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SCHRÖDINGER OPERATORS WITH ALMOST PERIODIC POTENTIALS IN NONSEPARABLE HILBERT SPACES

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We discuss the spectral properties of the Schrödinger operators $l = -\Delta + V$ with Stepanov almost periodic potential V in the nonseparable Hilbert space $B^2(\boldsymbol{R}^\nu)$ of Besicovitch almost periodic functions. The equality of spectra of l in $B^2(\boldsymbol{R}^\nu)$ and $L^2(\boldsymbol{R}^\nu)$ is proved, as well as the spectral mixing theorem. The spectra of l are shown to be essential.

0. Introduction

Almost periodic Schrödinger operator $l = -\Delta + V$ commands increasing interest (see for instance [1, 3, 33]). While an effective general theory is absent, analysis of many special cases has revealed subtle and often surprising spectral properties of l as well as of its discrete analogue. The present paper presents a different kind of contribution: we do not study specific examples but offer an alternative approach, based on the study of l both in the usual space $L^2(\boldsymbol{R}^\nu)$ and in the nonseparable Hilbert space of Besicovitch almost periodic functions $B^2(\boldsymbol{R}^\nu)$. For over 20 years now, since Burnat's basic paper [4], it has been known that $B^2(\boldsymbol{R}^\nu)$ is a natural space to study the periodic Schrödinger operator. Extending the results of [4–9, 20, 30] we give here basic properties of almost periodic operators in $B^2(\boldsymbol{R}^\nu)$. "Foundational" questions such as the characterization of the domain of self-adjointness of l, properties of some functions of l or the essentiality of the spectrum were settled in $L^2(\boldsymbol{R}^\nu)$ long ago, but had to be adressed directly here. It is hoped that further analysis of specific examples in $B^2(\boldsymbol{R}^\nu)$ will contribute to the understanding of the spectral properties of l. We mention below some of the open problems in this area.

The study of Schrödinger operators in nonseparable Hilbert spaces arose from the research of Burnat [4–9], who analyzed the spectral properties of l

with periodic potential V in $B^2(\boldsymbol{R}^3)$. This study is motivated by the desire to find such functional Hilbert spaces, called *spectral spaces*, in which l can be defined as a self-adjoint operator and which contain various classes of non-square integrable eigenfunctions of l as orthogonal eigenelements. Such "generalized" eigenfunctions, associated with points in the continuous spectrum, are of direct physical interest and are useful for the physicist even if their existence and precise properties have not been rigorously investigated. It is clear that spectral Hilbert spaces will in general depend on the potential, will usually be nonseparable and that it may be necessary in some situations, such as in multi-channel scattering, to consider whole families of them so as to classify various types of behaviour of eigenfunctions at infinity. For a number of cases, including multi-channel scattering, spectral spaces have been constructed by the author [20, 21]. The general problem is to find, for a given Schrödinger operator, "all" its spectral spaces in the sense that it is possible to obtain the eigenfunction expansion of any L^2 function in which there appear only eigenfunctions belonging to the considered spectral spaces.

This program is completed in four cases: (1) for V periodic, with one spectral space $B^2(\boldsymbol{R}^\nu)$ (see Appendix), (2) for $V(x, y)$ periodic in $x \in \boldsymbol{R}^k$, $V(x, y) \to \infty$ as $|y| \to \infty$, $y \in \boldsymbol{R}^l$ (cf. [20]), (3) for potential scattering, with two spectral spaces, $L^2(\boldsymbol{R}^\nu)$ for the bound states and $B^2(\boldsymbol{R}^\nu)$ for the perturbed plane waves (see [28]), and (4) for operators with compact resolvent, $V(x) \to \infty$ as $|x| \to \infty$ (this case is trivial from our point of view, since all eigenfunctions appearing in the expansion are L^2 functions). However, even if the eigenfunction expansion cannot be proved, the spectral analysis of l in the spectral spaces is of independent interest, as it may yield new results for $L^2(\boldsymbol{R}^\nu)$ theory. We present an example of such an application, the spectral mixing theorem (Section 3). This asserts that given Stepanov almost periodic potentials V_i, $i = 1, \ldots, N$, and appropriate cut-off functions ϱ_i, e.g. characteristic functions of disjoint open cones, the "mixture" defined by the potential $V_{\text{mix}} = \sum_{i=1}^{N} \varrho_i V_i$ satisfies

$$\bigcup_{i=1}^{N} \sigma(-\Delta + V_i) \subset \sigma(-\Delta + V_{\text{mix}}),$$

an inclusion with transparent physical meaning (spectra in $L^2(\boldsymbol{R}^\nu)$).

We also note that in his studies of almost periodic pseudodifferential operators Shubin [29–32] used the Hilbert space $B^2(\boldsymbol{R}^\nu)$ to derive an expression for the density of states and to analyze the spectral asymptotics. Perturbation properties of spectra of periodic Schrödinger operators, resulting from the B^2 analysis, are presented in [5, 10]. Moreover, the space $B^2(\boldsymbol{R}^\nu)$ and analogous spaces of almost periodic sequences $b^2(h\boldsymbol{Z}^\nu)$, $h > 0$, have been employed to study the approximation of the spectrum of l by the spectra of its finite difference analogues [22].

One of the main problems arising in the study of l in nonseparable Hilbert spaces is the relationship between the spectra of l in various spectral spaces. The first result in this direction was Burnat's proof of the equality of spectra of l in $B^2(\boldsymbol{R}^3)$ and $L^2(\boldsymbol{R}^3)$ for the case of periodic locally square integrable V (Burnat [4, 6]). Such an equality for almost periodic pseudodifferential operators was proved by Shubin [30]. For Schrödinger operators his result covers the case of smooth almost periodic potentials. We generalize these results by treating Stepanov almost periodic V. Note that in multichannel scattering the relationship between the spectra of l in the spectral spaces is given by the HVZ theorem [21].

Recently Krupa and Zawisza [24, 25] have shown that the relationship between $L^2(\boldsymbol{R}^\nu)$ and $B^2(\boldsymbol{R}^\nu)$ can be analyzed using the ultrapowers of Hilbert spaces and of self-adjoint operators acting in them. For an ultrafilter U, a Hilbert space H and a self-adjoint operator T in H let H_U and T_U denote the ultrapowers of H and T with respect to U. Furthermore, let A and $\mathfrak{A}$ denote the operators given formally by l in $L^2(\boldsymbol{R}^\nu)$ and $B^2(\boldsymbol{R}^\nu)$ respectively. Krupa and Zawisza show that $\mathfrak{A}$ is unitarily equivalent to A_U restricted to an invariant subspace of $L^2(\boldsymbol{R}^\nu)_U$. Moreover, A is unitarily equivalent to $\mathfrak{A}_U$ restricted to an invariant subspace of $B^2(\boldsymbol{R}^\nu)_U$. This unexpected symmetry puts the Burnat–Shubin theorem on the coincidence of spectra in a general abstract framework.

Another extension of that theorem is contained in a remarkable paper of Kozlov and Shubin [23], dealing with random elliptic operators. The nonseparable Hilbert spaces arising there are not the spaces of almost periodic functions.

When V is periodic, the analysis of l in $B^2(\boldsymbol{R}^\nu)$ is very much simplified. In particular, the almost periodic eigenfunctions of l, called Bloch waves, form an orthonormal basis in $B^2(\boldsymbol{R}^\nu)$, and thus there is a direct connection between the spectral resolutions of l in $L^2(\boldsymbol{R}^\nu)$ and $B^2(\boldsymbol{R}^\nu)$ provided by the Bloch waves expansion of any $L^2(\boldsymbol{R}^\nu)$ function (see Appendix). For almost periodic potentials this connection is certain to be much more difficult. Our results show that the resolvents of l in $L^2(\boldsymbol{R}^\nu)$ and $B^2(\boldsymbol{R}^\nu)$ are integral operators with the same kernel and, furthermore, that for any continuous function θ vanishing at infinity the bounded operators $\theta(l)$ in the two spaces may be "reconstructed" from each other (Section 2).

As was mentioned above, for periodic V the operator $\mathfrak{A}$ has pure point spectrum. No examples of almost periodic potentials are known for which the spectrum of $\mathfrak{A}$ in $B^2(\boldsymbol{R}^\nu)$ would have a continuous component, but presumably such potentials exist. Recently Chojnacki [15] exhibited examples with at least one point in the spectrum which is not an eigenvalue. Further study of such examples is very much needed.

The plan of the paper is as follows. In Section 1 we recall the definition of the space $B^2(\boldsymbol{R}^\nu)$ and define in this space the Schrödinger operator l. In

Section 2 we prove the equality of spectra and in Section 3 the mixing theorem. In the Appendix we discuss the cases of periodic potentials and of first order operators.

Results presented here extend a part of author's thesis [20], written under the supervision of Professor Marek Burnat. I am deeply grateful to Professor Burnat, who has been my teacher for many years, for his help and friendly criticism. His ideas form the backbone of this paper.

1. Spectral analysis in $B^2(\boldsymbol{R}^\nu)$

We start by recalling the definition of the Hilbert space $B^2(\boldsymbol{R}^\nu)$. Let $\mathrm{Trig}(\boldsymbol{R}^\nu)$ denote the space of *trigonometric polynomials*, i.e. of finite linear combinations of exponents $\exp(i\langle\lambda, x\rangle)$, $\lambda \in \boldsymbol{R}^\nu$, where $\langle\cdot,\cdot\rangle$ is the scalar product in $\boldsymbol{R}^\nu$, and let $CAP(\boldsymbol{R}^\nu)$ denote the space of *(uniformly) almost periodic functions* on $\boldsymbol{R}^\nu$, i.e. those functions f whose translates $T_\tau f$, $\tau \in \boldsymbol{R}^\nu$, $(T_\tau f)(x) = f(x-\tau)$, form a precompact set in the topology of uniform convergence. Clearly $\mathrm{Trig}(\boldsymbol{R}^\nu) \subset CAP(\boldsymbol{R}^\nu)$. Simply put, $B^2(\boldsymbol{R}^\nu)$ is the completion of $\mathrm{Trig}(\boldsymbol{R}^\nu)$ in the norm induced by the scalar product $(f, g)_{B^2} = M_x\{f(x)\overline{g(x)}\}$, where $M_x\{h(x)\}$, the *mean* of an almost periodic function h, is defined by

$$M_x\{h(x)\} = \lim_{T\to\infty} \frac{1}{\omega_\nu T^\nu} \int_{|x|<T} h(x)\,dx. \tag{1.1}$$

Here ω_ν denotes the volume of the unit ball in $\boldsymbol{R}^\nu$ (for the existence of the above limit and other properties of the space $CAP(\boldsymbol{R}^\nu)$ see, for instance, Levitan and Zhikov [26]). We shall follow Bass [2] in showing that $B^2(\boldsymbol{R}^\nu)$ is a space of functions.

Let $M^2(\boldsymbol{R}^\nu)$ be the Marcinkiewicz space of complex-valued locally L^2 functions for which the seminorm

$$|||f||| = \left(\limsup_{T\to\infty} \frac{1}{\omega_\nu T^\nu} \int_{|x|<T} |f(x)|^2\,dx\right)^{1/2} \tag{1.2}$$

is finite. In this seminorm $M^2(\boldsymbol{R}^\nu)$ is complete [27]. In order to obtain a Hilbert space with the scalar product mentioned above, we have to consider the set

$$P^2(\boldsymbol{R}^\nu) = \left\{f \in M^2(\boldsymbol{R}^\nu):\ \lim_{T\to\infty} \frac{1}{\omega_\nu T^\nu} \int_{|x|<T} |f(x)|^2\,dx \text{ exists}\right\},$$

which is not a linear subspace of $M^2(\boldsymbol{R}^\nu)$; however, as Bass has shown, the closure in the seminorm (1.2) of any linear subset of $P^2(\boldsymbol{R}^\nu)$ is again a linear subset of $P^2(\boldsymbol{R}^\nu)$. Thus we have some freedom in forming Hilbert spaces contained in $P^2(\boldsymbol{R}^\nu)$, which we shall employ in Section 3. Putting $E = \mathrm{Trig}(\boldsymbol{R}^\nu)$ or $E = CAP(\boldsymbol{R}^\nu)$, and writing $E_0 = \{f \in \bar{E}:\ |||f||| = 0\}$, where the closure is in the seminorm (1.2), we obtain the quotient space $B^2(\boldsymbol{R}^\nu) = \bar{E}/E_0$. Elements of $B^2(\boldsymbol{R}^\nu)$ are classes of functions; if $F \in B^2(\boldsymbol{R}^\nu)$ and $f \in F$, we say that f *represents* F and write $F = (f)$. The scalar product

$$(F, G)_{B^2} = \lim_{T \to \infty} \frac{1}{\omega_\nu T^\nu} \int_{|x|<T} f(x) \overline{g(x)}\, dx \tag{1.3}$$

for $F = (f)$, $G = (g)$ makes $B^2(\boldsymbol{R}^\nu)$ a Hilbert space, with the orthonormal Fourier basis consisting of the elements $(\exp(i\langle \lambda, x\rangle))$, $\lambda \in \boldsymbol{R}^\nu$. Every element F can be expanded in a Fourier series $F \sim \sum_n f_n \exp(i\langle \lambda_n, x\rangle)$ convergent in norm to F, $\|F\|_{B^2}^2 = \sum_n |f_n|^2$.

We say that a function on $\boldsymbol{R}^\nu$ is B^2*-almost periodic* if it is a limit of trigonometric polynomials in the norm $\|\cdot\|_{B^2}$, i.e. if it represents an element of $B^2(\boldsymbol{R}^\nu)$.

There is an alternative way of viewing $B^2(\boldsymbol{R}^\nu)$. With pointwise multiplication $CAP(\boldsymbol{R}^\nu)$ is a (commutative) Banach algebra. Its space of linear-multiplicative functionals, called the *Bohr compactification* of $\boldsymbol{R}^\nu$ and denoted by $b\boldsymbol{R}^\nu$, has a natural group structure. $\boldsymbol{R}^\nu$ can be densely embedded in $b\boldsymbol{R}^\nu$ by a group homomorphism. Any almost periodic function on $\boldsymbol{R}^\nu$ extends uniquely to a continuous function on $b\boldsymbol{R}^\nu$ and the characters of $b\boldsymbol{R}^\nu$ are extensions of the exponents $\exp(i\langle \lambda, x\rangle)$. Thus, clearly, $B^2(\boldsymbol{R}^\nu) \simeq L^2(b\boldsymbol{R}^\nu, d\mu)$, where $d\mu$ denotes the normalized Haar measure. This approach is often useful (cf. [15, 32]) but will not be employed here.

We shall now define the free Schrödinger operator $l_0 = -\Delta$ in $B^2(\boldsymbol{R}^\nu)$. Let

$$\mathscr{D}(\mathfrak{A}_0) = \{F\colon F \sim \sum_n f_n \exp(\langle \lambda_n, x\rangle),\ \sum_n |\lambda_n|^4 |f_n|^2 < +\infty\}$$

and put

$$\mathfrak{A}_0 F \sim \sum_n |\lambda_n|^2 f_n \exp(i\langle \lambda_n, x\rangle) \quad \text{for} \quad F \sim \sum_n f_n \exp(i\langle \lambda_n, x\rangle).$$

The operator $\mathfrak{A}_0$ is self-adjoint, with pure point spectrum $[0, +\infty)$. Moreover, $\mathscr{D} = \{F\colon F = (f), f \in \mathrm{Trig}(\boldsymbol{R}^\nu)\}$ is an essential domain of $\mathfrak{A}_0$, $\mathfrak{A}_0(f) = (-\Delta f)$ for $f \in \mathrm{Trig}(\boldsymbol{R}^\nu)$.

The natural question arises in what sense $\mathfrak{A}_0$ is a differential operator. Recall that if $A_0 = -\Delta$ in $L^2(\boldsymbol{R}^\nu)$, then the domain $\mathscr{D}(A_0)$ of A_0 consists precisely of functions $f \in L^2(\boldsymbol{R}^\nu)$ for which the distributional laplacian $-\Delta f$ is in $L^2(\boldsymbol{R}^\nu)$. The following lemma gives the analogous characterization of $\mathscr{D}(\mathfrak{A}_0)$.

LEMMA 1.1. (a) *If f is a B^2-almost periodic function whose distributional laplacian $g = -\Delta f$ is also B^2-almost periodic, then (f) is in $\mathscr{D}(\mathfrak{A}_0)$ and $\mathfrak{A}_0(f) = (g)$.*

(b) *If $F \in \mathscr{D}(\mathfrak{A}_0)$, then F is represented by a C^∞-function f such that $-\Delta f$ is B^2-almost periodic and $\mathfrak{A}_0 F = (-\Delta f)$.*

Proof. (a) Suppose that $-\Delta f = g$ in the sense of distributions. We have to show that for all $t \in \mathrm{Trig}(\boldsymbol{R}^\nu)$ we have $(\mathfrak{A}_0(t), (f))_{B^2} = ((t), (g))_{B^2}$. Let $\varphi \in C_0^\infty(\boldsymbol{R}^\nu)$, $\varphi(x) = 1$ for $|x| \leqslant 1$, $\varphi(x) = 0$ for $|x| \geqslant 2$, and put $\varphi_T(x) = T^{-\nu}\varphi(T^{-1}x)$, $T > 0$. Then

$$\begin{aligned}(\mathfrak{A}_0(t), (f))_{B^2} &= \lim_{T\to\infty} \int \varphi_T(x)(-\Delta t)(x)\overline{f(x)}\,dx \\ &= \lim_{T\to\infty} \int (-\Delta\varphi_T t)(x)\overline{f(x)}\,dx \\ &= \lim_{T\to\infty} \int \varphi_T(x)t(x)\overline{g(x)}\,dx = ((t), (g))_{B^2},\end{aligned}$$

where the first and last equalities follow from the definition of $B^2(\boldsymbol{R}^\nu)$, the second equality employs the fact that $\operatorname{supp}\operatorname{grad}\varphi_T \subset \{x\colon\ T < |x| < 2T\}$ and the third uses $\varphi_T t \in C_0^\infty(\boldsymbol{R}^\nu)$.

(b) Suppose $F \in \mathcal{D}(\mathfrak{A}_0)$, $\mathfrak{A}_0 F = G$. We shall construct a B^2-almost periodic function f such that $f \in C^\infty(\boldsymbol{R}^\nu)$, $(f) = F$, $-\Delta f$ is also B^2-almost periodic and $(-\Delta f) = G$. The proof is an extension of Marcinkiewicz's [27] proof of the completeness of $M^2(\boldsymbol{R}^\nu)$.

Observe that there exists a sequence t_n of trigonometric polynomials such that $\|(t_n) - F\|_{B^2} \to 0$, $\|(-\Delta t_n) - G\|_{B^2} \to 0$. Let

$$d_a(f) = \sup_{T \geqslant a} T^{-\nu} \int_{|x|<T} |f(x)|^2\,dx.$$

By passing to a subsequence of t_n we can assume that there exists a sequence of positive numbers a_n such that:

(i) $a_1 > 2$, $a_{n+1} > 2a_n$,

(ii) $d_{a_n}(t_n - t_{n+1}) < 2^{-2n}$, $d_{a_n}(-\Delta(t_n - t_{n+1})) < 2^{-2n}$,

(iii) $\displaystyle\lim_{n\to\infty} (a_n)^{-\nu} \sum_{j=1}^{n} \int_{a_n - 1 < |x| < a_n + 1} \{|t_n(x)|^2 + |\nabla t_n(x)|^2 + |\Delta t_n(x)|^2\}\,dx = 0.$

Let $\Omega_n = \{x\colon\ a_n < |x| < a_{n+1}\}$, $\Omega_n' = \{x\colon\ a_n + 1 < |x| < a_{n+1} - 1\}$, and let φ_n be a sequence of $C_0^\infty(\boldsymbol{R}^\nu)$ functions such that $\operatorname{supp}\varphi_n \subset \Omega_n$, φ_n equals 1 on Ω_n' and $\sup_n \|\partial^\alpha \varphi_n\|_\infty < +\infty$ for any multiindex α with $|\alpha| < 2$. Put $f(x) = \sum_n \varphi_n(x) t_n(x)$. We claim that this C^∞-function is B^2-almost periodic and satisfies $(f) = F$, $(-\Delta f) = G$.

Note first that if $g(x) = -\sum_n \varphi_n(x)\Delta t_n(x)$, then $|||-\Delta f - g||| = 0$. This follows from the properties of φ_n, (iii) above and the obvious inequality

$$T^{-\nu} \int_{|x|<T} |-\Delta f(x) - g(x)|^2\,dx \leqslant (a_n)^{-\nu} \sum_{j=1}^{n} \int |t_n \Delta\varphi_n - 2\langle \nabla t_n, \nabla\varphi_n\rangle|^2\,dx$$

for $a_n \leqslant T < a_{n+1}$.

We have to show that $|||t_n - f||| \to 0$, $|||-\Delta t_n - g||| \to 0$. Since f is built out of t_n's in the same way as g out of $-\Delta t_n$'s, these assertions are analogous; we consider only the first of them. Let $\tilde{f}(x)=\sum_n \chi_{\Omega_n}(x)\,t_n(x)$ where χ_{Ω_n} denotes the characteristic function of Ω_n. Property (iii) above implies that $|||f-\tilde{f}|||=0$. Following [27] we now show that $|||t_n-\tilde{f}|||\to 0$. Fix n and suppose $a_k \leqslant T < a_{k+1}$. Then assuming $k>n$ we have

$$T^{-\nu}\int_{|x|<T}|\tilde{f}(x)-t_n(x)|^2\,dx \leqslant T^{-\nu}\int_{|x|<a_n}|\tilde{f}(x)-t_n(x)|^2\,dx$$
$$+\sum_{j=n}^{k-1}T^{-\nu}\int_{\Omega_j}|t_j(x)-t_n(x)|^2\,dx+T^{-\nu}\int_{a_k<|x|<T}|t_k(x)-t_n(x)|^2\,dx.$$

The first term tends to 0 as $T\to\infty$. We shall estimate the second term (the last term is analogous). We have

$$\Big(\int_{\Omega_j}|t_j-t_n|^2\,dx\Big)^{1/2}\leqslant \sum_{s=n}^{j-1}\Big(\int_{\Omega_j}|t_s-t_{s+1}|^2\,dx\Big)^{1/2}$$
$$\leqslant \sum_{s=n}^{j-1}a_{j+1}^{\nu/2}\,d_{a_s}(t_s-t_{s+1})^{1/2}$$
$$\leqslant a_{j+1}^{\nu/2}\,2^{-n+1}$$

where we have used (ii) and $a_{j+1}>a_s$. Repeated use of (i) implies that $a_{j+1}<2^{j+1-k}a_k$, $j<k$, so using $a_k<T$ the second term can be estimated by

$$\sum_{j=n}^{k-1}T^{-\nu}a_{j+1}^{\nu}\,2^{-2n+2}\leqslant \sum_{j=n}^{k-1}2^{(j+1-k)\nu}\,2^{-2n+2}\leqslant 2^{-2n+3}$$

independently of k, i.e. of T. We can thus take the lim sup in T and the above estimates show that $|||t_n-\tilde{f}|||\to 0$. The proof of the lemma is accomplished.

Our aim now is to define the Schrödinger operator $\mathfrak{A}=\mathfrak{A}_0+V$ using the Kato–Rellich theorem. We assume that V is a real-valued Stepanov almost periodic function. Recall [25] that the space $S^pAP(\boldsymbol{R}^\nu)$ of *Stepanov almost periodic functions* is the completion of $\mathrm{Trig}(\boldsymbol{R}^\nu)$ or $CAP(\boldsymbol{R}^\nu)$ in the norm

$$\|f\|_p=\sup_{x\in\boldsymbol{R}^\nu}\Big(\int_Q|f(x+y)|^p\,dy\Big)^{1/p},$$

where $Q=\{x\colon -\frac{1}{2}\leqslant x_i<\frac{1}{2},\ i=1,\ldots,\nu\}$. Alternatively one may characterize S^pAP functions as $L^p_{\mathrm{loc}}(\boldsymbol{R}^\nu)$ functions for which $x\mapsto f(x+\cdot)$ is a (uniformly) almost periodic function on $\boldsymbol{R}^\nu$ with values in $L^p(Q)$. In this paper we consider the case $\nu\geqslant 3$, and assume $V\in S^pAP(\boldsymbol{R}^\nu)$ with $p>\nu/2$ ($p=2$ if $\nu=3$). To prove the relative boundedness of V with respect to $\mathfrak{A}_0$ we need the following lemma.

LEMMA 1.2. *Suppose that W is a uniformly locally $L^p(\boldsymbol{R}^\nu)$ function, $p>\nu/2$ ($p=2$ if $\nu=3$), and that $G(x,y)$ is measurable on $\boldsymbol{R}^\nu\times\boldsymbol{R}^\nu$ and satisfies the* *condition*

$$(1.4)\qquad |G(x, y)| \leqslant \begin{cases} C\exp(-\alpha|x-y|), & |x-y| > 1, \\ C\exp(-\alpha|x-y|)/|x-y|^{\nu-2}, & |x-y| < 1, \end{cases}$$

with $\alpha > 0$. *Then for any* $f \in M^2(\boldsymbol{R}^\nu)$ *the function*

$$u(x) = W(x) \int_{\boldsymbol{R}^\nu} G(x, y) f(y)\, dy$$

is also in $M^2(\boldsymbol{R}^\nu)$ *and* $|||u||| \leqslant C(\alpha)\|W\|_p |||f|||$, *where* $C(\alpha) \to 0$ *as* $\alpha \to \infty$.

Proof. Let

$$I_1(x) = W(x) \int_{|x-y|<1} G(x, y) f(y)\, dy$$

and let $I_2(x)$ be defined analogously with the integral over $\{y\colon |x-y| > 1\}$. We have

$$|I_1(x)| \leqslant |W(x)|\, C \int_{|y|<1} \frac{\exp(-\alpha|y|)}{|y|^{(\nu-\varepsilon)/2}} \frac{|f(y+x)|}{|y|^{(\nu-4+\varepsilon)/2}}\, dy$$

and therefore

$$|I_1(x)|^2 \leqslant C_1(\alpha)|W(x)|^2 \int_{|y|<1} \frac{|f(y+x)|^2}{|y|^{\nu-4+\varepsilon}}\, dy$$

where for sufficiently small $\varepsilon > 0$ the integral is almost everywhere convergent and $C_1(\alpha) \to 0$ as $\alpha \to \infty$. Integrating both sides over $\{x\colon |x| < T\}$ we obtain

$$\int_{|x|<T} |I_1(x)|^2\, dx \leqslant C_1(\alpha) \int_{|y|<1} \int_{|x-y|<T} |W(x-y)|^2 \frac{|f(x)|^2}{|y|^{\nu-4+\varepsilon}}\, dx\, dy.$$

Estimating now the inner integral by the integral over $\{x\colon |x| < T+1\}$, reversing the order of integration and applying Hölder's inequality to the integral over $\{y\colon |y| < 1\}$, we obtain, for sufficiently small ε depending on p,

$$\int_{|x|<T} |I_1(x)|^2\, dx \leqslant C_2(\alpha)\|W\|_p^2 \int_{|x|<T+1} |f(x)|^2\, dx$$

whence the estimate $|||I_1||| \leqslant C(\alpha)\|W\|_p |||f|||$ follows immediately. I_2 can be estimated in a similar way, which ends the proof of the lemma.

Let now $A_0 = -\Delta$ be self-adjoint in $L^2(\boldsymbol{R}^\nu)$, and for $\operatorname{Im}\lambda > 0$ let $G_0(x, y, \lambda^2)$ denote the Green function of A_0, i.e. the integral kernel of $(A_0 - \lambda^2)^{-1}$. As is well known, $G_0(x, y, \lambda^2)$ satisfies the condition (1.4) with $\alpha = \frac{1}{2}\operatorname{Im}\lambda$, and depends only on $|x-y|$. Moreover, for any $f \in C^1(\boldsymbol{R}^\nu) \cap L^\infty(\boldsymbol{R}^\nu)$, the function $u(x) = \int G_0(x, y, \lambda^2) f(y)\, dy$ is in $C^2(\boldsymbol{R}^\nu)$ and satisfies $-\Delta u(x) - \lambda^2 u(x) = f(x)$. Using this we can prove

LEMMA 1.3. *Let* $V \in S^p AP(\boldsymbol{R}^\nu)$, $p > \nu/2$ ($p = 2$ *if* $\nu = 3$). *In* $B^2(\boldsymbol{R}^\nu)$, *multiplication by* V *is an* $\mathfrak{A}_0$*-bounded operator, with relative bound* 0.

Proof. We first consider $(\mathfrak{A}_0-\lambda^2)^{-1}$. By Lemma 1.1, the operator $f \mapsto \int G_0(x, y, \lambda^2) f(y)\,dy$ is bounded in $M^2(\boldsymbol{R}^\nu)$, transforms $CAP(\boldsymbol{R}^\nu)$ into $CAP(\boldsymbol{R}^\nu)$ and so defines a bounded integral operator in $B^2(\boldsymbol{R}^\nu)$, which we shall denote by $R(\lambda^2)$. One shows easily that on $\mathscr{D} = \{F\colon F = (f), f \in \mathrm{Trig}(\boldsymbol{R}^\nu)\}$, $R(\lambda^2) = (\mathfrak{A}_0-\lambda^2)^{-1}$, so by continuity $(\mathfrak{A}_0-\lambda^2)^{-1}$ is an integral operator with kernel $G_0(x, y, \lambda^2)$. In the same way we show that the kernel $V(x)G_0(x, y, \lambda^2)$ defines a bounded integral operator in $B^2(\boldsymbol{R}^\nu)$ whose norm tends to 0 as $\operatorname{Im}\lambda \to \infty$. To define the (unbounded) operator V in $B^2(\boldsymbol{R}^\nu)$, let $\mathscr{D}(V) = \mathscr{D}(\mathfrak{A}_0)$ and given $F \in \mathscr{D}(V)$, put $VF = \big(V(x)\int G_0(x, y, \lambda^2) g(y)\,dy\big)$ for $(g) = (\mathfrak{A}_0-\lambda^2)F$. By the above, V is well defined and the norm of $V(\mathfrak{A}_0-\lambda^2)^{-1}$ is arbitrarily small for $\operatorname{Im}\lambda$ large. This proves the lemma.

The Kato–Rellich theorem and Lemma 1.3 show that $\mathfrak{A} = \mathfrak{A}_0 + V$ with $\mathscr{D}(\mathfrak{A}) = \mathscr{D}(\mathfrak{A}_0)$ is self-adjoint. We are justified in calling $\mathfrak{A}$ the Schrödinger operator in $B^2(\boldsymbol{R}^\nu)$ since for any smooth CAP function f, $\mathfrak{A}(f) = (-\Delta f + Vf)$. Since $S^p AP$ functions are uniformly locally L^p, V is also relatively bounded with respect to A_0 in $L^2(\boldsymbol{R}^\nu)$; let $A = A_0 + V$.

Suppose now that V satisfies the following regularity condition: outside a closed set S of measure 0 consisting of isolated points, curves, etc. V is continuous, and for any open domain $\Omega \subset \boldsymbol{R}^\nu$, $\bar{\Omega} \subset \boldsymbol{R}^\nu \setminus S$, V is a Lipschitz function on Ω (this condition is assumed throughout). In [10] it is proved that the Green function $G(x, y, \lambda^2)$ of A, i.e. the integral kernel of $(A-\lambda^2)^{-1}$, for sufficiently large $\operatorname{Im}\lambda$, $\operatorname{Im}\lambda > K(V)$, satisfies the estimate

$$|G(x, y, \lambda^2) - G_0(x, y, \lambda^2)| \leqslant C\frac{\exp(-\omega|x-y|)}{|x-y|^\beta},$$

with $0 < \omega < \frac{1}{2}(\operatorname{Im}\lambda - K(V))$, $\nu - 4 + \nu/p < \beta < \nu - 2$, the constant C depending on V, λ^2, ω and β. Moreover, the following estimate holds:

$$|G(x+\tau, y+\tau, \lambda^2) - G(x, y, \lambda^2)| \leqslant C\frac{\exp(-\omega|x-y|)}{|x-y|^\beta}\|V(\cdot+\tau) - V(\cdot)\|_p \tag{1.5}$$

with λ^2, ω, β and the constant C as above. (1.5) implies that for $V \in S^p AP(\boldsymbol{R}^\nu)$, $G(x+\tau, y+\tau, \lambda^2)$ is uniformly almost periodic as a function of τ, for fixed $x \neq y$. The regularity condition on V and the exponential decay of $G(x, y, \lambda^2)$ at infinity can be used to establish that for any $f \in C(\boldsymbol{R}^\nu) \cap C^1(\boldsymbol{R}^\nu \setminus S) \cap L^\infty(\boldsymbol{R}^\nu)$, the function $u(x) = \int G(x, y, \lambda^2) f(y)\,dy$ is in $C^2(\boldsymbol{R}^\nu \setminus S)$ and satisfies $-\Delta u(x) + V(x)u(x) - \lambda^2 u(x) = f(x)$ for $x \in \boldsymbol{R}^\nu \setminus S$. Therefore analogously to Lemma 1.3 we obtain

LEMMA 1.4. *Let V be as above. Then for* $\operatorname{Im}\lambda > K(V)$, $(\mathfrak{A}-\lambda^2)^{-1}$ *is an integral operator in $B^2(\boldsymbol{R}^\nu)$ with kernel $G(x, y, \lambda^2)$.*

We see that $(A-\lambda^2)^{-1}$ and $(\mathfrak{A}-\lambda^2)^{-1}$ are integral operators with the same kernel, though acting in quite different spaces. This property can be established for other functions of operators. For instance, if $V \geq 0$, $V \in C^{\infty}(\boldsymbol{R}^{\nu})$, then the results of Davies [17] show that $K_t(x, y)$, the integral kernel of $\exp(-tA)$ in $L^2(\boldsymbol{R}^{\nu})$, satisfies the condition (1.4), and so $\exp(-tA)$ and $\exp(-t\mathfrak{A})$ are integral operators with the same kernel. More general results of this type appear in [23, 25].

The spectral analysis of $\mathfrak{A}$ is a challenging mathematical problem: only relatively simple results are known. In particular, Burnat and Palczewski [11] have shown that the spectrum of $\mathfrak{A}$ is essential, i.e. isolated eigenvalues have infinite multiplicity. We generalize this result below.

Let $m(V)$ be the *almost periodicity module* of V, i.e. the additive subgroup of $\boldsymbol{R}^{\nu}$ generated by $\{\lambda: M_x\{V(x)\exp(-i\langle\lambda, x\rangle)\} \neq 0\}$, and let $\Gamma(V) = \boldsymbol{R}^{\nu}/m(V)$. Let B_γ^2, $\gamma \in \Gamma(V)$, be the natural mutually orthogonal separable subspaces of $B^2(\boldsymbol{R}^{\nu})$ invariant for $\mathfrak{A}$,

$$B_\gamma^2 = \{F: F \sim \sum_{\lambda\in\gamma} f_\lambda \exp(i\langle\lambda, x\rangle)\}.$$

We have

$$\bigoplus_{\gamma\in\Gamma(V)} B_\gamma^2 = B^2(\boldsymbol{R}^{\nu}).$$

Let $\mathfrak{A}_\gamma$ be the operator $\mathfrak{A}$ restricted to B_γ^2. If V is periodic then the Bloch analysis (see Appendix) implies that for any open subset U of $\boldsymbol{R}$ not disjoint from the spectrum of $\mathfrak{A}$, $U \cap \sigma(\mathfrak{A}) \neq \emptyset$, the range of the spectral projection of $\mathfrak{A}$ associated with U is nonseparable (in particular, this implies that isolated eigenvalues have uncountable multiplicity and the spectrum is essential). The same conclusion holds for general S^pAP potentials V, as the following lemma testifies.

LEMMA 1.5. *If V is not periodic then the spectrum $\sigma(\mathfrak{A}_\gamma)$ of $\mathfrak{A}_\gamma$ does not depend on γ and $\sigma(\mathfrak{A}_\gamma) = \sigma(\mathfrak{A})$. Moreover, for each $\gamma \in \Gamma(V)$ the spectrum of $\mathfrak{A}_\gamma$ is essential.*

Proof. Let γ_1, γ_2 be any two elements of $\Gamma(V)$. We shall show that $\sigma(\mathfrak{A}_{\gamma_1}) \subset \sigma(\mathfrak{A}_{\gamma_2})$. Suppose $E \in \sigma(\mathfrak{A}_{\gamma_1})$. Then there exists a sequence $F_n \in B_{\gamma_1}^2$, $\|F_n\|_{B^2} = 1$, $F_n \in \mathscr{D}(\mathfrak{A})$, satisfying $\|(\mathfrak{A}-E)F_n\|_{B^2} \to 0$. We shall construct a sequence G_n with the same properties except that $G_n \in B_{\gamma_2}^2$; this will complete the proof of the first assertion of the lemma. Observe that there exists a sequence $p_n \in \boldsymbol{R}^{\nu}$ such that $|p_n| \to 0$ as $n \to \infty$ and $p_n + m(V) + \gamma_1 = \gamma_2$; the existence of p_n follows from the fact that for nonperiodic V, $m(V)$ is dense in $\boldsymbol{R}^{\nu}$. Let $G_n = E_{p_n}F_n$, where E_λ is the (unitary) operator of multiplication by $\exp(i\langle\lambda, x\rangle)$. We clearly have $\|G_n\|_{B^2} = 1$, $G_n \in \mathscr{D}(\mathfrak{A})$, $G_n \in B_{\gamma_2}^2$, so we have to show that $\|(\mathfrak{A}-E)G_n\|_{B^2} \to 0$. Suppose that $F_n \sim \sum_k f_{nk}\exp(i\langle\lambda_{nk}, x\rangle)$, where $\lambda_{nk} \in \gamma_1$.

Then

$$(\mathfrak{A}-E)\,G_n - E_{p_n}(\mathfrak{A}-E)\,F_n \sim \sum_k f_{nk}(2\langle\lambda_{nk}, p_n\rangle + |p_n|^2)\exp(i\langle\lambda_{nk}+p_n, x\rangle),$$

hence

$$\|(\mathfrak{A}-E)\,G_n\|_{B^2} \leqslant \|(\mathfrak{A}-E)\,F_n\|_{B^2} + |p_n|^2 + |p_n|\Big(\sum_k |f_{nk}|^2\,|\lambda_{nk}|^2\Big)^{1/2}$$

and we have to show that the series in the last term is bounded uniformly in n. But

$$\sum_k |f_{nk}|^2\,|\lambda_{nk}|^2 \leqslant \sum_k |f_{nk}|^2(|\lambda_{nk}|^4+1) = \|\mathfrak{A}_0 F_n\|_{B^2}^2 + 1,$$

and using Lemma 1.3 we see that there exists a constant $b>0$ such that $\|VF_n\|_{B^2} \leqslant b(\|\mathfrak{A}F_n\|_{B^2}+1)$. Hence

$$\|\mathfrak{A}_0 F_n\|_{B^2} \leqslant (b+1)\,\|\mathfrak{A}F_n\|_{B^2} + b$$

which is bounded in n as $\|(\mathfrak{A}-E)\,F_n\|_{B^2} \to 0$.

Now suppose that E is an isolated point of the spectrum of $\mathfrak{A}_\gamma$, $\gamma\in\Gamma(V)$ fixed, and let F be any corresponding normalized eigenelement, $F\in B_\gamma^2$, $\mathfrak{A}F = EF$. Choose a sequence $p_n\in m(V)$ with $|p_n|\to 0$, $p_n\neq 0$. Arguing as above we deduce that $F_n = E_{p_n}F$ is a normalized sequence in B_γ^2 satisfying $\|\mathfrak{A}-E)\,F_n\|_{B^2}\to 0$. The second assertion of the lemma follows when we notice that $F_n\to 0$ weakly, so E cannot be of finite multiplicity. The lemma is proved.

When V is periodic we have the direct integral decomposition $L^2(\boldsymbol{R}^\nu) \simeq \int_{\Gamma(V)} B_\gamma^2\,d\gamma$ (cf. [28]). For nonperiodic V the difficult problem of "noncommutative" integration over $\Gamma(V)$ is discussed by Bellisard and Testard [3], who do not obtain the results of the above lemma. This points to a relative simplicity of nonseparable space approach.

We end this section with two simple results on the spectral properties of $\mathfrak{A}$. We say that the interval (a, b) is a *gap* in the spectrum of $\mathfrak{A}$ if $(a, b)\cap\sigma(\mathfrak{A})\neq\emptyset$. Since, as we prove in the next section, $\sigma(\mathfrak{A}) = \sigma(A)$, the following result is included in the more general result of Wong [34]. However, with the theory we have developed so far its proof is immediate.

LEMMA 1.5. *Suppose (a, b) is a gap in the spectrum of $\mathfrak{A}$, with $a\geqslant -\|V\|_{B^2}$. Then $b-a\leqslant 2\|V\|_{B^2}$.*

Proof. Let $E\geqslant 0$, and choose $\lambda\in\boldsymbol{R}^\nu$ such that $|\lambda|^2 = E$. Putting $F = (\exp(i\langle\lambda, x\rangle))$, we have $\|F\|_{B^2} = 1$, and $\|(\mathfrak{A}-E)\,F\|_{B^2}\leqslant\|V\|_{B^2}$ which guarantees that $\sigma(\mathfrak{A})\cap(E-\|V\|_{B^2}, E+\|V\|_{B^2})$ is not empty. This proves the lemma.

Our final result in this section concerns eigenfunctions. We say that u_E is a *classical eigenfunction* of l associated with the eigenvalue E iff

$u_E \in C^1(\boldsymbol{R}^\nu) \cap C^2(\boldsymbol{R}^\nu \backslash S)$ and $-\Delta u_E(x) + V(x) u_E(x) = E u_E(x)$ for $x \in \boldsymbol{R}^\nu \backslash S$. In general we cannot expect that every eigenelement of $\mathfrak{A}$ is represented by a classical eigenfunction. However, the following weaker result holds.

LEMMA 1.6. *Suppose that F is a normalized eigenelement of $\mathfrak{A}$ associated with the eigenvalue $E \in \sigma(\mathfrak{A})$. If F is represented by an $S^2 AP(\boldsymbol{R}^\nu)$ function, then it is also represented by a classical almost periodic eigenfunction.*

Proof. Let $F = (f)$, $f \in S^2 AP(\boldsymbol{R}^\nu)$. Clearly, $(\mathfrak{A} - \lambda^2)^{-1} F = (E - \lambda^2)^{-1} F$, hence the function

$$g(x) = (E - \lambda^2)^k \int G_k(x, y, \lambda^2) f(y) dy$$

represents F, $F = (g)$, for any natural k, where G_k is the kth iteration of the Green function $G(x, y, \lambda^2)$. For sufficiently large k, G_k has no singularity at $x = y$, and since it vanishes exponentially for $|x-y|$ large, and $G_k(x+\tau, y+\tau, \lambda^2)$ is a CAP function of τ, we conclude that $g \in CAP(\boldsymbol{R}^\nu)$. Applying the integral operator once again, we find that the function

$$\tilde{g}(x) = (E - \lambda^2) \int G(x, y, \lambda^2) g(y) dy$$

is also in $CAP(\boldsymbol{R}^\nu)$, and $(\tilde{g}) = (g)$. No element of $B^2(\boldsymbol{R}^\nu)$ can be represented by two distinct CAP functions, so $\tilde{g} = g$ everywhere. We conclude that $g(x) = (E - \lambda^2) \int G(x, y, \lambda^2) g(y) dy$, hence g is a classical eigenfunction of l representing the element F.

2. Equality of spectra

In this section we prove the main result of the paper.

THEOREM 2.1. *Suppose $V \in S^p AP(\boldsymbol{R}^\nu)$, $p > \nu/2$ ($p = 2$ if $\nu = 3$). Then $\sigma(\mathfrak{A}) = \sigma(A)$.*

Proof. We do not go into all the details of the proof since, as was noted in the introduction, several variants of this theorem already exist in the literature. Consider first the inclusion $\sigma(\mathfrak{A}) \subset \sigma(A)$. Let φ_T, for $T > 0$, be a family of C_0^∞ functions equal to $(\omega_\nu T^\nu)^{-1/2}$ on $\{x: |x| < T\}$ and 0 on $\{x: |x| > T+1\}$, with equibounded derivatives. For $F = (f)$, $f \in \mathrm{Trig}(\boldsymbol{R}^\nu)$, we have $\varphi_T f \in C_0^\infty(\boldsymbol{R}^\nu)$, and $\|\varphi_T f\|_{L^2} \to \|F\|_{B^2}$, $\|(A-E)\varphi_T f\|_{L^2} \to \|(\mathfrak{A} - E) F\|_{B^2}$ as $T \to \infty$, for any real E. Therefore if $E \notin \sigma(A)$, i.e. if there exists a constant C_E such that for any $g \in \mathcal{D}(A)$, $\|(A-E)g\|_{L^2} \geqslant C_E \|g\|_{L^2}$, we have by the above $\|(\mathfrak{A} - E) F\|_{B^2} \geqslant C_E \|F\|_{B^2}$ for the elements F represented by trigonometric polynomials. Hence $E \notin \sigma(\mathfrak{A})$.

To show the inverse inclusion, let ψ_n be a family of $CAP(\boldsymbol{R}^\nu)$ functions satisfying the following conditions:

(a) $\psi_n \geqslant 0$, and $\forall \varepsilon > 0\ \exists N\ \forall n > N\ \forall x \in \boldsymbol{R}^\nu$ $\psi_n(x) \neq 0$ implies that x satisfies $\|V(\cdot + x) - V(\cdot)\|_p < \varepsilon$.

(b) Writing $\theta_n(\alpha) = M_x\{\psi_n(x-\alpha)\psi_n(x)\}$, we have for any $W \in C_0^\infty(\boldsymbol{R}^\nu)$

$$\lim_{n\to\infty} \int W(\alpha)\theta_n(\alpha)\,d\alpha = W(0).$$

The explicit construction of such ψ_n's is given in [19]. We now proceed analogously to the first part of the proof. For $f \in C_0^\infty(\boldsymbol{R}^\nu)$, let $f_n = f * \psi_n$, where $*$ denotes convolution. f_n is in $CAP(\boldsymbol{R}^\nu)$ and represents an element $F_n = (f_n) \in \mathscr{D}(\mathfrak{A})$. It is enough to show that

$$\lim_{n\to\infty} \|F_n\|_{B^2} = \|f\|_{L^2} \quad \text{and} \quad \lim_{n\to\infty} \|(\mathfrak{A}-E)F_n\|_{B^2} = \|(A-E)f\|_{L^2}.$$

The first of these follows from (b), and the second will follow if we show that

$$\lim_{n\to\infty} (H_n - \mathfrak{A}F_n, F_n)_{B^2} = 0 \quad \text{and} \quad \lim_{n\to\infty} (H_n - \mathfrak{A}F_n, \mathfrak{A}F_n)_{B^2} = 0,$$

where $H_n = ((Af) * \psi_n)$. Consider, for example, the first of these. We have

$$(H_n - \mathfrak{A}F_n, F_n)_{B^2}$$
$$= \lim_{T\to\infty} \frac{1}{\omega_\nu T^\nu} \int_{|x|<T} \int_{\boldsymbol{R}^\nu} \int_{\boldsymbol{R}^\nu} f(t)\overline{f(t')}\,[V(t)-V(x)]\,\psi_n(x-t')\psi_n(x-t)\,dt\,dt'\,dx.$$

Let $S > 0$ be such that $\operatorname{supp} f \subset \{x\colon |x| < S\}$. Using the boundedness of f we have

$$|(H_n - \mathfrak{A}F_n, F_n)_{B^2}| \leqslant C \lim_{T\to\infty} \frac{1}{\omega_\nu T^\nu} \int_{|t|<S} \int_{|t'+t|<S} \int_{|x+t|<T} |V(t) - V(t+x)|$$
$$\times \psi_n(x)\psi_n(x-t')\,dx\,dt'\,dt.$$

Estimating the inner integrals by the integrals over $\{t'\colon |t'| < 2S\}$ and $\{x\colon |x| < T+S\}$, and using Hölder's inequality for the integral over $\{t\colon |t| < S\}$, we have the bound

$$\leqslant C_1 \lim_{T\to\infty} \frac{1}{\omega_\nu T^\nu} \int_{|t|<2S} \int_{|x|<T+S} \|V(\cdot) - V(\cdot+x)\|_p\,\psi_n(x)\psi_n(x-t')\,dx\,dt'.$$

By (a) and (b), this tends to 0 as n tends to infinity. The theorem is proved.

The case of periodic V shows that while $\sigma(\mathfrak{A})$ and $\sigma(A)$ are equal as sets, their spectral nature may be quite distinct (see Appendix). The lemma below suggests that the spectral measures of $\mathfrak{A}$ and A for a given finite interval are closely related. φ_T and ψ_n are as in the above proof, except that now we do not need the smoothness of φ_T (unlike Theorem 2.1, Lemma 2.2 requires the regularity condition on V).

LEMMA 2.2 (Reconstruction lemma). *Assume V is as in the above theorem, and satisfies the regularity condition of Section 1. Let θ be any continuous* *function on* $\boldsymbol{R}$ *vanishing at infinity. For any* $F_i = (f_i) \in B^2(\boldsymbol{R}^\nu)$, $i = 1, 2$, *we have*

$$(2.1) \qquad (F_1, \theta(\mathfrak{A}) F_2)_{B^2} = \lim_{T \to \infty} (\varphi_T f_1, \theta(A)(\varphi_T f_2))_{L^2}.$$

For any $g_i \in C_0^\infty(\boldsymbol{R}^\nu)$, $i = 1, 2$, *we have*

$$(2.2) \qquad (g_1, \theta(A) g_2)_{L^2} = \lim_{n \to \infty} ((g_1 * \psi_n), \theta(\mathfrak{A})(g_2 * \psi_n))_{B^2}.$$

Proof. We consider only (2.2); (2.1) can be treated analogously. We shall employ Lemma 1.3. Indeed, observe that both $\mathfrak{A}$ and A are bounded below, say by μ_0. Choose $\mu < \mu_0$ such that the resolvents $(\mathfrak{A} - \mu)^{-1}$ and $(A - \mu)^{-1}$ are integral operators with the same kernel $G(x, y, \mu)$. By the Stone–Weierstrass theorem, the function $\theta(t)$ can be approximated uniformly on $\sigma(\mathfrak{A})$ and $\sigma(A)$ by polynomials in $(t - \mu)^{-1}$, so it is enough to prove (2.2) when θ is such a polynomial. We treat the case $\theta(t) = (t - \mu)^{-1}$, since higher powers of $(t - \mu)^{-1}$ are easier to deal with (the kernels of the operators $(\mathfrak{A} - \mu)^{-k}$ and $(A - \mu)^{-k}$, the iterated Green functions $G_k(x, y, \mu)$, have weaker singularities at $x = y$ and satisfy (1.5) with lesser β).

Given $g_i \in C_0^\infty(\boldsymbol{R}^\nu)$, $i = 1, 2$, let $g_3 = (A - \mu)^{-1} g_2$, and let $G_{i,n} = (g_i * \psi_n)$, $i = 1, 2, 3$. Since $(G_{1,n}, G_{3,n})_{B^2} \to (g_1, (A - \mu)^{-1} g_2)_{L^2}$ as $n \to \infty$, we have to prove only that $(G_{1,n}, (\mathfrak{A} - \mu)^{-1} G_{2,n} - G_{3,n})_{B^2}$ tends to 0 as $n \to \infty$. We have

$$\begin{aligned} &|(G_{1,n}, (\mathfrak{A} - \mu)^{-1} G_{2,n} - G_{3,n})_{B^2}| \\ &\leqslant \lim_{T \to \infty} \frac{1}{\omega_\nu T^\nu} \int_{|x| < T} \iiint |g_1(x - s)|\, |g_2(y)|\, |G(x, y + s', \mu) - G(x - s', y, \mu)| \\ &\qquad\qquad \times \psi_n(s) \psi_n(s')\, ds'\, ds\, dy\, dx \end{aligned}$$

Fix $\varepsilon > 0$; then for sufficiently large n we have, by (b) (in the proof of Theorem 2.1) and (1.5)

$$|G(x, y + s', \mu) - G(x - s', y, \mu)|\, \psi_n(s') \leqslant \varepsilon C \frac{\exp(-\omega |x - s' - y|)}{|x - s' - y|^\beta} \psi_n(s').$$

Therefore putting

$$h_1(x) = |g_1(x)|, \qquad h_2(x) = C \int \frac{\exp(-\omega |x - y|)}{|x - y|^\beta} |g_2(y)|\, dy,$$

and $H_{i,n} = (h_i * \psi_n)$, $i = 1, 2$, we obtain

$$|(G_{1,n}, (\mathfrak{A} - \mu)^{-1} G_{2,n} - G_{3,n})_{B^2}| \leqslant \varepsilon (H_{1,n}, H_{2,n})_{B^2} \to \varepsilon (h_1, h_2)_{L^2}$$

as $n \to \infty$, which by the arbitrariness of ε proves the lemma.

Using the Reconstruction Lemma, we can give a simple proof that $\sigma(A)$ is essential (an L^2 proof for V uniformly almost periodic is given in Avron and Simon [1]).

LEMMA 2.3. *If V satisfies the conditions of Lemma 2.2, $\sigma(A)$ is essential.*

Proof. Suppose E is an isolated point of $\sigma(\mathfrak{A}) = \sigma(A)$, of finite multiplicity in $L^2(\boldsymbol{R}^\nu)$. Let $F = (f)$ be the corresponding eigenelement in $B^2(\boldsymbol{R}^\nu)$, and g_i, $i = 1, \ldots, N$, the corresponding eigenfunctions in $L^2(\boldsymbol{R}^\nu)$. Let θ be a continuous function with $\theta(E) = 1$ and $\operatorname{supp}\theta \cap \sigma(A) = \{E\}$. Assuming F, g_i are normalized we have

$$(F, \theta(\mathfrak{A})F)_{B^2} = 1, \quad (\varphi_T f, \theta(A)(\varphi_T f))_{L^2} = \sum_{i=1}^{N} |(\varphi_T f, g_i)_{L^2}|^2,$$

so the Reconstruction Lemma implies that

$$\lim_{T\to\infty} \sum_{i=1}^{N} |(\varphi_T f, g_i)_{L^2}|^2 = 1,$$

which is impossible since $\lim_{T\to\infty} |(\varphi_T f, g_i)_{L^2}|^2 = 0$ for $i = 1, \ldots, N$. The lemma is proved.

3. The spectral mixing theorem

The spectral mixing theorem proved in this section is a generalization of the spectral cut-off theorem of Burnat [9] and together with perturbation results [5, 10] is an example of applications of nonseparable Hilbert spaces to L^2-theory. We want to find conditions for cut-off functions ϱ_i under which for any $V_i \in S^p AP(\boldsymbol{R}^\nu)$, $i = 1, \ldots, N$, the inclusion

$$\bigcup_{i=1}^{N} \sigma(-\Delta + V_i) \subset \sigma\Big(-\Delta + \sum_{i=1}^{N} \varrho_i V_i\Big)$$

holds, where the spectra of operators are understood as spectra in $L^2(\boldsymbol{R}^\nu)$. The physical intuition behind this inclusion is as follows. Suppose V_i describe some infinite media (crystals or alloys). We want to find conditions under which the energy properties of an electron moving in the medium composed of large parts of the given N media cut out and "glued" in some way include the properties of the electron in any of these (infinite) media. We note that for $\nu = 1$, V periodic, Carmona [12] proved a similar inclusion (moreover, in his case $\sigma(-d^2/dx^2 + V) \subset \sigma_{ac}(-d^2/dx^2 + \varrho V)$).

We say that a cut-off function ϱ is *admissible* if it is $C^2(\boldsymbol{R}^\nu)$ and the following requirements are met:

(a) $\varrho \geqslant 0$, $\varrho, \varrho^2 \in P^2(\boldsymbol{R}^\nu)$, $|||\varrho(1-\varrho)||| = 0$, $|||D^\alpha \varrho||| = 0$ for every multiindex α with $1 \leqslant |\alpha| \leqslant 2$, and $|||\varrho||| > 0$.

(b) There exist constants $K_1, K_2 > 0$ such that for any CAP function f we have $K_1 |||\varrho f||| \leqslant |||f||| \leqslant K_2 |||\varrho f|||$.

It follows that for every admissible function ϱ and every $f \in CAP(\boldsymbol{R}^\nu)$ we have $\varrho f \in P^2(\boldsymbol{R}^\nu)$. A typical admissible function may be obtained as follows.

For any open cone $\Gamma \subset \boldsymbol{R}^\nu$ and any $r > 0$ let $\Gamma_r = \{x\colon\ x \in \Gamma,\ |x| > r\}$, and let χ_{Γ_r} denote the characteristic function of Γ_r. Then if j is a nonnegative C_0^∞ function with $\int j(x)\,dx = 1$, $\varrho = j * \chi_{\Gamma_r}$ is admissible. Clearly, no function with compact support is admissible.

We say that a finite family of admissible cut-off functions $\{\varrho_i\}$, $i = 1, \ldots, N$, is *admissible* if $|||\varrho_i \varrho_j||| = 0$ for any $i \neq j$. It follows that for an admissible family $\{\varrho_i\}$ and any $f_i \in CAP(\boldsymbol{R}^\nu)$, $i = 1, \ldots, N$, we have $\sum_{i=1}^N \varrho_i f_i \in P^2(\boldsymbol{R}^\nu)$.

If in the above construction of admissible cut-off functions we take disjoint cones Γ_i and any r_i, then the resulting ϱ_i will form an admissible family. A family consisting of one admissible cut-off function is admissible.

We will prove the following theorem.

THEOREM 3.1. *Suppose $V_i \in S^p AP(\boldsymbol{R}^\nu)$, and $\{\varrho_i\}$ is an admissible family of cut-off functions, $i = 1, \ldots, N$. Let V_0 be a locally L^p potential vanishing at infinity, and put*

$$V_{\text{mix}} = \sum_{i=1}^N \varrho_i V_i + V_0.$$

Let $A_i = -\Delta + V_i$, $A_{\text{mix}} = -\Delta + V_{\text{mix}}$ be self-adjoint in $L^2(\boldsymbol{R}^\nu)$. Then for any i, $i = 1, \ldots, N$, $\sigma(A_i) \subset \sigma(A_{\text{mix}})$.

Proof. Given an admissible family $\{\varrho_i\}$ we shall construct Hilbert spaces B^2_{mix} and B^2_i, $i = 1, \ldots, N$, and we shall verify for them the following assertions:

(i) $B^2_{\text{mix}} = \bigoplus_i B^2_i$,

(ii) B^2_i are invariant subspaces for the self-adjoint operator $\mathfrak{A}_{\text{mix}} = -\Delta + V_{\text{mix}}$ in B^2_{mix},

(iii) the spectrum of $\mathfrak{A}_{\text{mix}}$ reduced to the subspace B^2_i is equal to the spectrum of $\mathfrak{A}_i = -\Delta + V_i$ in $B^2(\boldsymbol{R}^\nu)$,

(iv) $\sigma(\mathfrak{A}_{\text{mix}}) \subset \sigma(A_{\text{mix}})$.

The result of the theorem will then follow upon application of Theorem 2.1.

We shall exploit Bass's scheme discussed in Section 1. Let

$$E_{\text{mix}} = \{f = \sum_{i=1}^N \varrho_i t_i\colon\ t_i \in \text{Trig}(\boldsymbol{R}^\nu)\}, \qquad E_i = \{f = \varrho_i t\colon\ t \in \text{Trig}(\boldsymbol{R}^\nu)\}.$$

We have $E_{\text{mix}}, E_i \subset P^2(\boldsymbol{R}^\nu)$, so we obtain as in Section 1 the Hilbert spaces B^2_{mix} and B^2_i, $i = 1, \ldots, N$. Assertion (i) above follows because $\{\varrho_i\}$ is an admissible family. Let $\mathfrak{A}_{\text{mix}} = -\Delta + V_{\text{mix}}$ with domain $\mathscr{D}(\mathfrak{A}_{\text{mix}}) = \{F \in B^2_{\text{mix}}\colon F = (f), f \in E_{\text{mix}}\}$ be defined by $\mathfrak{A}_{\text{mix}}(f) = ((-\Delta + V_{\text{mix}})f)$ for $f \in E_{\text{mix}}$. This operator is well defined. Indeed, multiplication by V_0 is a zero operator in B^2_{mix}, and for $f = \sum_{i=1}^N \varrho_i t_i \in E_{\text{mix}}$ we see, using property (a) of admissible functions that

$$\mathfrak{A}_{\mathrm{mix}}(f) = \sum_{i=1}^{N} ((-\Delta + V_{\mathrm{mix}})\varrho_i t_i) = \sum_{i=1}^{N} (\varrho_i(-\Delta + V_i)t_i).$$

This also shows that B_i^2 are invariant for $\mathfrak{A}_{\mathrm{mix}}$. Let now J_i: $B^2(\boldsymbol{R}^\nu) \to B_i^2$ be defined by $J_i(f) = (\varrho_i f)$. Property (b) of admissible functions implies that J_i are bounded operators with bounded inverses. Moreover, $J_i^{-1}\mathscr{D}(\mathfrak{A}_{\mathrm{mix}}) \subset \mathscr{D}(\mathfrak{A}_i)$ and $J_i\mathfrak{A}_i J_i^{-1}F = \mathfrak{A}_{\mathrm{mix}}F$ for $F \in B_i^2$, $F \in \mathscr{D}(\mathfrak{A}_{\mathrm{mix}})$. Hence $\mathfrak{A}_{\mathrm{mix}}$ reduced to the subspace B_i^2 is essentially self-adjoint, so the same is true of $\mathfrak{A}_{\mathrm{mix}}$ in B_{mix}^2. Denote its self-adjoint closure again by $\mathfrak{A}_{\mathrm{mix}}$. We thus obtain assertion (ii). The operators J_i can now be used to establish assertion (iii). The final assertion (iv) can be proved just as the easier inclusion of spectra in the proof of Theorem 2.1. The theorem is proved.

When $N = 1$ we recover the spectral cutting theorem of Burnat [9]. If the potentials V_i are periodic, then $\mathfrak{A}_{\mathrm{mix}}$ reduced to B_i^2, and hence also $\mathfrak{A}_{\mathrm{mix}}$ in B_{mix}^2, have a complete set of eigenelements (cf. Appendix and the properties of J_i). It would be interesting to prove an analogue of Lemma 1.6 and using it find the classical eigenfunctions representing these eigenelements.

Appendix

Here we discuss two simpler cases when the analysis of differential operators with almost periodic coefficients in $B^2(\boldsymbol{R}^\nu)$ can be carried much further than for the Schrödinger operator with general $S^p AP$ potential. These are the case of the Schrödinger operator with periodic potential and the case of the first order operator $i^{-1}d/dx + q$, $q \in CAP(\boldsymbol{R})$, in $B^2(\boldsymbol{R})$. They are meant to illustrate the range of possibilities to be expected in the $B^2(\boldsymbol{R}^\nu)$ analysis.

Let V be periodic with periodicity matrix $\mathscr{A}$, $V(x + \mathscr{A}m) = V(x)$ a.e. for all $m \in \boldsymbol{Z}^\nu$, where $\boldsymbol{Z}$ is the group of integers. Let Q be the *Seitz–Wigner cell* of the periodicity lattice $\Lambda = \{x\colon x = \mathscr{A}m,\ m \in \boldsymbol{Z}^\nu\}$, i.e. $Q = \{x\colon x \text{ is closer to } 0$ than to any other element of $\Lambda\}$, and let $|Q|$ denote its measure. The *Brillouen zone* Γ is the Seitz–Wigner cell of the inverse lattice $\Lambda' = \{k\colon \langle k, x\rangle \in 2\pi\boldsymbol{Z}$ for all $x \in \Lambda\}$. Putting $M = 2\pi(\mathscr{A}^*)^{-1}$ we see that $\Lambda' = \{k\colon k = Mm,\ m \in \boldsymbol{Z}^\nu\}$. We assume that V is locally L^p, $p > \nu/2$ ($p = 2$ if $\nu = 3$), so

$$V(x) = \sum_{m \in \boldsymbol{Z}^\nu} v_m \exp(i\langle x, Mm\rangle)$$

with the uniformly locally L^2 convergence.

For $\gamma \in \Gamma$ consider the subspace B_γ^2 of $B^2(\boldsymbol{R}^\nu)$,

$$B_\gamma^2 = \{F \in B^2(\boldsymbol{R}^\nu)\colon F \sim \sum_{m \in \boldsymbol{Z}^\nu} f_m \exp(i\langle x, Mm + \gamma\rangle)\}.$$

B^2_γ are mutually orthogonal, their direct sum is $B^2(\boldsymbol{R}^\nu)$, they are separable and invariant for $\mathfrak{A}$. Moreover, under the isomorphism J_γ: $B^2_\gamma \to L^2(Q)$ given by

$$J_\gamma F = \frac{1}{|Q|^{1/2}} \sum_{m \in \boldsymbol{Z}^\nu} f_m \exp(i\langle x, Mm\rangle) \quad \text{for } F \sim \sum_{m \in \boldsymbol{Z}^\nu} f_m \exp(i\langle x, Mm+\gamma\rangle),$$

the operator $\mathfrak{A}$ reduced to B^2_γ is unitarily equivalent to A_γ in $L^2(Q)$,

$$A_\gamma u = -\Delta u - 2i\langle\gamma, \nabla u\rangle + |\gamma|^2 u + Vu,$$

with periodic boundary conditions. A_γ is an analytic type A family of self-adjoint operators with compact resolvents. The following theorem demonstrates that nonseparable Hilbert spaces provide a full alternative theory of periodic Schrödinger operators (for the L^2-theory see Reed and Simon [28]).

THEOREM A1. *There exist continuous on Γ real-valued functions $E_n(\gamma)$ and $B^2(\boldsymbol{R}^\nu)$-valued functions $F_n(\gamma)$ such that:*

(i) $F_n(\gamma) \in B^2_\gamma \cap \mathscr{D}(\mathfrak{A})$,

(ii) $\mathfrak{A}F_n(\gamma) = E_n(\gamma) F_n(\gamma)$,

(iii) *$F_n(\gamma)$ form an orthonormal basis in $B^2(\boldsymbol{R}^\nu)$,*

(iv) *$E_n(\gamma) \to \infty$ as $n \to \infty$ uniformly on Γ,*

(v) *$F_n(\gamma)$ are represented by classical eigenfuctions, called Bloch waves, of the form $f_{n,\gamma}(x) = \exp(i\langle\gamma, x\rangle) w_{n,\gamma}(x)$, where $w_{n,\gamma}$ are periodic with periodicity matrix $\mathscr{A}$.*

The operator $\mathfrak{A}$ has band spectrum. Moreover, the transformation $\hat{\ }$: $\mathscr{S}(\boldsymbol{R}^\nu) \to \sum_{n=1}^{\infty} L^2(\Gamma)$ *defined by*

$$\hat{\varphi}_n(\gamma) = (2\pi)^{-\nu/2} \int \varphi(x) \overline{f_{n,\gamma}(x)}\, dx$$

extends to a unitary operator on $L^2(\boldsymbol{R}^\nu)$ with inverse

$$\varphi(x) = \text{l.i.m.} \sum_{n=1}^{\infty} \int_\Gamma \hat{\varphi}_n(\gamma) f_{n,\gamma}(x)\, d\gamma$$

which diagonalizes A: $(A\varphi)^{\hat{}}_n(\gamma) = E_n(\gamma)\hat{\varphi}_n(\gamma)$.

Proof. We briefly outline the proof. To obtain the first part of the theorem we need only show that $F_n(\gamma)$ can be represented by Bloch waves; this can be performed with the use of Lemma 1.6, since the elements of B^2_γ are represented by $S^2 AP$ functions. The second part of the theorem follows if we expand $\exp(i\langle\lambda, x\rangle)$ in terms of the Bloch basis and $F_n(\gamma)$ in terms of the Fourier basis, and apply these expansions in the Fourier inversion formula (see [20] for details).

We shall now discuss the case of the first order operator $h = i^{-1} d/dx + q$, $q \in CAP(\boldsymbol{R})$. Defined as a self-adjoint operator H in $L^2(\boldsymbol{R})$, it has absolutely continuous spectrum $\sigma(H) = \sigma_{ac}(H) = \boldsymbol{R}$, its non-square integrable eigenfunctions are

$$u_\mu(x) = \exp\Big(-i\int_0^x q(u)\,du + i\mu x\Big), \qquad hu_\mu = \mu u_\mu,$$

$\mu \in \boldsymbol{R}$, and multiplication by u_0 in $L^2(\boldsymbol{R})$ provides a unitary equivalence between H and $H_0 = i^{-1}d/dx$. As we now demonstrate, the situation in $B^2(\boldsymbol{R})$ is more interesting.

Let $\mathfrak{H} = i^{-1}d/dx + q$ in $B^2(\boldsymbol{R})$ be a self-adjoint operator with domain

$$\mathscr{D}(\mathfrak{H}) = \Big\{F \sim \sum_n a_n \exp(i\lambda_n x)\colon \sum_n |\lambda_n a_n|^2 < +\infty\Big\}.$$

If u_μ is in $CAP(\boldsymbol{R})$, which happens exactly when

$$x \mapsto \int_0^x q(u)\,du - M_t\{q(t)\}\,x$$

is in $CAP(\boldsymbol{R})$, then u_μ represents an eigenelement of $\mathfrak{H}$ in $B^2(\boldsymbol{R})$. However, the uniform almost periodicity of u_μ is not necessary for this: it is enough that u_μ is a Besicovitch almost periodic function, i.e. that it is a $\|\cdot\|_{B^2}$-limit of trigonometric polynomials. Helson [18] (see also Chojnacki [13]) proved that $\mathfrak{H}$ has either pure point or purely continuous spectrum, and established criteria to distinguish these two cases. Our aim here is to rederive these results in a very simple way. For every $t \in \boldsymbol{R}$, let

$$\Phi_t(x) = \exp\Big(i\int_x^{x+t} q(u)\,du\Big).$$

Since Φ_t is in $CAP(\boldsymbol{R})$ for any $t \in \boldsymbol{R}$, we can define $\varphi(t) = M_x\{\Phi_t(x)\}$. Let T_t denote the translation, $T_t f(x) = f(x+t)$; T_t extends to a group of unitary operators in $B^2(\boldsymbol{R})$. Putting $\mathfrak{H}_0 = i^{-1}d/dx$ with $\mathscr{D}(\mathfrak{H}_0) = \mathscr{D}(\mathfrak{H})$, we see that $\exp(it\mathfrak{H}_0) = T_t$.

THEOREM A2. *The spectrum $\sigma(\mathfrak{H}) = \boldsymbol{R}$ of $\mathfrak{H}$ is either purely continuous or pure point according as the limit*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\varphi(t)|^2\,dt \tag{A.1}$$

vanishes or not. The latter case occurs exactly when $\varphi \in CAP(\boldsymbol{R})$.

Proof. For any element $G = (g(x))$ of $B^2(\boldsymbol{R}^\nu)$ and $\mu \in \boldsymbol{R}$, let $G_\mu = (g(x)\exp(i\mu x))$. Write $I = (1)$.

Suppose $E \in \sigma(\mathfrak{H})$, and let G_n be a sequence of normalized elements such that $\|(\mathfrak{H}-E)G_n\|_{B^2} \to 0$ as $n \to \infty$. Then for any $\mu \in \boldsymbol{R}$ we have $\|G_{n,\mu}\|_{B^2} = 1$ and $\|(\mathfrak{H}-E-\mu)G_{n,\mu}\|_{B^2} \to 0$, showing that $E+\mu \in \sigma(\mathfrak{H})$. Hence $\sigma(\mathfrak{H}) = \boldsymbol{R}$.

We shall now show that if $\mathfrak{H}$ has an eigenelement, then its eigenelements span $B^2(\boldsymbol{R})$. Indeed, suppose that $\mathfrak{H}F = EF$, $\|F\|_{B^2} = 1$. Then F_μ, for $\mu \in \boldsymbol{R}$, are the eigenelements of $\mathfrak{H}$ associated with the eigenvalues $E+\mu$, so they are orthonormal.

We will show that P, the projection onto the subspace of $B^2(\boldsymbol{R})$ spanned by F_μ, $\mu \in \boldsymbol{R}$, is the identity operator by proving that for any $\lambda \in \boldsymbol{R}$, $\|PI_\lambda\|_{B^2} = 1$. Indeed,

$$\|PI_\lambda\|_{B^2}^2 = \sum_\mu |(F_\mu, I_\lambda)_{B^2}|^2 = \sum_\mu |(F, I_{\lambda-\mu})_{B^2}|^2 = \|F\|_{B^2}^2 = 1.$$

Using the Kato–Trotter formula we obtain $\exp(it\mathfrak{H}) = \Phi_t T_t$ (this group of unitary operators has the same form as $\exp(itH)$ in $L^2(\boldsymbol{R})$).

Suppose that $\mathfrak{H}$ has pure point spectrum. Then for any element G in $B^2(\boldsymbol{R})$ the trajectory $\exp(it\mathfrak{H})G$ is precompact in norm, so the function $t \mapsto (\exp(it\mathfrak{H})G, G)_{B^2}$ is CAP for any G. Putting $G = I$ we obtain $\varphi \in CAP(\boldsymbol{R})$, and since $\varphi(0) = 1$, the limit (A.1) exists and is nonzero.

Suppose now that the spectrum of $\mathfrak{H}$ is purely continuous. Then the RAGE theorem (Reed and Simon [28]) shows that for any element G and for any compact operator K we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|K\exp(it\mathfrak{H})G\|_{B^2}^2\, dt = 0$$

so putting in the above $G = I$ and K the projection onto the one-dimensional subspace of $B^2(\boldsymbol{R})$ spanned by I, we conclude that the limit (A.1) exists and is zero. The theorem is proved.

If u_μ is Besicovitch almost periodic, it represents an element U_μ of $B^2(\boldsymbol{R})$, and since clearly $\Phi_t T_t U_\mu = \exp(i\mu t) U_\mu$ for any $t \in \boldsymbol{R}$, it follows that U_μ is an eigenelement and thus $\mathfrak{H}$ has pure point spectrum. The inverse implication is not true; moreover, if for a given q with pure point spectrum all eigenfunctions u_μ for all potentials Q in the hull of q are B^2-almost periodic then they are all uniformly almost periodic [14]. There are Q, therefore, for which an eigenelement of $\mathfrak{H}$ is not represented by a classical eigenfunction and presumably this may also be the case for Schrödinger operators $\mathfrak{H}$. The subtle not quite almost periodic but close to almost periodic behaviour of eigenfunctions at infinity may require the introduction, for different almost periodic potentials, of distinct nonseparable Hilbert spaces based on invariant means on $\boldsymbol{R}^\nu$ (see [16]).

Returning to the case of first order operators we note an interesting open problem of connection between the eigenfunctions u_μ and the spectral measure of $\mathfrak{H}$ when its spectrum is purely continuous. Finally, we note that Chojnacki [13] exhibited examples of $q \in CAP(\boldsymbol{R})$ for which the spectrum of $\mathfrak{H}$ is purely continuous and also of q with pure point spectrum but with the eigenfunctions u_μ not in $CAP(\boldsymbol{R})$.

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Added in proof (June 1987). It can be deduced from a result of [3] and from an unpublished result of W. Chojnacki that if V is a real almost periodic function on $\mathbf{R}$ such that for almost all ω in the hull of V, the operator $-d^2/dx^2+V_\omega$ in $L^2(\mathbf{R})$ has an eigenvector, then the operator $-d^2/dx^2+V$ in $B^2(\mathbf{R})$ has a nonzero continuous component in the spectrum.