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ON EQUATIONS OF CLONES

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A clone is a set of functions which is closed under composition, permutations of variables and identifying variables. E. Post used this concept to analyse the clones of functions on the set $\{0, 1\}$ which consists of the 2-valued logic functions. The lattice of these clones is described in detail in the monograph of Jablonski, Gawrilow and Kudrjawzew [6]. The next important step was the analysis of all maximal clones for the set $\{0, 1, 2\}$ in Jablonski [5] by which a completeness criterion was given for the 3-valued logic functions. This result could be extended further to the general case of the general case of the k-valued logic functions by I. G. Rosenberg [12].

A. I. Mal'cev [8] gave the definition of a clone as an abstract algebra and also studied their congruence relations. We are following these ideas and are considering varieties of clones. The main result states the relation between a variety of algebras and the variety generated by the clones of term functions of these algebras. For these varieties of clones a new kind of equations arises which are studied under the form of hyperidentities. We illustrate this abstract approach by the varieties of clones connected to varieties of lattices and varieties of abelian groups.

Then we show that the congruence relations of clones are related to the fully invariant congruence relations of free algebras. Denecke and Lau [3] have shown that one can use this method to prove a theorem of Oates-Williams on the subvarieties of the non-finitely based variety of Murskii's groupoid [11].

Finally we study clone isomorphisms which give rise to consideration on canonical simplifyers for terms. The computation of terms can sometimes be transformed by clone isomorphisms to other varieties where the reductions are well known.

Our approach is related in some aspects also to Lawvere's concept of algebraic theories [9]. But here we use methods of universal algebra to derive hyperidentities, to study congruences and to consider reductions of terms.

1. Definitions and basic facts

For the convenience of the reader we define most of the concepts which are used in this paper. We rely on the definition of a clone as a universal algebra. This approach was already developed by A. I. Mal'cev [8].

DEFINITION 1.1. Let H be a set of functions on A. The clone $\boldsymbol{H} = (H; *, \xi, \tau, \Delta, e)$ is an algebra of type (2, 1, 1, 1, 0) where the operations are defined in the following way:

(1) $(f*g)(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n-1}) = f(g(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1})$ for an n-ary function f and an m-ary function g;

(2) $(\xi f)\ (x_1, \ldots, x_n) = f(x_2, \ldots, x_n, x_1)$ for an n-ary function f, $n > 1$, (ξf) $(x_1) = f(x_1)$ for any 1-ary function f;

(3) $(\tau f)\ (x_1, \ldots, x_n) = f(x_2, x_1, x_3, \ldots, x_n)$ for an n-ary function f, $n > 1$, (τf) $(x_1) = f(x_1)$;

(4) $(\Delta f)\ (x_1, \ldots, x_{n-1}) = f(x_1, x_1, x_2, \ldots, x_{n-1})$ for an n-ary function f, $n > 1$, $(\Delta f)\ (x_1) = f(x_1)$;

(5) $e(x_1, x_2) = x_1$.

We like to remark that in $\boldsymbol{H}$ every projection e_i^n, $e_i^n(x_1, \ldots, x_n) = x_i$, $i = 1, \ldots, n$ is generated. Throughout the paper the arity of functions will play an essential role. Hence we will use the following extended version of Definition 1.1.

DEFINITION 1.2. Let H be a set of functions on A. The algebra $\boldsymbol{H} = (H; *, \xi, \tau, \Delta, e, \Box_n\ (n \in N))$ type (2, 1, 1, 1, 0, 1, ...) is called a *clone* with arity, where the operations $*$, ξ, τ, Δ, e are defined as in 1.1 and the operations $\Box_n (n \in N)$ are defined by

(6) $(\Box_n f)\ (x_1, \ldots, x_n) = f(x_1, \ldots, x_k)$ if f is a k-ary function with $k \leqslant n$ and $(\Box_n f)\ (x_1, \ldots, x_n) = f(x_1, \ldots, x_1, x_2, \ldots, x_n)$ if f is a k-ary function with $k > n$.

Definition 1.2 is equivalent to Definition 1.1 in the sense that every function $(\Box_n f)$ can be generated by $*$, ξ, τ, Δ, e. If f is a k-ary function with $k \leqslant n$ we consider $e_1^m * f$ which gives

$$(e_1^m * f)(x_1, \ldots, x_{m-k+1}) = e_1^m(f(x_1, \ldots, x_k), x_{k+1}, \ldots, x_{m-k+1}).$$

For $m = n+k-1$ we have $\Box_n f = (e_1^m * f)$. If f is a k-ary function with $k > n$ we apply $\Delta(k-n)$times and we get $\Box_n f = (\Delta^{k-n} f)$.

Furthermore the composition $\circ$ defined by $(f \circ g)(x_1, \ldots, x_n) = f(g(x_1, \ldots, x_m), x_2, \ldots, x_n)$ (f n-ary and g m-ary) offers a lot of advantages compared to $*$ whenever one is considering varieties of clones. This composition $\circ$ can be generated by $*$, ζ, τ, Δ, e and vice versa $\circ$, ζ, τ, Δ, e generate the clone operation $*$. We write

$$f^k(x_1, \ldots, x_n) = f(f^{k-1}(x_1, \ldots, x_n), x_2, \ldots, x_n).$$

To avoid lengthy notations clones with arity are called *clones*. This may be justified be the fact that most of the results hold for both kinds of clones. The advantages of Definition 1.2 is that hyperidentities can be linked to clone equations in a simple way. As a clone is defined like a universal algebra it is natural to construct varieties of clones by the operators H (homomorphic images), S (subalgebras) and P (direct products).

Remark 1.3. Every subclone D of a clone C of functions on a set is again a clone of functions.

Remark 1.4. A countable power of a clone C of finitary functions gives rise to a clone which contains infinitary functions. For this consider

$$C_1^\infty = \{(f_1, f_2, f_3, \ldots) | f_i \in C, \; i = 1, 2, 3, \ldots\}.$$

The sequence $(e_1^1, e_1^2, e_1^3, \ldots)$ with $e_1^i(x_1, \ldots, x_i) = x_1$, $i = 1, 2, 3$ can be considered as a function but of infinite arity,

Remark 1.5. Not every homomorphic image of a clone of functions is again isomorphic to a clone of functions. If one defines a clone $(\{a\}; *, \zeta, \tau, \Delta, e, \square_n (n \in N))$ by $a * a = \zeta a = \tau a = \Delta a = \square_n a = a$, then this clone is a homomorphic image of every clone of functions.

These remarks show that the above definition of a clone has to be extended. Henceforth we will understand by a clone an algebra which is a member of the variety generated by all clone of functions. In [15] we have looked for a definition by equations for this variety.

2. Varieties of algebras and equations of clones

The concept of a clone of functions can be used to define term functions and polynomial functions of an algebra. We denote by $F(A)$ the clone of all functions of a set A.

Definition 2.1. Let $\mathcal{A} = (A, \Omega)$ be an algebra. The clone $T(\mathcal{A})$ of the term functions of $\mathcal{A}$ is the *subclone of* $F(A)$ which is generated (by the projections and) by the operations of $\mathcal{A}$.

The clone $P(\mathcal{A})$ of the polynomial functions of $\mathcal{A}$ is the *subclone of* $F(A)$ which is generated (by the projections and) by the operations of $\mathcal{A}$ and the constant functions c_a^n, $c_a^n(x_1, \ldots, x_n) = a$, $a \in A$, $n \in N$.

We denote terms by $t(x_1, \ldots, x_n)$ and the corresponding term functions by $t\colon A^n \to A$ defined by $(a_1, \ldots, a_n) \to t(a_1, \ldots, a_n)$. For a lattice $\mathscr{L} = (L; \wedge, \vee)$ we have for instance a term $t(x_1, x_2, x_3) = x_1 \wedge (x_2 \vee x_3)$ and the corresponding term function $t\colon L^3 \to L$ with $(a_1, a_2, a_3) \to (a_1 \wedge (a_2 \vee a_3))$.

NOTATIONS 2.2. $T\colon A \to C$ denotes the mapping from a class of algebras to the class of clones of term functions. T maps every algebra A to its clone $T(A)$ of term functions.

The following result (Schweigert [13]) shows that the mapping T can be extended to a mapping (also denoted by T) $T\colon V \to C^*$ from the class V of all varieties to the class C^* of all varieties of clones.

THEOREM 2.3. *Let V be a variety generated by the family $\{A_i \mid i \in I\}$ of algebras. Let $T(V)$ be the variety of clones which is generated by the family $\{T(A_i) \mid i \in I\}$ of clones. If A is an algebra of V, then $T(A)$ is a clone of $T(V)$.*

Equations which hold in varieties of clones are called *clone equations.* We denote the variables in these equations by X, Y, Z, X_1, X_2, X_3, ... We are considering the following example $\Delta(\square_2 X) = \Delta e$. Obviously this clone equation holds for every term function of a lattice.

Following W. Taylor [17] we define a hyperidentity to be formally the same as an identity. Consider for example $F(x, x) = x$. A variety V is said to *satisfy a hyperidentity* ε if whenever the operation symbols of ε are replaced by any choice of terms of V of the same arity as the corresponding operation symbols of ε, then the resulting identity holds in V in the usual sense. For example the variety of lattices satisfies the above hyperidentity.

As hyperidentities are much easier to read we usually translate clone equations into hyperidentities or sets of hyperidentities. On the other hand these hyperidentities can also be translated back into clone equations. (Such a translation is not unique.)

To illustrate Theorem 2.3 we give examples from lattice theory and the theory of abelian groups.

THEOREM 2.4.1. *If V is a variety of lattices, then the following hyperidenties holds:*

(1) $F(x, x) = x$,

(2) $F(x, F(y, z)) = F(F(x, y), z)$.

2.4.2. *A (non-trivial) variety of lattices is the variety of distributive lattice if and only if (one of) the following hyperidentities hold:*

(3) $F^2(x_1, \ldots, x_k) = F(x_1, \ldots, x_k)$, $k \geqslant 4$, $k \in N$,

(4) $F(x, G(y, z)) = G(F(x, y), F(x, z))$,

(5) $F(G(x, y), z)) = G(F(x, z), F(y, z))$.

THEOREM 2.5. *If V is the variety of all groups of exponent 2, then the following hyperidentities hold:*

(1) $F(x, F(y, z)) = F(F(x, y), z)$,

(2) $F(F(x, x), x) = x$,

(3) $F(F(x, y), F(y, x)) = F(x, x)$,

(4) $F(G(x, x), G(y, y)) = G(F(x, y), F(x, y))$.

Remark. By (1) and (3) of Theorem 2.5 it follows that

(5) $F^3(x, y) = F(x, y)$ holds.

The proofs of the hyperidentities can partly be found in Schweigert [13], [14], [15] or derived by the methods developed in these papers.

THEOREM 2.6. *If V is the variety of Boolean algebras, then the following hyperidentities hold*:

(1) $F^3(x_1, \dots, x_k) = F(x_1, \dots, x_k)$, $k \in N$,

(2) $F(F(x, x), F(y, y)) = F(F(x, y), F(x, y))$.

3. Congruence relations of clones

An important congruence relation on every clone of functions is the relation $\varkappa$ which is defined by $(f, g) \in \varkappa$ if and only if the arity of the function f is equal of the arity of the function g (A. I. Mal'cev [8]). We call a congruence relation θ of a clone of functions an arity-congruence if $\theta \subseteq \varkappa$.

Let θ_F be the congruence relation of the free algebra $F(X)$ of a variety $\mathrm{HSP}(A)$ such that θ_F is generated by a pair of terms $(t(x_1, \dots, x_n), s(x_1, \dots, x_m))$. Now we can define term functions t', s' by

$$t'(x_1, \dots, x_k) = t(x_1, \dots, x_n) \quad \text{and} \quad s'(x_1, \dots, x_k) = s(x_1, \dots, x_m),$$
$$k = \max\{n, m\}$$

such that t', s' are given by these terms and their arities are equal. (t', s') generate a congruence relation on the clone $T(A)$ of term functions of A.

THEOREM 3.1. *Let $A = (A, \Omega)$ be an algebra, $\mathrm{HSP}(A)$ be the variety generated by A and let $F(X)$ be the free algebra of $\mathrm{HSP}(A)$. Every arity-congruence relation of $T(A)$ corresponds to a fully invariant congruence relation of the free algebra $F(X)$. Furthermore the lattice of arity congruences is antiisomorphic to the lattice of subvarieties of $\mathrm{HSP}(A)$.*

Remark 3.2. For every subvariety U of a variety V there is an hyperidentity which holds for U but not in V. An example of this fact is Theorem 2.4.2 (see also [13], [14]).

Remark 3.3. In paper [3] (Denecke, Lau) the congruence relations of clones are discussed in great detail and the above theorem is used to derive Oates-Williams' result that Murskii's algebra does not satisfy Min. [11] (i.e., that the variety generated by Murskii's algebra contains an infinite chain of subvarieties).

4. Clone isomorphisms

DEFINITION 4.1. Two algebras $A = (A, \Omega_1)$, $B = (B, \Omega_2)$ not necessarily of the same type are *clone-isomorphic* to each other if the clones of term functions $T(A)$ und $T(B)$ are isomorphic.

NOTATIONS 4.2. Let h: $T(A) \to T(B)$ a clone isomorphism and let ω be an n-ary fundamental operation of B. Then $h^{-1}(\omega)$ is a term function of $T(A)$ and may be presented by a term $\psi(x_1, \ldots, x_n)$ of the free algebra of HSP(A). Now $h(\psi)$ is a term function of B and may be presented by a term $\varphi(x_1, \ldots, x_n)$ of the free algebra of HSP(B). Obviously the equation $\varphi(x_1, \ldots, x_n) = \omega(x_1, \ldots, x_n)$ holds for B. We denote this equation by π_ω and consider the set $\{\pi_\omega | \ \omega \in \Omega_2\}$ of equations. Furthermore for a set Σ of equations (s, t) we denote by $h(\Sigma) := \{(h(s), h(t)) | \ (s, t) \in \Sigma\}$.

EXAMPLE. Let $A = \{\{0, 1\}; \ \wedge, \vee, ', 0, 1\}$ be the Boolean algebra on the set $\{0, 1\}$. Let $B = \{\{0, 1\}; +, 0, \cdot, 1\}$ be the commutative ring on the set $\{0, 1\}$, where addition is modulo 2. Then A and B are clone-isomorphic. We have

$$\pi_+: \ \big(((x+1)\cdot y)+(x\cdot(y+1))\big)+\big(((x+1)\cdot y)\cdot(x\cdot(y+1))\big) = x+y,$$

$$\pi_\cdot: \ x\cdot y = x\cdot y$$

(which need not to be considered because of triviality) for h with $x + y \leftrightarrow (x' \wedge y) \vee (x' \wedge y)$, $x \cdot y \leftrightarrow x \wedge y$.

The following is an extension of the result in [16].

THEOREM 4.3. *Let Σ_1 be an equational basis for the equational theory of the algebra $A = (A, \Omega_1)$. If $B = (B, \Omega_2)$ (not necessarily of the same type) is clone isomorphic to A, then $\Sigma_2 = h(\Sigma_1) \cup \{\Pi_\omega | \ \omega \in \Omega_2\}$ is an equational basis for the equational theory of the algebra B.*

Remark 4.4. Let $A = (A, \Omega_1)$ and $B = (B, \Omega_2)$ be of finite type and clone isomorphic to each other. The equational theory of B is finitely based if and only if the equational theory of A is finitely based.

With the clone isomorphism h: $T(A) \to T(B)$ a proof in the equational theory of A can be transformed in a proof in the equational theory of B [16].

NOTATIONS 4.5. ([2], p. 11.) Let A be an algebra and $F(X)$ the free algebra for $X = \{x_1, x_2, \ldots\}$ in the variety HSP(A). Let S be an effective procedure, S: $F(X) \to F(X)$, with the following properties:

(4.5.1) $S(t) = t$ for term functions $S(t), t$ on A induced by the terms $S(t)(x_1, \ldots, x_n)$, $t(x_1, \ldots, x_n)$ of $F(X)$.

(4.5.2) If $t = s$ for the term functions t, s on A induced by terms $t(x_1, \ldots, x_n)$, $s(x_1, \ldots, x_n)$ of $F(x)$, then $S(t) = S(s)$ for the term functions $S(t)$, $S(s)$ on A.

Then S is called a *canonical simplifyer* and $S(t)(x_1, \ldots, x_n)$ is called a *canonical form* for the term $t(x_1, \ldots, x_n)$.

Remark 4.6. If the algebra A has a canonical simplifyer S and A is clone isomorphic to the algebra B (not necessarily of the same type), then B has also a canonical symplifyer.

EXAMPLE 4.7. We consider some reductions of $A = \{\{0, 1\}; \wedge, \vee, ', 0, 1\}$ and their transformation to $B = (\{0, 1\}; +, 0, \cdot, 1)$

$$\begin{array}{ll} x \vee (x \wedge a) \to x, & x + ((x \cdot y) + x \cdot (x \cdot y)) \to x, \\ x \wedge 1 \to x, & x \cdot 1 \to x, \end{array}$$

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