are given by

$$\boldsymbol{u}\cdot\boldsymbol{n} \leqslant 0, \qquad \sigma_n \leqslant 0, \qquad \sigma_n\, \boldsymbol{u}\cdot\boldsymbol{n} = 0, \qquad \boldsymbol{\sigma}_\tau = 0 \qquad \text{on } \Gamma_2. \tag{3.7}$$

Here σ_n, σ_τ denote the normal and tangential components of the stress tensor σ, respectively.

Now let u denote a specific displacement field of the body, which corresponds to an equilibrium state of the body for given data: body forces f and tractions P. The displacement field u is a unique weak solution of the following nonlinear system:

$$-\big(c_{ijkl}(x)\,u_{k,l}(x)\big)_{,j} = f_i(x) \quad \text{in } \Omega,\ i = 1, 2, \tag{3.8}$$

$$u_i = 0 \quad \text{on } \Gamma_0,\ i = 1, 2, \tag{3.9}$$

$$c_{ijkl}\,u_{k,l}\,n_j = P_i \quad \text{on } \Gamma_1,\ i = 1, 2, \tag{3.10}$$

$$\begin{aligned} &u_i n_i \leqslant 0, \quad \sigma_n = c_{ijkl}\,u_{k,l}\,n_j n_i \leqslant 0, \quad \sigma_n u_i n_i = 0 \quad \text{and} \\ &\qquad c_{ijkl}\,u_{k,l}\,n_j = \sigma_n n_i \quad \text{on } \Gamma_2. \end{aligned} \tag{3.11}$$

Let us recall [5] that a weak solution $u \in H^1(\Omega; \boldsymbol{R}^2)$ to the system (3.8)–(3.11) satisfies the variational inequality

$$\begin{cases} u \in U, \\ a(u, \varphi - u) \geqslant (F, \varphi - u), \quad \forall \varphi \in U, \end{cases} \tag{3.12}$$

where the bilinear form $a(\cdot, \cdot)$, the element F, and the convex closed set $U \subset H^1(\Omega; \boldsymbol{R}^2)$ are defined as follows:

$$a(z, \varphi) = \int_\Omega Dz \ldots C \ldots D\varphi\, dx = \int_\Omega c_{ijkl}(x)\, z_{i,j}(x)\, \varphi_{k,l}(x)\, dx, \qquad \forall z, \varphi \in H^1(\Omega; \boldsymbol{R}^2), \tag{3.13}$$

$$(F, \varphi) = \int_\Omega f \cdot \varphi\, dx + \int_{\Gamma_1} P \cdot \varphi\, d\Gamma, \quad \forall \varphi \in H^1(\Omega; \boldsymbol{R}^2), \tag{3.14}$$

$$U = \{\varphi \in H^1(\Omega; \boldsymbol{R}^2) \mid \varphi = 0 \text{ on } \Gamma_0,\ \varphi \cdot n \leqslant 0 \text{ on } \Gamma_2\}. \tag{3.15}$$

We assume here that $f \in L^2(\Omega; \boldsymbol{R}^2)$, $P \in L^2(\Gamma_1; \boldsymbol{R}^2)$ are given elements, and that meas $\Gamma_0 > 0$; therefore there exists [2] a unique weak solution of the variational inequality (3.12).

We combine Example 3, Example 4 and Theorem 2 in order to obtain the form of the conical differential of the solution u to (3.12) with respect to the functions

$$\{c_{ijkl} \in L^\infty(\Omega), f_i \in L^2(\Omega),\ P_i \in L^2(\Gamma_1),\ i, j, k, l = 1, 2\}. \tag{3.16}$$

To this end we assume that there are given elements

$$c^\varepsilon_{ijkl},\ c'_{ijkl} \in L^\infty(\Omega), \quad i, j, k, l = 1, 2,\ \varepsilon \in [0, \delta), \tag{3.17}$$

$$f_i^\varepsilon, f_i' \in L^2(\Omega), \quad i = 1, 2, \varepsilon \in [0, \delta), \tag{3.18}$$

$$P_i,\ P_i' \in L^2(\Gamma_1), \quad i = 1, 2,\ \varepsilon \in [0, \delta), \tag{3.19}$$

and we write

(3.20) $$a^\varepsilon(z, \varphi) = \int_\Omega c^\varepsilon_{ijkl}(x) z_{i,j}(x) \varphi_{k,l}(x)\,dx, \quad \varepsilon \in [0, \delta),\ \forall z,\ \varphi \in H^1(\Omega; \boldsymbol{R}^2),$$

(3.21) $$a'(z, \varphi) = \int_\Omega c'_{ijkl}(x) z_{i,j}(x) \varphi_{k,l}(x)\,dx, \quad \forall z,\ \varphi \in H^1(\Omega; \boldsymbol{R}^2),$$

(3.22) $$(F^\varepsilon, \varphi) = \int_\Omega f_i^\varepsilon(x) \varphi_i(x)\,dx + \int_{\Gamma_1} P_i(x) \varphi_i(x)\,d\Gamma, \quad \forall \varphi \in H^1(\Omega; \boldsymbol{R}^2),$$

(3.23) $$(F', \varphi) = \int_\Omega f_i'(x) \varphi_i(x)\,dx + \int_{\Gamma_1} P_i'(x) \varphi_i(x)\,d\Gamma, \quad \forall \varphi \in H^1(\Omega; \boldsymbol{R}^2).$$

We assume that

(i) the elements $c^\varepsilon_{ijkl} \in L^\infty(\Omega)$, $i, j, k, l = 1, 2$, $\varepsilon \in [0, \delta)$, satisfy the conditions (3.2), (3.3), furthermore

(3.24) $$\lim_{\varepsilon \downarrow 0} \|(c^\varepsilon_{ijkl} - c^0_{ijkl})/\varepsilon - c'_{ijkl}\|_{L^\infty(\Omega)} = 0.$$

(ii)

(3.25) $$\lim_{\varepsilon \downarrow 0} \|(f_i^\varepsilon - f_i^0)/\varepsilon - f_i'\|_{L^2(\Omega)} = 0, \quad i = 1, 2.$$

(iii)

(3.26) $$\lim_{\varepsilon \downarrow 0} \|(P_i - P_i^0)/\varepsilon - P_i'\|_{L^2(\Gamma_1)} = 0, \quad i = 1, 2.$$

Let $u^\varepsilon \in H^1(\Omega; \boldsymbol{R}^2)$ be a unique solution of the variational inequality

(3.27) $$\begin{cases} u^\varepsilon \in U, \\ a^\varepsilon(u^\varepsilon, \varphi - u^\varepsilon) \geqslant (F^\varepsilon, \varphi - u^\varepsilon), \quad \forall \varphi \in U. \end{cases}$$

THEOREM 3. *Assume that* meas $\Gamma_0 > 0$ *and the conditions* (i)–(iii) *are satisfied. Then for* $\varepsilon > 0$, ε *small enough,*

(3.28) $$u^\varepsilon = u^0 + \varepsilon Q + o(\varepsilon) \quad \text{in } H^1(\Omega; \boldsymbol{R}^2)$$

where $\|o(\varepsilon)\|_{H^1(\Omega; \boldsymbol{R}^2)}/\varepsilon \to 0$ *as* $\varepsilon \to 0$. *The element* $Q \in H^1(\Omega; \boldsymbol{R}^2)$ *is a unique solution of the variational inequality*

(3.29) $$\begin{cases} Q \in S, \\ a^0(Q, \varphi - Q) \geqslant -a'(u^0, \varphi - Q) + (F', \varphi - Q), \quad \forall \varphi \in S, \end{cases}$$

where the cone $S \subset H^1(\Omega; \boldsymbol{R}^2)$ *is given by*

(3.30) $$S = \{\varphi \in H^1(\Omega; \boldsymbol{R}^2) \mid \varphi \cdot n \leqslant 0 \text{ a.e. on } Z(u^0), a^0(u^0, \varphi) = (F^0, \varphi)\},$$

(3.31) $$Z(u^0) = \{x \in \Gamma_2 \mid u^0(x) \cdot n(x) = 0\}.$$

Proof. We assume for simplicity that the outward unit normal n on Γ_2

takes the form

$$n = (1, 0) \quad \text{on } \Gamma_2. \tag{3.32}$$

We write

$$W = \{\varphi \in H^1(\Omega; R^2) \mid \varphi = 0 \text{ on } \Gamma_0\}, \tag{3.33}$$

$$H = \{h \in L^2(\Gamma_2) \mid \exists \psi \in H^1(\Omega)\colon \psi|_{\Gamma_0} = 0,\ \psi|_{\Gamma_2} = h\}. \tag{3.34}$$

Then the set (3.15) takes the form

$$U = \{\varphi \in W \mid R\varphi \in K\} \tag{3.35}$$

where

$$R\varphi = \varphi_1|_{\Gamma_2}, \quad \forall \varphi \in W, \tag{3.36}$$

and K is defined by (3.41) below.

Let

$$W_1 = \ker R, \tag{3.37}$$

$$W_2 = \{\varphi \in W \mid a^0(z, \varphi) = 0,\ \forall z \in W_1\}. \tag{3.38}$$

The inverse operator $R^{-1} \in L(H; W_2)$ exists, therefore the bilinear form

$$b(h_1, h_2) = a(R^{-1} h_1, R^{-1} h_2), \quad \forall h_1, h_2 \in H, \tag{3.39}$$

is well defined. We define a scalar product in the space H:

$$((h_1, h_2))_H = b(h_1, h_2), \quad \forall h_1, h_2 \in H. \tag{3.40}$$

It can be verified [6] that $\{H, b(\cdot, \cdot)\}$ is a Dirichlet space, thus by the results of Mignot [5] it follows that the projection in the scalar product (3.40) onto the set

$$K = \{h \in H \mid h(x) \leqslant 0 \text{ a.e. on } \Gamma_2\} \tag{3.41}$$

is conically differentiable and by Lemma 2 and Theorem 2 it follows that (3.28) holds. After some calculations, taking into account (1.40), (1.41) and (2.29), it follows that the element $Q \in W$ in (3.28) is a unique solution of the variational inequality (3.29) [6].

If the condition (3.32) is not satisfied we can use the following transformation:

$$\psi_1 = N_1 \varphi_1 + N_2 \varphi_2, \quad \psi_2 = -N_2 \varphi_1 + N_1 \varphi_2,$$

where

$$n(x) = N(x) \quad \text{for a.a. } x \in \Gamma_2,$$

$$N_1, N_2 \in W^{1,\infty}(\Omega), \quad N_1^2(x) + N_2^2(x) \geqslant c > 0 \quad \text{a.e. on } \Omega.$$

Then $\varphi \cdot n = \psi_1$ on Γ_2.

Concluding remarks

In this paper we derive the form of the conical differential of the solution to the Signorini variational inequality [6] with respect to the right-hand side and the functional coefficients of the elliptic operator. Using this result the local behaviour of those solutions under deformations of the domain of integration is investigated in [8, 9]. For related results concerning sensitivity analysis of optimal control problems for distributed parameter systems we refer the reader to [7].

Reference

[1] G. Duvaut et J.-L. Lions, *Les inéquations en mécanique et en physique*, Dunod, Paris 1972.

[2] G. Fichera, *Boundary value problems of elasticity with unilateral constraints*, Handbuch der Physik, Band 6a/2, Springer, New York 1972.

[3] A. Haraux, *How to differentiate the projection on a convex set in Hilbert space. Some applications to variational inequalities*, J. Math. Soc. Japan 29 (4) (1977), 615–631.

[4] J.-L. Lions et E. Magenes, *Problèmes aux limites non homogènes et applications*, Vol. 1, Dunod, Paris 1968.

[5] F. Mignot, *Contrôle dans les inéquations variationnelles elliptiques*, J. Funct. Anal. 22 (1976), 130–185.

[6] J. Sokolowski, *Conical differentiability of projection on convex sets – an application to sensitivity analysis of Signorini variational inequality*, technical report, Institute of Mathematics, University of Genoa, 1981.

[7] –, *Differential stability of solutions to constrained optimization problems*, Appl. Math. Optim. 13 (2) (1985), 97–115.

[8] J. Sokołowski et J.-P. Zolésio, *Dérivée par rapport au domaine de la solution d'un problème unilatéral*, C. R. Acad. Sci. Paris Sér. I 301 (1985), 103–106.

[9] –, –, *Shape design sensitivity analysis of plates and plane elastic solids under unilateral constraints*, J. Optim. Theory Appl. 54 (2) (1987), 361–382.