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TWO EXAMPLES OF AN ALTERNATIVE APPROACH TO SYSTEMS ANALOGOUS TO THE HEAT POLYNOMIALS

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P. C. Rosenbloom and D. V. Widder [8] studied a system $(v_n)_{n=0,1,2,\ldots}$ of polynomial solutions of the heat equation $u_t = u_{xx}$. These "heat polynomials" are given by

$$v_n(x, t) = n! \sum_{k=0}^{[n/2]} \frac{t^k}{k!} \frac{x^{n-2k}}{(n-2k)!}$$

($n = 0, 1, 2, \ldots$, $(x, t) \in \boldsymbol{R}^2$, $[n/2]$ = largest integer $\leqslant n/2$). They proved that expansions $\sum a_n v_n$ are convergent for $x \in \boldsymbol{R}$, $|t| < \sigma$ if

$$\limsup_{n\to\infty} (2n/e)|a_n|^{2/n} = 1/\sigma.$$

This result has a remarkable consequence: D. V. Widder [13] showed that a solution of $u_t = u_{xx}$ which is analytic in a neighbourhood of (0, 0) can be continued into a strip $\boldsymbol{R} \times \{t \in \boldsymbol{R}: |t| < \sigma\}$ (see also Colton [1]).

The results on the system of heat polynomials have been the model for investigations on equations of second order with singular coefficients, with small parameters, of higher order, etc. (see e.g. Haimo [3], Lo [6], [7], Kemnitz [4], [5], Givens–Lo [2], a survey is given in [10]). Usually a system of polynomial solutions u_n is defined explicitly and the convergence result for $\sum a_n u_n$ follows from appropriate estimates for the functions u_n. In [11] the polynomials are not given in an explicit form, they are only defined via a recursion formula, but the convergence result is still based on an estimate for the polynomials.

If more general differential equations are studied it is not easy to give an explicit representation of the solutions u_n and to find appropriate estimates for these functions. Therefore it seems to be desirable to have an alternative method which is based essentially on the differential operator belonging to the equation. In [12] such a method is described for differential equations of the form $u_t = Lu$ with

$$L = L_b := D_x^r + \sum_{j=1}^{r-1} b_j D_x^j \quad (r \in \{2, 3, \ldots\}, b_j \in \boldsymbol{R}), \tag{1}$$

$$L = L_\lambda := D_x^r + \sum_{j=1}^{r-1} \lambda_j x^{-j} D_x^{r-j} \quad (r \in \{2, 3, \ldots\}, \lambda_j \in \boldsymbol{R}). \tag{2}$$

(We use the notation $D_x = \partial/\partial x$, $D_t = \partial/\partial t$.)

In both cases L^m $(m \in \boldsymbol{N})$ is regarded as a bounded linear operator in an appropriate scale of Banach spaces. Estimates for $\|L^m\|$ allow to define a family of bounded operators $T(t)$ which is used for the definition of a system (u_n) generalizing the heat polynomials. It is sufficient to know that the functions u_n are given in the form $T(t)x^n$.

In the following the main features of this method are described for the equation $D_t^p = L_b u$ with $p < r$ and L_b as in (1), and it is shown that the equation

$$\sum_{j=1}^{r-p} \eta_j D_t^{p+j} u + D_t^p u = D_x^r u + \sum_{j=1}^{r-1} \lambda_j x^{-j} D_x^{r-j} u \quad (p \geqslant 1, r > p, \eta_j, \lambda_j \in \boldsymbol{R}) \tag{3}$$

interpreted as a perturbation of $D_t^p u = L_\lambda u$ can also be discussed on the basis of a family of bounded linear operators. C. R. Givens and C. Y. Lo [2] studied equation (3) in the case $p = 1$, $r = 2$ by considering it as a singular perturbation of the generalized heat equation $u_{xx} + 2\nu x^{-1} u_x = u_t$ (for this equation see e.g. Haimo [3]). The case $\lambda_1 = \ldots = \lambda_{r-1} = 0$ (interpreted as a perturbation of $D_t^p u = D_x^r u$) was discussed in [11].

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Let $l^1_{r,p,s}$ $(r, p \in \boldsymbol{N}, r > p, s > 0)$ be the space of complex sequences $(a_k)_{k=0,1,2,\ldots}$ with

$$\|(a_k)\|_{1,r,p,s} := \sum_{k=0}^{\infty} |a_k| \left((pk)!\right)^{-1/r} s^{pk/r} < \infty.$$

An injective map $j\colon l^1_{r,p,s} \to C^\infty(\boldsymbol{R})$ can be defined by

$$j\left((a_k)\right)(x) := \sum_{k=0}^{\infty} a_k (k!)^{-1} x^k \quad \text{for } x \in \boldsymbol{R}.$$

The space $X_{r,p,s} := j(l^1_{r,p,s})$ then becomes a Banach space if the norm

$$\|f\|_{r,p,s} := \|j^{-1}(f)\|_{1,r,p,s}$$

is used. (Spaces of this type also appear in connection with the Ovsyannikov theorem, see Steinberg–Treves [9].)

LEMMA 1. *If* $f \in X_{r,p,s}$, $m \in N$, *then* $f^{(m)} \in X_{r,p,\sigma}$ *for* $\sigma \in (0, s)$ *and*

$$\|f^{(m)}\|_{r,p,\sigma} \leqslant (pm/e)^{pm/r}(s/\sigma)^{1/r}(s-\sigma)^{-pm/r}\|f\|_{r,p,s}. \tag{4}$$

Proof. As in the case $p = 1$ (see [12]), the proof is based on the inequality

$$\frac{m!}{(m-n)!} \leqslant \left(\frac{s}{\sigma}\right)^{m+1}\left(\frac{\sigma}{s-\sigma}\right)^n\left(\frac{n}{e}\right)^n \quad \text{for } m,\ n \in N,\ m \geqslant n,\ s,\ \sigma > 0,\ s > \sigma. \tag{5}$$

If $(a_k) \in l^1_{r,p,s}$, the inequality (5) implies

$$\sum_{k=0}^{\infty} |a_{k+m}|\,((pk)!)^{-1/r}\sigma^{pk/r} = \sum_{k=m}^{\infty} |a_k|\,((pk)!)^{-1/r}s^{pk/r}\left(\frac{(pk)!}{(pk-pm)!}\right)^{1/r}\sigma^{-pm/r}(\sigma/s)^{pk/r}$$

$$\leqslant \|(a_k)\|_{1,r,p,s}(pm/e)^{pm/r}(s/\sigma)^{1/r}(s-\sigma)^{-pm/r}.$$

If L_b is the operator defined by (1), the following estimate for $\|L_b^m f\|_{r,p,\sigma}$ is a consequence of Lemma 1:

LEMMA 2. *For* $\sigma \in (0, s)$ *and* $\varepsilon > 0$ *there exists* $m_0 \in N$ *such that*

$$\|L_b^m f\|_{r,p,\sigma} \leqslant (rpm/e)^{pm}(s/\sigma)^{1/r}(s-\sigma)^{-pm}(1+\varepsilon)^m\|f\|_{r,p,s}$$

for $m \geqslant m_0$, $f \in X_{r,p,s}$.

Proof. The assertion follows from (4) and

$$\|L_b^m f\|_{r,p,\sigma} \leqslant \sum_{|\alpha|=m} \frac{m!}{\alpha!}|b_1|^{\alpha_1}|b_2|^{\alpha_2}\cdot\ldots\cdot|b_{r-1}|^{\alpha_{r-1}}\|D_x^{\alpha_1+2\alpha_2+\ldots+r\alpha_r}f\|_{r,p,\sigma}.$$

Now L_b^m $(m \in N)$ can be interpreted as bounded linear operators L_b^m: $X_{r,p,s} \to X_{r,p,\sigma}$, and it is possible to define operators $T_q(t) \in B(X_{r,p,s}, X_{r,p,\sigma})$ by

$$T_q(t) = \sum_{m=0}^{\infty} \frac{t^{pm+q}}{(pm+q)!}L_b^m \quad (q \in \{0, 1, \ldots, p-1\})$$

for $|t| < (s-\sigma)/r$ (L_b^0 denotes the injection $X_{r,p,s} \subset X_{r,p,\sigma}$, $B(X_{r,p,s}, X_{r,p,\sigma})$ the Banach space of bounded linear operators with the operator norm).

THEOREM 1. (a) *If* $0 < \sigma < s$, $q \in \{0, 1, \ldots, p-1\}$ *and* $(a_k) \in l^1_{r,p,s}$, *then the series*

$$\sum_{k=0}^{\infty} a_k(k!)^{-1}\,T_q(t)\,x^k \tag{6}$$

is convergent in $X_{r,p,\sigma}$ *for* $|t| < (s-\sigma)/r$.

(b) *If* $(a_k) \in l^1_{r,p,r\varrho}$, *then the series* (6) *is absolutely convergent for* $(x, t) \in \boldsymbol{R} \times (-\varrho, \varrho)$. *The convergence is uniform on compact subsets of* $\boldsymbol{R} \times (-\varrho, \varrho)$.

Proof. Let

$$f = j\left((a_k)\right) \in X_{r,p,s}, \quad f_n(x) = \sum_{k=0}^{n} a_k (k!)^{-1} x^k.$$

Then $\|f - f_n\|_{r,p,s} \to 0$ as $n \to \infty$ and therefore $\|T_q(t) f - T_q(t) f_n\|_{r,p,\sigma} \to 0$ as $n \to \infty$. This gives (a). Part (b) follows from the fact that for $s > 0$ and $R > 0$ there exists $C > 0$ such that $\sup_{|x| \leqslant R} |f(x)| \leqslant C \|f\|_{r,p,s}$ for $f \in X_{r,p,s}$.

The functions $T_q(t) x^k$ are polynomial solutions of the equation $D_t^p u = L_b u$. It is easy to see that the polynomials $T_0(t) x^n$ are the heat polynomials if $p = 1$, $r = 2$, $b_1 = 0$.

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As mentioned before C. R. Givens and C. Y. Lo [2] studied series expansions of solutions of

$$u_{xx} + (2\nu/x) u_x + \varepsilon^2 u_{tt} = u_t \qquad (\varepsilon > 0,\ \nu > 0).$$

For the system $(P_{n,\nu,\varepsilon})_{n=0,1,2,\dots}$ given by

$$P_{0,\nu,\varepsilon} = 1,$$

$$P_{n,\nu,\varepsilon}(x, t) = x^{2n} + \sum_{k=1}^{n} x^{2n-2k} 2^{2k} \binom{n}{k} \frac{\Gamma(\nu + 1/2 + n)}{\Gamma(\nu + 1/2 + n - k)} \sum_{m=0}^{k-1} \frac{(k+m-1)!\, t^{k-m} \varepsilon^{2m}}{m!\,(k-m-1)!} \qquad (n \in \boldsymbol{N}),$$

they proved the convergence of $\sum_{n=0}^{\infty} a_n P_{n,\nu,\varepsilon}(x, t)$ in $\boldsymbol{R}^2$ if

$$\limsup_{n \to \infty} (2n/e) |a_n|^{1/(2n)} = \sigma < 1/(2\varepsilon).$$

Combining the technique used above and an estimate proved in [11] it is now easy to get an analogous result for equation (3).

Polynomial solutions of (3) which correspond to the polynomials $P_{n,\nu,\varepsilon}$ can be defined by

$$U_{n,q,\eta,\lambda}(x, t) = \sum_{k=0}^{n} A_{k,q}(t) L_\lambda^k x^{rn} \tag{7}$$

($n = 0, 1, 2, \dots$, $q = 0, 1, \dots, p-1$, L_λ the operator (2)), with $A_{k,q}$ uniquely determined by

$$A_{0,q}(t) = t^q/q!, \qquad A_{k,q}(t) = \sum_{j=p}^{kp+q} c_{j,k,q} t^j/j!,$$

$$\sum_{j=1}^{r-p} \eta_j D_t^{p+j} A_{k+1,q} + D_t^p A_{k+1,q} = A_{k,q} \quad \text{for } k = 0, 1, 2, \ldots$$

The system (7) was discussed in [11] if $\lambda_1 = \ldots = \lambda_{r-1} = 0$. From [11] we now take the following estimate for the polynomials $A_{k,q}$: If $|\eta_j| \leqslant \varepsilon^j$ for $j = 1, \ldots, r-p$, then

$$(8) \qquad |A_{k,q}(t)| \leqslant \left(\frac{r-p}{p}\varepsilon\right)^{pk+q} (p+1)^{(p+1)k+q} \exp\left(\frac{p|t|}{(r-p)(p+1)\varepsilon}\right)$$

for $t \in \boldsymbol{R}$, $k = 0, 1, 2, \ldots$, $q = 0, 1, \ldots, p-1$.

The polynomials $U_{n,q,\eta,\lambda}$ can be represented in the form

$$U_{n,q,\eta,\lambda}(x, t) = S_q(t)\, x^{rn}$$

with $S_q(t) \in B(Y_s, Y_s)$ for appropriate Banach spaces Y_s $(s > 0)$. For the construction of Y_s we use the space l_s^1 of complex sequences $(a_k)_{k=0,1,2,\ldots}$ with

$$\|(a_k)\|_{1,s} := \sum_{k=0}^{\infty} |a_k|\, s^k < \infty.$$

An injective map $j_r\colon\ l_s^1 \to C^\infty(\boldsymbol{R})$ is defined by

$$j_r\big((a_k)\big)(x) := \sum_{k=0}^{\infty} a_k \big((rk)!\big)^{-1} x^{rk},$$

and the space $Y_s := j_r(l_s^1)$ becomes a Banach space if the norm

$$\|f\|_s := \|j_r^{-1}(f)\|_{1,s}$$

is used. (The notation $Y_{r,s}$ would be more precise, but there is no danger of misunderstanding.)

Lemma 3. *Let L_λ be the operator* (2). *For $\delta > 0$ there exists $C > 0$ such that*

$$(9) \qquad \|L_\lambda^m f\|_s \leqslant C s^{-m}(1+\delta)^m \|f\|_s \quad \textit{for } m \in N, f \in Y_s,\ s > 0.$$

Proof. If $f = j_r\big((a_k)\big)$, then

$$(L_\lambda^m f)(x) = \sum_{k=0}^{\infty} a_{k+m} M(k, m, \lambda)\big((rk)!\big)^{-1} x^{rk}$$

with

$$M(k, m, \lambda) = \prod_{\nu=0}^{m-1} \Big(1 + \sum_{j=1}^{r-1} \lambda_j (rk+r\nu)!\,\big((rk+r\nu+j)!\big)^{-1}\Big).$$

It is easy to see that $M(k, m, \lambda)$ can be estimated in the form

$$|M(k, m, \lambda)| \leqslant \text{const} \cdot (1+\delta)^m \quad \text{for } k = 0, 1, 2, \ldots,\ m \in N.$$

This implies

$$\|L_\lambda^m f\|_s \leqslant C(1+\delta)^m \sum_{k=0}^{\infty} |a_{k+m}| s^k \leqslant C(1+\delta)^m s^{-m} \sum_{k=0}^{\infty} |a_k| s^k = C(1+\delta)^m s^{-m} \|f\|_s.$$

A combination of the estimates (8) and (9) gives

LEMMA 4. *If* $|\eta_j| \leqslant \varepsilon^j$ *for* $j = 1, \ldots, r-p$ *and*

$$\left(\frac{r-p}{p}\varepsilon\right)^p (p+1)^{p+1} < s,$$

then an operator $S_q(t) \in B(Y_s, Y_s)$ $(q \in \{0, 1, \ldots, p-1\})$ *can be defined by*

$$S_q(t) := \sum_{m=0}^{\infty} A_{m,q}(t) L_\lambda^m$$

for $t \in \boldsymbol{R}$.

THEOREM 2. *If* $|\eta_j| \leqslant \varepsilon^j$ *for* $j = 1, \ldots, r-p$,

$$\left(\frac{r-p}{p}\varepsilon\right)^p (p+1)^{p+1} < s$$

and $(a_k) \in l_s^1$, *then*:

(a) *The series*

$$\sum_{k=0}^{\infty} a_k \big((rk)!\big)^{-1} S_q(t) x^{rk} \qquad (q \in \{0, 1, \ldots, p-1\})$$

is convergent in Y_s *for* $t \in \boldsymbol{R}$.

(b) *The series*

$$\sum_{k=0}^{\infty} a_k \big((rk)!\big)^{-1} U_{k,q,\eta,\lambda}(x, t) \qquad (q \in \{0, 1, \ldots, p-1\})$$

is absolutely convergent in $\boldsymbol{R}^2$.

Proof. Similar to the proof of Theorem 1.

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