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A NOTE ON TRIEBEL-LIZORKIN SPACES

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In this note we discuss characterizations of Triebel–Lizorkin spaces F_α^{pq} and relate them to spaces of DeVore and Sharpley [3] and Christ [1] defined by means of maximal functions. These characterizations allow to generalize known extension theorems in the framework of F_α^{pq}-spaces. We simultaneously consider isotropic and anisotropic spaces. Let us first fix some notations and definitions. Let $(A_t) = (t^P)$ be a one-parameter group of dilations in $\boldsymbol{R}^n$, trace $(P) = \nu$, the real parts of the eigenvalues of P being larger than $a_0 > 0$. Let further $\varrho^* \in C^\infty(\boldsymbol{R}_0^n)$ be an A_t^*-homogeneous distance function in $\boldsymbol{R}^n$ (for a discussion of these notions we refer to the survey [2] of Dappa and Trebels, this volume). Let $\varphi \in C_0^\infty(\boldsymbol{R}_+)$, $\operatorname{supp}\varphi \subset (\frac{1}{2}, 2)$, $\sum_{k\in\boldsymbol{Z}} \varphi(2^k s) = 1$, if $s > 0$, and

$\eta_k = \mathscr{F}^{-1}\left[\varphi \circ \frac{\varrho^*}{2^k}\right]$; set $\psi_k = \eta_k$ for $k \geqslant 1$, and $\psi_0 = 1 - \sum_{k\geqslant 1} \eta_k$.

DEFINITION. Let $0 < p < \infty$, $0 < q \leqslant \infty$, $-\infty < \alpha < \infty$. We define $F_\alpha^{pq}(P, \boldsymbol{R}^n)$ as the subspace of tempered distributions f for which

$$\|f\|_{F_\alpha^{pq}} = \Big\|\Big(\sum_{k=0}^{\infty} |2^{k\alpha}\psi_k * f|^q\Big)^{1/q}\Big\|_p$$

is finite. Similarly $\dot{F}_\alpha^{pq}(P, \boldsymbol{R}^n)$ is the subspace of tempered distributions modulo polynomials, which consists of those f for which

$$\|f\|_{\dot{F}_\alpha^{pq}} = \Big\|\Big(\sum_{k=-\infty}^{\infty} |2^{k\alpha}\eta_k * f|^q\Big)^{1/q}\Big\|_p$$

is finite.

In this paper we assume throughout $\alpha > 0$; then

$$\|f\|_{F_\alpha^{pq}} \approx \|f\|_p + \|f\|_{\dot{F}_\alpha^{pq}}.$$

The connection with the notation in [2] is that for $q = 2$ we have by Littlewood–Paley theory $F_\alpha^{p2}(P) = \mathscr{L}_{\alpha,\varrho^*}^2$.

A main tool in proving characterizations for F^{pq}_α are maximal functions. Let ϱ be an A_t-homogeneous distance function and

$$\mathcal{M}f(x) = \sup_{t>0} \fint_{\varrho(y)\leqslant t} |f(x-y)|\, dy,$$

the associated anisotropic Hardy–Littlewood maximal function. ($\fint_S f$ denotes the mean value $|S|^{-1}\int_S f$.) Further let

$$\sigma^*_{k,N} f(x) = \sup_{z\in\mathbb{R}^n} \frac{|\eta_k * f(x-z)|}{[1+2^k\varrho(z)]^N}$$

be an anisotropic version of Peetre's maximal function. The basic vector valued inequalities for $\mathcal{M}$ are due to Fefferman and Stein [6], for $\sigma^*_{k,N}$ due to Peetre [10]: For $1 < p,\ q < \infty$ there holds

$$\|\{\mathcal{M}f_k\}\|_{L^p(l^q)} \leqslant C\|\{f_k\}\|_{L^p(l^q)} \tag{1}$$

and for $0 < p < \infty$, $0 < q \leqslant \infty$, $N > \nu/\min(p, q)$

$$\|\{2^{k\alpha}\sigma^*_{k,N} f\}\|_{L^p(l^q)} \leqslant C\|\{2^{k\alpha}\eta_k * f\}\|_{L^p(l^q)}. \tag{2}$$

The easiest way to prove (1) is by interpreting the vector valued maximal operator as a singular integral operator (see [12]). Our anisotropic analogue follows using an anisotropic variant of singular integral theory as one can find e.g. in Rivière [11]. We also use continuous versions of (1) which are proved in the same way (or which can be deduced from (1) by limiting arguments). (2) is proved by straightforward modifications of the isotropic case considered in [10] and [16, Ch. 1].

To give our characterizations define $\Delta_h f(x) = f(x+h) - f(x)$ and the higher differences $\Delta_h^m f$ by iteration. Further let

$$S^\alpha_{q,r,m} f(x) = \left(\int_0^\infty \Big[\fint_{\varrho(h)\leqslant t} |\Delta_h^m f(x)|^r\, dh\Big]^{q/r} \frac{dt}{t^{1+\alpha q}}\right)^{1/q}. \tag{3}$$

For a fixed ϱ-ball $Q = Q_x(t) = \{y,\ \varrho(y-x) \leqslant t\}$ we define the *oscillation*

$$\operatorname{osc}^m_r(f, Q) = \operatorname{osc}^m_r(f, x, t) = \inf_{P\in\mathcal{P}^m} \Big(\fint_Q |f(y) - P(y)|^r\, dy\Big)^{1/r}$$

where the infimum is taken over all polynomials of degree $\leqslant m$. Further we set

$$\mathcal{S}^\alpha_{q,r,m} f(x) = \left(\int_0^\infty [\operatorname{osc}^{m-1}_r(f, x, t)]^q \frac{dt}{t^{1+\alpha q}}\right)^{1/q}. \tag{4}$$

For $q = \infty$ or $r = \infty$ we have the usual modifications and replace integrations by sup-norms.

In (3), (4) we assume $m > \alpha/a_0$ and usually omit the subscript m. The following theorem generalizes and improves results from [13], [14], [15], [8], [16], [4], [2].

THEOREM 1. *Suppose that* $0 < p < \infty$, $0 < q \leqslant \infty$, $m > \alpha/a_0$, $r \geqslant 1$. *If*

$$\alpha > \sigma_{p,q,r} = \max\left\{0,\ \nu\left(\frac{1}{p}-\frac{1}{r}\right),\ \nu\left(\frac{1}{q}-\frac{1}{r}\right)\right\},$$

then $\|f\|_{F_\alpha^{pq}} \approx \|S^\alpha_{q,r,m} f\|_p \approx \|\mathcal{D}^\alpha_{q,r,m} f\|_p$.

Proof. The proof is carried out for the cases $q,\ r \neq \infty$, the remaining cases are similar. The first step is to prove the pointwise inequality

$$S^\alpha_{q,r} f(x) \leqslant c\mathcal{D}^\alpha_{q,r} f(x). \tag{5}$$

Now we choose best approximants $P_t f$ in f in $L^1(Q_x(t))$. Since $\Delta_h^m P \equiv 0$ for all polynomials P of degree less than m, we may split

$$(-1)^m \Delta_h^m f(x) = f(x) - P_t f(x) + \sum_{j=1}^{m} (-1)^j \binom{m}{j} (f(x+jh) - P_t f(x+jh))$$

$$=: I_0(x,\ t) + \sum_{j=1}^{m} I_j(x,\ t,\ h).$$

Clearly, if $j \geqslant 1$,

$$\left(\int_0^\infty \Big[\int_{Q_0(t)} [I_j(x,\ h,\ t)]^r\, dh\Big]^{q/r} \frac{dt}{t^{1+\alpha q}}\right)^{1/q} \leqslant c\mathcal{D}^\alpha_{q,r}(f)$$

(here $Q_0(t) = \{y,\ \varrho(y) \leqslant t\}$).

Since $\lim\limits_{l\to\infty} P_{t2^{-l}} f(x) = f(x)$ a.e., $|P_t f| \leqslant \int_{Q_x(t)} |f|$ (see [3]),

$$|f(x) - P_t f(x)| \leqslant \sum_{l=0}^{\infty} |P_{2^{-l-1}t} f(x) - P_{2^{-l}t} f(x)|$$

$$\leqslant c \sum_{l=0}^{\infty} \int_{Q_x(2^{-l}t)} |f(y) - P_{2^{-l}t} f(y)|\, dy.$$

Hence, if $q \leqslant 1$,

$$\left(\int_0^\infty \Big[\int_{Q_0(t)} |I_0(x,\ t)|^r\, dh\Big]^{q/r} \frac{dt}{t^{1+\alpha q}}\right)^{1/q}$$

$$\leqslant c\left(\sum_l \int_0^\infty \Big[\int_{Q_x(2^{-l}t)} |f(y) - P_{2^{-l}t} f(y)|\, dy\Big]^q \frac{dt}{t^{1+\alpha q}}\right)^{1/q}$$

$$\leqslant c\left(\int_0^\infty \Big[\int_{Q_x(t)} |f(y) - P_t f(y)|\, dy\Big]^q \frac{dt}{t^{1+\alpha q}}\right)^{1/q} \leqslant c\mathcal{D}^\alpha_{q,1} f(x) \leqslant c\mathcal{D}^\alpha_{q,r} f(x).$$

If $q \geqslant 1$, we apply Minkowski's inequality to get the same result. The main step of the proof is to prove the inequality

$$\|\mathscr{D}^{\alpha}_{q,r} f\|_p \leqslant c\,\|f\|_{\dot F^{pq}_{\alpha}}. \tag{6}$$

We use a pointwise Sobolev inequality due to DeVore and Sharpley to reduce matters to the case $r < p$, q; then we can apply maximal inequalities. Choose β and $\sigma < p$, q such that $\alpha > \beta = \nu(1/\sigma - 1/r)$, then

$$\inf_{P \in P^{m-1}} \Big(\int_{Q_x(t)} |f-P|^r\,dy\Big)^{1/r} \leqslant c\Big(\int_{Q_x(t)} [\sup\{|R|^{-\beta/\nu} \operatorname{osc}^{m-1}_{\tau}(f, R)\}]^{\sigma}\,dy\Big)^{1/\sigma}, \tag{7}$$

where $\tau > 0$ (we assume $\tau < \min(1, \sigma)$), and the supremum is taken over all ϱ-balls R with $y \in R \subset Q_x(t)$, see [3, p. 23]. If $y \in R$, let R^* be a ϱ-ball with center y containing R, with $\operatorname{rad} R^* \leqslant 2b \operatorname{rad} R$ (b being the constant in the triangle inequality $\varrho(x+y) \leqslant b[\varrho(x)+\varrho(y)]$), then

$$\operatorname{osc}^{m-1}_{\sigma}(f, R) \leqslant c \operatorname{osc}^{m-1}_{\sigma}(f, R^*).$$

Using (7) we obtain

$$\mathscr{D}^{\alpha}_{q,r} f(x) \leqslant c\left(\int_0^{\infty} \Big[t^{\nu/\sigma - \nu/r}\Big(\int_{Q_x(t)} \big[\sup_{s \leqslant 2bt} s^{-\beta} \operatorname{osc}^{m-1}_{\tau}(f, y, s)\big]^{\sigma}\,dy\Big)^{1/\sigma}\Big]^q \frac{dt}{t^{1+\alpha q}}\right)^{1/q}$$

$$\leqslant c\left(\int_0^{\infty} \Big[\int_{Q_x(t)} \Big(\sum_{l \geqslant 0} 2^{l\beta} \operatorname{osc}^{m-1}_{\tau}(f, 2^{-l}t, y)\Big)^{\sigma} dy\Big]^{q/\sigma} \frac{dt}{t^{1+\alpha q}}\right)^{1/q}$$

which is majorized in the case $\sigma \leqslant 1$ by

$$c\left(\sum_l 2^{l\beta\sigma}\left(\int_0^{\infty} \Big[\int_{Q_x(t)} (\operatorname{osc}^{m-1}_{\tau}(f, 2^{-l}t, y))^{\sigma}\,dy\Big]^{q/\sigma} \frac{dt}{t^{1+\alpha q}}\right)^{\sigma/q}\right)^{1/\sigma}$$

$$\leqslant c\left(\sum_l 2^{l(\beta-\alpha)\sigma}\left(\int_0^{\infty} \Big[\int_{Q_x(t)} (\operatorname{osc}^{m-1}_{\tau}(f, t, y))^{\sigma}\,dy\Big]^{q/\sigma} \frac{dt}{t^{1+\alpha q}}\right)^{\sigma/q}\right)^{1/\sigma}$$

$$\leqslant c\left(\int_0^{\infty} \big[\mathscr{M}([\operatorname{osc}^{m-1}_{\tau}(f, t, \cdot)]^{\sigma})\big]^{q/\sigma} \frac{dt}{t^{1+\alpha q}}\right)^{1/q}$$

If $\sigma \geqslant 1$, we use instead Minkowski's inequality to obtain the same result. Since $\sigma < p$, q we may apply the Fefferman–Stein inequality (1) to obtain

$$\|\mathscr{S}^{\alpha}_{q,r} f\|_p \leqslant c\,\|\mathscr{S}^{\alpha}_{q,\tau} f\|_p.$$

Now we decompose

$$f = f_{0,t} + f_{1,t}, \qquad f_{0,t} = \sum_{2^k t \geqslant 1} \eta_k * f.$$

Then

$$\mathscr{S}^{\alpha}_{q,\tau} f(x) \leqslant \mathrm{I}(x) + \mathrm{II}(x)$$

with

$$\mathrm{I}(x)=\left(\int_0^\infty [\mathrm{osc}_r^{m-1}(f_{0,t},t,x)]^q \frac{dt}{t^{1+\alpha q}}\right)^{1/q}$$

and $\mathrm{II}(x)$ analogously defined.

Since $\mathrm{osc}_\tau^{m-1}(f_{0,t},t,y)\leqslant[\mathcal{M}(f_{0,t}^\tau)]^{1/\tau}$ and $\tau<q$ we may apply once more the Fefferman–Stein inequality to get

$$\|\mathrm{I}\|_p\leqslant c\left\|\left(\int_0^\infty\Big[\sum_{2^kt\geqslant 1}\eta_k*f\Big]^q\frac{dt}{t^{1+\alpha q}}\right)^{1/q}\right\|_p\leqslant c\Big\|\Big(\sum_k 2^{k\alpha q}|\eta_k*f|^q\Big)^{1/q}\Big\|_p.$$

For the latter inequality we use Hölder's inequality (if $q>1$):

$$\Big[\sum_{2^kt\geqslant 1}\eta_k*f\Big]^q\leqslant c\sum_{2^kt\geqslant 1}(2^kt)^{\varepsilon q}|\eta_k*f|^q,\qquad 0<\varepsilon<\alpha,$$

and interchange summation and integration. The case $q\leqslant 1$ is simpler. In order to estimate II, let $\tilde\eta_k=2^{k\nu}\tilde\eta(A_{2^k}\cdot)$ be similarly defined as η_k, but with $\hat{\tilde\eta}_k(\xi)=1$, if $\xi\in\operatorname{supp}\hat\eta_k$, hence $\eta_k*\tilde\eta_k=\eta_k$. Then we have the inequality

$$\begin{aligned}&\mathrm{osc}_\tau^{m-1}(f_{1,t},y,t)\\&\leqslant\left(\int_{Q_y(t)}\left|\sum_{2^kt\leqslant 1}\left[\eta_k*f(z)-\sum_{j=0}^{m-1}\frac{1}{j!}((z-y)\cdot\nabla)^j\eta_k*f(y)\right]\right|^\tau dz\right)^{1/\tau}\\&\leqslant\Big(\int_{Q_y(t)}\Big|\sum_{2^kt\leqslant 1}\int_0^1[(z-y)\cdot\nabla]^m\tilde\eta_k*\eta_k*f(y+\omega(z-y))\,d\omega\Big|^\tau dz\Big)^{1/\tau}.\end{aligned}$$

Since η is a Schwartz function, it holds for $z\in Q_y(t)$

$$\begin{aligned}|[(z-y)\cdot\nabla]^m\tilde\eta_k(\cdot)|&=|[(A_{2^k}(z-y)\cdot\nabla)^m\tilde\eta](A_{2^k}\cdot)|\\&\leqslant c_N(2^kt)^{ma_0}\big(1+2^k\varrho(\cdot)\big)^{-N-2\nu}\end{aligned}$$

which implies

$$\mathrm{osc}_\tau^{m-1}(f_{1,t},y,t)\leqslant c\Big(\int_{Q_y(t)}\Big|\sum_{2^kt\leqslant 1}(2^kt)^{ma_0}\sigma_{k,N}^*f(z)\Big|^\tau dz\Big)^{1/\tau}.$$

We integrate and obtain by the Fefferman–Stein inequality

$$\begin{aligned}\|\mathrm{II}\|_p&\leqslant c\left\|\left(\int_0^\infty\Big|\sum_{2^kt\leqslant 1}(2^kt)^{ma_0-\alpha}2^{k\alpha}\sigma_{k,N}^*f\Big|^q\frac{dt}{t}\right)^{1/q}\right\|_p\\&\leqslant c\Big\|\Big(\sum_k 2^{k\alpha q}[\sigma_{k,N}^*f]^q\Big)^{1/q}\Big\|_p,\end{aligned}$$

since $\alpha<ma_0$. Now we choose $N>\nu/\min(p,q)$ and by Peetre's maximal inequality (2) the right side is majorized by the $\dot F_\alpha^{pq}$-norm of f, hence

$$\|\mathcal{D}_{q,\tau}^\alpha f\|_p\leqslant c\|f\|_{\dot F_\alpha^{pq}}.$$

It remains to show the inequality

$$\|f\|_{F_\alpha^{pq}} \leqslant c\|S_{q,r}^\alpha f\|_p. \tag{8}$$

This follows by an approximation procedure with functions of exponential type. We define the operator $T_t f$ by convolution with the kernel

$$\sum_{j=1}^{m}(-1)^{j+1}\binom{m}{j}j^{-n}t^{-\nu}\psi_0\left(A_{1/t}\frac{y}{j}\right),$$

then

$$f - T_t f = \int t^{-\nu}\psi_0(A_{1/t}h)\Delta_{-h}^m f\,dh.$$

Using this representation, the proof of (8) follows the lines of Triebel [16, pp. 82, 103], so we omit the details. ∎

Triebel-Lizorkin spaces on domains

Definition. Let Ω be a subdomain of $\boldsymbol{R}^n$. The space $F_\alpha^{pq}(\Omega, P)$ is defined as the *space of restrictions of* $F_\alpha^{pq}(\boldsymbol{R}^n, P)$ *to* Ω, normed by

$$\|f\|_{F_\alpha^{pq}(\Omega,P)} = \inf\{\|g\|_{F_\alpha^{pq}(\boldsymbol{R}^n,P)},\ g|_\Omega = f \text{ in } \Omega\}.$$

In order to give inner descriptions of those spaces it is necessary to derive extension theorems. Kalyabin [9] proved that for Lipschitz domains the Stein extension operator is a universal extension operator for the isotropic F_α^{pq}-spaces ($1 < p,\ q < \infty,\ \alpha > 0$).

Let $C_{\alpha,r}^{pq}(\Omega, P)$ be the space of $L_{\mathrm{loc}}^p(\Omega)$-functions, for which

$$\|f\|_{C_{\alpha,r}^{pq}} = \left\|\left(\int_0^{\delta(x)}[\operatorname{osc}_r^{m-1}(f,\cdot,t)]^q\frac{dt}{t^{1+\alpha q}}\right)^{1/q}\right\|_p,$$

here $\delta(x) = \dfrac{1}{2b}\inf\{\varrho(x-y);\ y\in\partial\Omega\} =: \dfrac{1}{2b}\varrho(x, \partial\Omega)$.

DeVore and Sharpley [3] and Christ [1] prove extension theorems for isotropic $C_{\alpha,r}^{p\infty}$-spaces; in [1] this is done for the largest known class of extension domains, the (ε, δ)-domains, introduced by Jones [7].

Definition. Let ϱ be an A_t-homogeneous distance function and let $0 < \varepsilon < 1$, $0 < \delta \leqslant \infty$. A domain $\Omega \subset \boldsymbol{R}^n$ is called an $(\varepsilon, \delta, \varrho)$*-domain*, if for given $x, y\in\Omega$ with $\varrho(x-y) < \delta$ there is an arc $\Gamma\subset\Omega$ joining x to y such that the following conditions are satisfied:

(i) $$\sup_{z,z'\in\Gamma}\varrho(z-z') \leqslant \varrho(x-y)/\varepsilon,$$

(ii) $$\varrho(z, \partial\Omega) \geqslant \varepsilon\min\{\varrho(x-z), \varrho(y-z)\}, \qquad z\in\Gamma.$$

A similar definition was given by Fajn [5]. The appropriate extension operator for $(\varepsilon, \delta, \varrho)$-domains is of Whitney type; we have to use the Whitney decomposition theorem [14] in the following form:

PROPOSITION. *Let Ω be an open set in $\mathbf{R}^n$ with nonvoid complement Ω_c. Let ϱ be an A_t-homogeneous distance function, and let $d \geqslant 4b$. Then there is a covering $\mathscr{W}(\Omega) = (Q_j)$ of ϱ-balls with dyadic radii, which has the following properties*

(i) *If $Q_j = \{x, \varrho(x-x_0) \leqslant 2^k\}$, $Q_j^* = \{x, \varrho(x-x_0) \leqslant 2^{k+1}\}$ then we have for $x \in Q_j^*$*

$$\frac{d}{2b} \operatorname{rad} Q_j \leqslant \varrho(x, \partial\Omega) \leqslant 4bd \operatorname{rad} Q;$$

(ii) *There is a constant N such that each $x \in \Omega$ lies in at most N of the Q_j^*.*

As usual we may associate to $\mathscr{W}(\Omega)$ a partition of unity $(\varphi_{Q_j}^*)$;

$$\varphi_{Q_j}^* = \varphi_{Q_j} \Big[\sum_k \varphi_{Q_k}\Big]^{-1},$$

where $\varphi_Q(x) = \chi(A_{1/t}(x-y))$, if $Q = Q_y(t)$, and $\chi \in C_0^\infty(\mathbf{R}^n)$; $\chi(\xi) = 1$ if $\varrho(\xi) \leqslant 1$ and $\operatorname{supp}\chi \subset \{\varrho \leqslant 2\}$.

The key property mf $(\varepsilon, \delta, \varrho)$-domains is (see [7], [1]): There are constants b_0, b_1, b_2, n_0 such that for every $\mathscr{W}(\mathbf{R}^n \setminus \bar{\Omega})$ with $\operatorname{rad}(Q) \leqslant b_0$ there is a 'reflected' ball $\tilde{Q} \in \mathscr{W}(\Omega)$ such that

$$b_1\, \varrho(Q, \tilde{Q}) \leqslant \operatorname{rad} Q, \qquad \operatorname{rad} \tilde{Q} \leqslant b_2\, \varrho(Q, \tilde{Q});$$

here $\varrho(Q, \tilde{Q})$ denotes the ϱ-distance between Q and $\tilde{Q}$; the number of reflected balls of a fixed Q does not exceed n_0.

Now we fix for every $Q_j \in \mathscr{W}(\mathbf{R}^n \setminus \bar{\Omega})$ with $\operatorname{rad} Q \leqslant b_0$ a reflected ball $\tilde{Q}_j \in \mathscr{W}(\Omega)$. Let $\bar{r} = \min(1, p, q)$ and let $\pi_{\tilde{Q}_j} f$ be a best $L^{\bar{r}}(\tilde{Q}_j)$-approximant of f. Then a *Whitney type extension operator* is defined by

$$Ef(x) = \begin{cases} f(x), & x \in \Omega, \\ \sum_j (\pi_{\tilde{Q}_j} f)\, \varphi_{\tilde{Q}_j}^*, & x \in \mathbf{R}^n \setminus \bar{\Omega}. \end{cases}$$

THEOREM 2. *Let Ω be a $(\varepsilon, \delta, \varrho)$-domain, and $0 < p < \infty$, $0 < q$, $r < \infty$, $\alpha > \sigma_{p,q,r}$. Then*

$$\|Ef\|_{C_{\alpha,r}^{pq}(\mathbf{R}^n, P)} \leqslant c\, \|f\|_{C_{\alpha,r}^{pq}(\Omega, P)}.$$

We do not give a detailed proof here. If $r = 1$, $1 < p \leqslant \infty$, $1 < q \leqslant \infty$, it is a straightforward but tedious modification of Christ's proof. The maximal inequalities in [1] must be replaced by vector valued versions, if $q < \infty$. For the modifications needed in the case $p<1$ or $q<1$ see DeVore and Sharpley [3, ch. 12]. If $r>\bar{r}=\min(1, p, q)$ we use, as in the proof of Theorem 1, the pointwise Sobolev inequality (7). ∎

COROLLARY 1. *Let $f\in F_\alpha^{pq}(\Omega, P)$, $\alpha>\sigma_{p,q,r}$, $r\geqslant 1$, $m>\alpha/a_0$. Then*

$$\|f\|_{F_\alpha^{pq}(\Omega,P)}\approx\|f\|_p+\left\|\left(\int_0^{\delta(\cdot)}[\operatorname{osc}_r^{m-1}(f,\cdot,t)]^q\frac{dt}{t^{1+\alpha q}}\right)^{1/q}\right\|_p.$$

This follows immediately by a combination of Theorems 1 and 2.

If p, $q>1$ there is a simpler version of Theorem 1 valid for arbitrary domains Ω. We define

$$B_m=\{x;\ \varrho(sx)\leqslant 1 \text{ for } s\in[-m, m]\}.$$

THEOREM 3. *Let $1<p<\infty$, $1<q\leqslant\infty$, $\alpha>0$, $m\in N$ fixed and*

$$\frac{1}{\min(p,q)}-\frac{\alpha}{v}<\frac{1}{r},\ \frac{1}{s}\leqslant 1.$$

Then for $f\in L^1_{\mathrm{loc}}(\Omega)$ the following seminorms are equivalent:

(i) $$\|f\|_{1,r}=\left\|\left(\int_0^{\delta(\cdot)}[\operatorname{osc}_r^{m-1}(f,\cdot,t)]^q\frac{dt}{t^{1+\alpha q}}\right)^{1/q}\right\|_p,$$

(ii) $$\|f\|_{2,s}=\left\|\left(\int_0^{\delta(\cdot)}\Big[\int_{A_tB_m}|\Delta_h^m f|^s\,dh\Big]^{q/s}\frac{dt}{t^{1+\alpha q}}\right)^{1/q}\right\|_p.$$

Sketch of the proof. We will use an approximation of the identity introducted by Kalyabin [9]. Let $\varphi\in C_0^\infty(\boldsymbol{R}^n)$, $\int\varphi=1$, $\operatorname{supp}\varphi\subset A_{1/2b}B_{m^2}$, and define the kernels $K_t=t^{-v}K(A_{1/t}y)$ by

$$K(y)=\frac{(-1)^{m+1}}{m!}\sum_{j=1}^m\sum_{k=1}^m(-1)^{j+k}k^m\binom{m}{j}\binom{m}{k}\frac{1}{(jk)^n}\varphi\left(\frac{-y}{jk}\right).$$

Then

(9) $$f(x)-K_t*f(x)=\int t^{-v}\tilde{K}(A_th)\,\Delta_h^m f(x)\,dh,$$

(10) $$(y\cdot\nabla)^m K_t*f(x)=\int t^{-v}[(A_{1/t}y\cdot\nabla)^m K_*](A_{1/t}h)\,\Delta_h^m f(x)\,dh$$

where $\tilde{K}$, $K_*\in C_0^\infty(\boldsymbol{R}^n)$ are supported in $A_{1/2b}B_m$, b being the constant in the triangle inequality for ϱ.

The inequalities $\|f\|_{2,1}\leqslant c\|f\|_{2,r}\leqslant c\|f\|_{1,r}$ follows as in the first step of the proof of Theorem 1. To prove $\|f\|_{1,r}\leqslant c\|f\|_{2,1}$, we start with the Sobolev inequality (7) and choose there $\tau=1$, $1<\sigma<p, q$, $\alpha>\beta>v\left(\frac{1}{\sigma}-\frac{1}{r}\right)$. Now let

$$I(y, \omega, x) = \sup \{\mathrm{osc}_1^{m-1}(f, R);\ y \in R \subset Q_x(t),\ \omega/4 \leqslant \mathrm{rad}\, R \leqslant \omega\}$$

and

$$\Omega(t) = \{x, \delta(x) \leqslant t\}.$$

Then by (7)

$$(11) \quad \left(\int_0^{\delta(x)} [\mathrm{osc}_r^{m-1}(f, x, t)]^q \frac{dt}{t^{1+\alpha q}}\right)^{1/q}$$

$$\leqslant c\left(\int_0^{\delta(x)} \Big[\int_{Q_x(t)} \big(\sup_{l \geqslant 0} 2^{l\beta} I(y, 2^{-l}t, x)\big)^\sigma dy\Big]^{q/\sigma} \frac{dt}{t^{1+\alpha q}}\right)^{1/q}$$

$$\leqslant c \sum_{l \geqslant 0} 2^{l(\beta-\alpha)} \left(\int_0^{\infty} \Big[\int_{Q_x(t)} (I(y, t, x))^\sigma dy\Big]^{q/\sigma} \chi_{\Omega(2^{-l}t)}(x) \frac{dt}{t^{1+\alpha q}}\right)^{1/q}.$$

If $R \subset Q_x(t)$ is a ϱ-ball with center x_R and $t/4 \leqslant \mathrm{rad}\, R \leqslant t \leqslant \delta(x)$ then we have

$$\inf_P \int_R |f(z) - P(z)|\, dz \leqslant c \int_R |f(z) - K_t * f(z)|\, dz + \int_R |K_t * f(z) - \Pi_{t,x_R}(z)|\, dz$$

where Π_{t,x_R} is the Taylor polynomial of $K_t * f$ of degree $m-1$ around x_R. We observe for $z \in R$

$$\delta(z) \geqslant \frac{\delta(x)}{b} - \frac{t}{2b} \geqslant \frac{t}{2b},$$

hence

$$\chi_{\Omega(2^{-l}t)}(x) \leqslant \chi_{\Omega(t/2b)}(z), \qquad l \geqslant 0.$$

Using (9), (10) we majorize (11) by

$$c\left(\int_0^{\infty} \Big[\int_{Q_x(t)} \Big(\sup_{\substack{y \in R \subset Q_x(t)\\ \mathrm{rad} R \leqslant t}} \int_R \chi_{\Omega(t/2b)}(z) \int_{A_{t/2b}B_m} |\Delta_h^m f(z)|\, dh dz\Big)^\sigma dy\Big]^{q/\sigma} \frac{dt}{t^{1+\alpha q}}\right)^{1/q}$$

$$\leqslant c\left(\int_0^{\infty} \Big[\mathcal{M}\big[\mathcal{M}\{\chi_{\Omega(t)} \int_{A_tB_m} |\Delta_h f|\, dh\}\big]^\sigma\Big]^{q/\sigma} \frac{dt}{t^{1+\alpha q}}\right)^{1/q}.$$

An application of the Fefferman–Stein inequality (1) concludes the proof. ∎

COROLLARY 2. *Let Ω be an $(\varepsilon, \delta, \varrho)$-domain and $\alpha > \sigma_{p,q,r}$, $m > \alpha/a_0$, $r \geqslant 1$, $1 < p < \infty$, $1 < q \leqslant \infty$. Then*

$$\|f\|_{F_\alpha^{pq}(\Omega, P)} \approx \|f\|_p + \left\|\left(\int_0^{\delta(\cdot)} \Big[\int_{A_tB_m} |\Delta_h^m f|^r\, dh\Big]^{q/r} \frac{dt}{t^{1+\alpha q}}\right)^{1/q}\right\|_p.$$

Remark. For the homogeneous spaces $\dot{F}_\alpha^{pq}$, analogues of Corollaries 1 and 2 are valid in $(\varepsilon, \infty, \varrho)$-domains.

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