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ENTROPY AND n-WIDTHS OF OPERATORS IN BANACH SPACES

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In recent times, much attention has been paid to the study of (bounded linear) operators in Hilbert spaces and Banach spaces by means of geometric quantities, such as entropy numbers, approximation numbers, n-widths etc. The almost classical theory in Hilbert spaces may be found in the book by I. C. Gochberg and M. G. Krein [G–K] while a comprehensive description in the Banach space setting is given in the monographs by A. Pietsch [P], A. Pinkus [Pi], and H. König [Kö]. In the eighties research activity in this area grew considerably. A great deal of classical problems were solved, interesting new developments started, and deep connections between Banach space geometry and other areas of mathematics were discovered. The purpose of this article is to present a survey of main results and current research directions in entropy and s-numbers of operators in Banach spaces. With the help of recent progress in Banach space geometry various examples of special operators are studied.

Throughout the paper E and F always denote Banach spaces. The dual Banach space and the closed unit ball of E are denoted by E' and $\mathscr{U}_E$, respectively. For the class of all (bounded linear) operators from E into F we shall write $\mathfrak{L}(E, F)$, and for $\mathfrak{L}(E, E)$ simply $\mathfrak{L}(E)$. Let us start by defining the entropy and s-numbers we are going to use. Given an operator $T \in \mathfrak{L}(E, F)$, the *n-th entropy number* is defined by

$$\varepsilon_n(T) := \inf\{\varepsilon > 0 : \exists\ y_1, \ldots, y_n \in F \text{ such that } T(\mathscr{U}_E) \subseteq \bigcup_1^n \{y_i + \varepsilon\mathscr{U}_F\}\},$$

the *n-th dyadic entropy number* by

$$e_n(T) := \varepsilon_{2^{n-1}}(T),$$

the *n-th entropy modulus* by

$$g_n(T) := \inf_{k=1,2,\ldots} k^{1/n} \varepsilon_k(T)$$

provided that the underlying Banach spaces are real ones, and

$$g_n(T) := \inf_{k=1,2,\ldots} k^{1/(2n)} \varepsilon_k(T)$$

provided that the underlying Banach spaces are complex ones.

The *n-th approximation number* is defined by

$$a_n(T) := \inf\{\|T-L\|:\ \operatorname{rank}(L) < n\},$$

the *n-th Gelfand number* by

$$c_n(T) := \inf\{\|TJ_M^E\|:\ M \subseteq E,\ \operatorname{codim} M < n\},$$

where J_M^E is the natural embedding from M into E, and the *n-th Kolmogorov number* by

$$d_n(T) := \inf\{\|Q_N^F T\|:\ N \subseteq F,\ \dim N < n\},$$

where Q_N^F is the quotient map from F onto F/N. Furthermore, we use the notation $s := (s_n)$ for an s-number function. Given two sequences of positive real numbers (α_n) and (β_n) we shall write $\alpha_n \asymp \beta_n$ if $\alpha_n \leqslant c\beta_n$ and $\beta_n \leqslant d\alpha_n$ for some positive constants c and d and all $n = 1, 2, \ldots$

I. Inequalities between entropy and s-numbers

This section is devoted to basic inequalities between (dyadic) entropy numbers and s-numbers for operators. They may be interpreted as counterparts to the classical Bernstein–Jackson inequalities for real functions in approximation theory. At first glance we observe that certain analogies exist between

entropy numbers of operators and the modulus of continuity of functions,

and between

approximation numbers of operators and the so-called Bernstein numbers of functions.

Inequalities in which the entropy numbers are estimated by approximation numbers are called *inequalities of Bernstein type*, while reverse estimates are called *inequalities of Jackson type*. The first inequality of Bernstein type may be found in [C1].

THEOREM 1. *Let* $s \in \{a, c, d\}$, $0 < p < \infty$, *and* $T \in \mathfrak{L}(E, F)$. *Then*

$$\sup_{1 \leqslant k \leqslant n} k^{1/p} e_k(T) \leqslant C(p) \sup_{1 \leqslant k \leqslant n} k^{1/p} s_k(T)$$

for $n = 1, 2, \ldots,$ *where* $C(p)$ *is a constant only depending on* p.

Very recently A. Pajor and N. Tomczak-Jaegermann [P–T2] proved the following reverse inequality of Jackson type.

THEOREM 2. *Let* $0 < p < 2$ *and* $T \in \mathfrak{L}(E, H)$, *H being a Hilbert space. Then*

$$\sup_{1 \leqslant k < \infty} k^{1/p} c_k(T) \leqslant C(p) \sup_{1 \leqslant k < \infty} k^{1/p} e_k(T),$$

where $C(p)$ *is a constant only depending on* p.

For operators $T \in \mathfrak{L}(H, K)$ between (real) Hilbert spaces H and K one may derive from a result of Y. Gordon, H. König and C. Schütt [G–K–S] that

$$\sup_{1 \leqslant k < \infty} 2^{-n/k} \Big(\prod_1^k a_i(T)\Big)^{1/k} \leqslant e_k(T) \leqslant 14 \sup_{1 \leqslant k < \infty} 2^{-n/k} \Big(\prod_1^k a_i(T)\Big)^{1/k}.$$

In the case of complex Hilbert spaces one has to replace $2^{-n/k}$ by $2^{-n/(2k)}$.

From this inequality one immediately gets for $T \in \mathfrak{L}(H, K)$, $0 < p < \infty$,

$$\sup_{1 \leqslant k \leqslant n} k^{1/p} e_k(T) \asymp \sup_{1 \leqslant k \leqslant n} k^{1/p} a_k(T).$$

This equivalence is still valid for operators $T \in \mathfrak{L}(E, F)$ acting between two Banach spaces E and F with the property that E and F' are of (Rademacher) type 2 (cf. [C2] and [G–K–S]). We say a Banach space E is of (*Rademacher*) *type p*, $1 < p \leqslant 2$, if there is a constant $C > 0$ such that for all finite families $x_1, \ldots, x_n \in E$ the inequality

$$\Big(E\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\|^2\Big)^{1/2} \leqslant C\Big(\sum_{i=1}^n \|x_i\|^p\Big)^{1/p}$$

holds, where (ε_i) is a sequence of independent random variables, each taking the values $+1$ and -1 with probability $\frac{1}{2}$. The type p constant of E is defined by $\tau_p(E) := \inf C$. As an example let us mention that the function spaces L_p (over an arbitrary σ-finite measure space) are of type $\min(p, 2)$ if $1 \leqslant p < \infty$. For the Rademacher type $\min(p, 2)$ constant one has $\tau_{\min(p,2)}(L_p) \leqslant \sqrt{p}$.

II. Inequalities between entropy moduli and s-numbers

Entropy moduli are very convenient quantities to investigate compactness properties as well as eigenvalue distribution of operators. First we formulate some results given in [C–S2].

THEOREM 3. *Let* $s \in \{c, d\}$ *and* $T \in \mathfrak{L}(E, F)$. *Then*

$$n^{-1}\Big(\prod_{k=1}^{n} s_k(T)\Big)^{1/n} \leqslant g_n(T) \leqslant 48n\Big(\prod_{k=1}^{n} s_k(T)\Big)^{1/n}$$

for $n = 1, 2, \ldots$

Furthermore, if $T \in \mathfrak{L}(H, K)$ acts between Hilbert spaces one may prove the following inequalities:

$$\Big(\prod_{k=1}^{n} a_k(T)\Big)^{1/n} \leqslant g_n(T) \leqslant 12\Big(\prod_{k=1}^{n} a_k(T)\Big)^{1/n}$$

for $n = 1, 2, \ldots$ Related inequalities in the context of type and cotype of Banach spaces may also be found in [C2] and [G–K–S].

Now we turn to some interesting formulas between entropy moduli and approximation numbers. For this purpose we mention the following multiplicative property of the entropy moduli:

$$g_n(ST) \leqslant g_n(S)\, g_n(T).$$

This implies by a little algebraic calculation that for $T \in \mathfrak{L}(E)$ the limit

$$\lim_{k \to \infty} g_n^{1/k}(T^k)$$

exists for each n, $n = 1, 2, \ldots$

Owing to the estimate

$$d_n(T) \leqslant a_n(T) \leqslant (2n)^{1/2} d_n(T)$$

we conclude from Theorem 3 the following formulas.

THEOREM 4. *Let* $T \in \mathfrak{L}(E)$. *Then*

$$\lim_{k \to \infty} g_n^{1/k}(T^k) = \Big(\prod_{i=1}^{n} \lim a_i^{1/k}(T^k)\Big)^{1/n}, \quad \textit{for } n = 1, 2, \ldots$$

In the next section we shall give a meaning to the previous limit formulas in connection with eigenvalues.

Finally, in contrast to the previous formulas we check from the obvious estimates

$$\|T\| = g_1(T) \leqslant n\varepsilon_n(T) \leqslant n\|T\|,$$

$n = 1, 2, \ldots$, that for the single entropy numbers we always have

$$\lim_{k \to \infty} \varepsilon_n^{1/k}(T^k) = \lim_{k \to \infty} \|T^k\|^{1/k}$$

for each $n = 1, 2, \ldots$ Thus the notion of entropy moduli was motivated also from this point of view.

III. Spectral properties and eigenvalues

We describe the spectral properties and eigenvalue behaviour of operators in $\mathfrak{L}(E)$, E being a complex Banach space in this section, in terms of entropy moduli. At the very beginning let us briefly explain the notations we are going to use. Given an operator $T \in \mathfrak{L}(E)$ consider the coset $T + \mathscr{K}(E)$ as an element of the Calkin algebra $\mathfrak{L}(E)/\mathscr{K}(E)$, where $\mathscr{K}$ denotes the two-sided ideal of compact operators in $\mathfrak{L}$. The spectral radius of this element is called the *essential spectral radius of* T, $r_{\text{ess}}(T)$. Let $\sigma(T)$ denote the usual spectrum of T: then for every $r > r_{\text{ess}}(T)$ the set

$$\{\lambda \in C\colon\ \lambda \in \sigma(T), |\lambda| \geqslant r\}$$

consists of only a finite number of points, each being an eigenvalue of T of a finite algebraic multiplicity. Thus we can order all eigenvalues λ of T with $|\lambda| > r_{\text{ess}}(T)$ in such a way that

$$|\lambda_1(T)| \geqslant |\lambda_2(T)| \geqslant \ldots \geqslant r_{\text{ess}}(T),$$

where each eigenvalue is counted according to its algebraic multiplicity. If there are only n $(n = 0, 1, 2, \ldots)$ eigenvalues with $|\lambda| > r_{\text{ess}}(T)$, then we put $\lambda_{n+1}(T) = \lambda_{n+2}(T) = \ldots = r_{\text{ess}}(T)$. So we have assigned to every $T \in \mathfrak{L}(E)$ the sequence $(\lambda_n(T))$.

In this way, as a consequence of H. König's (cf. [Kö], [Z]) spectral radius formula

$$\lim_{k \to \infty} a_n^{1/k}(T^k) = |\lambda_n(T)|,$$

we immediately get from Theorem 4 the result of E. Makai and J. Zemánek [M–Z]:

$$\lim_{k \to \infty} g_n^{1/k}(T^k) = \Big(\prod_1^n |\lambda_i(T)|\Big)^{1/n}.$$

Both formulas are generalizations of the Beurling–Gelfand spectral radius formula. Furthermore, from $g_n(T^k) \leqslant g_n^k(T)$ we get the inequality [C–T]

$$\Big(\prod_{i=1}^n |\lambda_i(T)|\Big)^{1/n} \leqslant g_n(T)$$

which was the starting point for investigating the eigenvalue distributions of various operators with the use of entropy quantities (cf. e.g. [C–K]).

IV. Inequalities between Gelfand numbers and ideal norms

For an operator $T \in \mathfrak{L}(l_2^n, E)$, we define

$$l(T) := \Big(\int_{R^n} \|Tx\|^2 \, d\gamma_n(x) \Big)^{1/2},$$

where γ_n denotes the canonical (normalized) Gaussian measure on the Euclidean space R^n (cf. e.g. [F–L–M]). For any operator $T \in \mathfrak{L}(l_2, E)$ we define the *l-norm of T* by

$$l(T) := \sup \{ l(TX) \colon \|X \colon l_2^n \to l_2\| \leqslant 1, \ n = 1, 2, \ldots \}.$$

The following striking result has been proved by A. Pajor and N. Tomczak-Jaegermann [P–T].

THEOREM 5. *Let E be a Banach space and let $T \in \mathfrak{L}(E, l_2)$ with $l(T') < \infty$. Then*

$$\sup_{1 \leqslant k < \infty} k^{1/2} c_k(T) \leqslant C \cdot l(T'),$$

where C is a universal constant.

Owing to the inequality in Theorem 1:

$$\sup_{1 \leqslant k < \infty} k^{1/2} e_k(T) \leqslant C \sup_{1 \leqslant k < \infty} k^{1/2} c_k(T),$$

Theorem 5 is an improved operator version of Sudakov's minorization theorem for Gaussian processes:

$$\sup_{1 \leqslant k < \infty} k^{1/2} e_k(T) \leqslant C \cdot l(T'),$$

where C is a universal constant.

A converse inequality (called Dudley's majorization theorem [D]) is

$$l(T') \leqslant C \sum_{1}^{\infty} k^{-1/2} e_k(T),$$

where C is a universal constant.

The result of A. Pajor and N. Tomczak-Jaegermann is closely related with the following problem: given an n-dimensional Banach space E, a Euclidean norm $\|\cdot\|_2$ on E and $0 < \lambda < 1$, find a subspace M of E with $\dim M \geqslant \lambda n$ such that

$$\|x\|_2 \leqslant M_* f(1-\lambda) \|x\| \quad \text{for } x \in M.$$

Here M_* denotes the Lévy mean of the dual norm of E:

$$M_* = \Big(\int_S \|x\|_*^2 \, d\mu(x)\Big)^{1/2},$$

where μ is the normalized rotation invariant measure on $S := \{x \in R^n: \|\cdot\|_2 = 1\}$. This problem was considered by V. Milman, who proved in [M] that $f(1-\lambda) \leqslant C/(1-\lambda)$, where C is a universal constant. The estimate was improved by A. Pajor and N. Tomczak-Jaegerman [P–T]: $f(1-\lambda) \leqslant C/\sqrt{1-\lambda}$.

In connection with Banach spaces of weak cotype 2 we should also refer to the following result of V. Milman and G. Pisier [M–P]:

COROLLARY. *Let E be a Banach space of weak cotype 2 and $T \in \mathfrak{L}(l_2, E)$. Then*

$$\sup_{1 \leqslant k < \infty} k^{1/2} a_k(T) \leqslant C \cdot l(T),$$

where C is a constant only depending on E.

Now we use a modified notion of l-norm by using, instead of Gaussian variables, the Rademacher sequence (ε_i). For $T \in \mathfrak{L}(l_2^n, E)$ we define

$$r(T) := \inf\Big(E\Big\|\sum_{i=1}^{n} \varepsilon_i T(e_i)\Big\|^2\Big)^{1/2},$$

where the infimum is taken over all orthonormal bases (e_i) of l_2^n. The following very recent result can be found in [C–P]:

THEOREM 6. *Let $T \in \mathfrak{L}(E, l_2^n)$ $(n = 1, 2, \ldots)$. Then*

$$k^{1/2} c_k(T) \leqslant C \cdot r(T') \log^{1/2}(n/k + 1)$$

for $k = 1, 2, \ldots, n$. Here C is a universal constant.

This inequality may be interpreted as a minorization theorem for the Rademacher sequence (ε_i).

Next we give inequalities between Gelfand numbers and absolutely 2-summing norms, π_2. Let $T \in \mathfrak{L}(E, F)$, $0 < p < \infty$. We will say that T is *absolutely p-summing* if there is a constant C such that for all finite families $x_1, \ldots, x_n \in E$ we have

$$\Big(\sum_{i=1}^{n} \|Tx_i\|^p\Big)^{1/p} \leqslant C \sup\Big\{\Big(\sum_{i=1}^{n} |\langle x_i, a\rangle|^p\Big)^{1/p}: \|a\| \leqslant 1\Big\}.$$

We will denote by $\pi_p(T)$ the smallest C satisfying this inequality. In [C–P] one may find the following striking results.

THEOREM 7. (i) *Let E be a Banach space such that the dual E' is of (Rademacher) type 2 and $T \in \mathfrak{L}(E, F)$ with $\pi_2(T) < \infty$. Then*

$$\sup_{1 \leqslant k < \infty} k^{1/2} c_k(T) \leqslant C\tau_2(E')\pi_2(T),$$

where C is a universal constant.

(ii) *Let $n = 1, 2, \ldots,$ and $T \in \mathfrak{L}(l_1^n, F)$. Then*

$$k^{1/2} c_k(T) \leqslant C\pi_2(T)\log^{1/2}(n/k+1)$$

for $k = 1, 2, \ldots, n$.

As an immediate corollary of Theorem 7 (ii), we obtain for $T \in \mathfrak{L}(l_1^n, H)$, H being a Hilbert space, by using the "little" Grothendieck theorem: $\pi_2(T) \leqslant C\|T\|$ (where $C \leqslant \sqrt{\pi/2}$ for real spaces and $C \leqslant 2/\sqrt{\pi}$ for complex spaces), the following striking result [C–P]:

THEOREM 8. *Let $n = 1, 2, \ldots$ and $T \in \mathfrak{L}(l_1^n, H)$, H being a Hilbert space. Then*

$$k^{1/2} c_k(T) \leqslant C\|T\|\log^{1/2}(n/k+1)$$

for $k = 1, 2, \ldots, n$, where C is a universal constant.

In particular, for the identity operators $I_n\colon l_1^n \to l_2^n$ we obtain the asymptotically optimal estimate of A.-Yu. Garnaev and E. D. Gluskin [G–G] (cf. also B. S. Kashin [K]):

$$k^{1/2} c_k(I_n) \leqslant C\log^{1/2}(n/k+1).$$

In this direction we also have to refer to estimates of Gelfand numbers for identity operators between symmetric spaces obtained by Y. Gordon, H. König and C. Schütt [G–K–S] via probabilistic arguments.

V. Integral operators

Since 1959 theorems about the entropy of embedding maps between function spaces over compact metric spaces have been obtained. We mention the work of A. N. Kolmogorov and V. M. Tikhomirov [K–T], G. G. Lorentz [L], A. F. Timan [T], and S. Heinrich and T. Kühn [H–K]. The main purpose of this section is to give some new results in this direction by using the results of the previous sections. We determine the asymptotic behaviour of entropy, Kolmogorov and Gelfand numbers of integral operators generated by Hölder continuous kernels on metric compacta. The results will show the interaction between functional-analytic quantities and the metric topology of the underlying compact metric space. Following [H–K] we denote by

(X, d) a compact metric space and by

$$\varepsilon_n(X) := \inf\{\varepsilon > 0\colon X \text{ has an } \varepsilon\text{-net of cardinality} < n\}$$

the nth entropy number of X with respect to the metric d. As usual, $C(X)$ stands for the space of continuous functions on X equipped with the supremum norm while $C^\alpha(X)$ is the space of continuous functions f on X satisfying a Hölder condition

$$|f(x_1) - f(x_2)| \leqslant M \cdot d(x_1, x_2)^\alpha \quad \text{with } 0 < \alpha \leqslant 1.$$

$C^\alpha(X)$ is a Banach space with respect to the norm

$$\|f\|_\alpha := \max\left\{\sup_{x \in X} |f(x)|,\ \sup_{\substack{x_1, x_2 \in X \\ x_1 \neq x_2}} \frac{|f(x_1) - f(x_2)|}{d(x_1, x_2)^\alpha}\right\}.$$

The embedding map $I_\alpha\colon C^\alpha(X) \to C(X)$ of $C^\alpha(X)$ into $C(X)$ turns out to be compact. The asymptotic behaviour of the approximation numbers, Kolmogorov numbers and Gelfand numbers can be described by ([H–K]):

$$a_n(I_\alpha) \asymp d_n(I_\alpha) \asymp c_n(I_\alpha) \asymp \varepsilon_n^\alpha(X).$$

S. Heinrich and T. Kühn [H–K] also gave a precise characterization for the asymptotic behaviour of the entropy numbers. However, their result is rather involved in the general case. Owing to the estimate $e_n(I_\alpha) \geqslant \frac{1}{4}\varepsilon_n^\alpha(X)$ and $a_n(I_\alpha) \asymp \varepsilon_n^\alpha(X)$ we may easily check by Theorem 1 the following characterization ([C–S1]):

Let $0 < p < \infty$. Then

$$\sup_{1 \leqslant k \leqslant n} k^{1/p} e_k(I_\alpha) \asymp \sup_{1 \leqslant k \leqslant n} k^{1/p} \varepsilon_k^\alpha(X).$$

If one has the additional condition

$$\varepsilon_k(X) \leqslant \varrho (n/k)^\sigma \varepsilon_n(X) \quad \text{for } 1 \leqslant k \leqslant n,\ n = 1, 2, \ldots,$$

with appropriate constants $\varrho \geqslant 1$ and $\sigma > 0$, then one may derive from the previous formula that

$$e_n(I_\alpha) \asymp \varepsilon_n^\alpha(X).$$

In the case of a connected compact metric space (X, d) the previous additional assumption is fulfilled with $\varrho = 16$ and $\sigma = 1$ (cf. [H–K]). In this way the original result of Timan [T] is recovered. But also the case of a compact metric space (X, d) with $\varepsilon_n(X) \asymp n^{-\gamma}$, $0 < \gamma < \infty$, is comprehended by this condition, since then $\varepsilon_k(X) \leqslant \varrho (n/k)^\gamma \varepsilon_n(X)$. This condition does not say anything about the connectedness of metric compacta. We refer to the cube $X = [0, 1]^N$ of the N-dimensional Euclidean space whose entropy numbers satisfy $\varepsilon_n(X) \asymp n^{-\gamma}$ with $\gamma = 1/N$ on the one hand, and to Cantor's disconnected ternary set $X \subset [0, 1]$ on the other hand, whose degree of compactness can be described by $\varepsilon_n(X) \asymp n^{-\gamma}$ with $\gamma = \log 3/\log 2$.

Now we turn to entropy, Kolmogorov and Gelfand numbers of integral operators acting between L_p spaces which can be factored through the embedding map I_α: $C^\alpha(X) \to C(X)$. The result is much stronger than one would expect. Indeed, the entropy and s-number behaviour of the integral operator is influenced not only by the entropy and s-number behaviour of I_α. Let μ be a positive Radon measure on X with $\mu(X) \leqslant 1$ and $K(x, y)$ a continuous kernel on $X \times X$ which belongs to $C^\alpha(X)$ for all $y \in X$ and fulfils the condition

$$\sup \|K(\cdot, y)\|_\alpha = \|K\|_{\alpha,\infty} < \infty .$$

Then the integral operator

$$(T_{K,\mu} f)(x) := \int_X K(x, y)\, d\mu(y)$$

generated by $K(x, y)$ can be regarded as an operator

$$T^{(\alpha)}_{K,\mu}\colon L_1(X, \mu) \to C^\alpha(X)$$

from $L_1(X, \mu)$ into $C^\alpha(X)$ with a norm estimate

$$\|T^{(\alpha)}_{K,\mu}\| \leqslant \|K\|_{\alpha,\infty}.$$

Unfortunately, we are not able to determine the degree of compactness of the operator $T^{(\alpha)}_{K,\mu}$: $L_1(x, \mu) \to C^\alpha(X)$. But the situation changes if $T_{K,\mu}$ is considered as an operator from $L_r(X, \mu)$ in $L_s(X, \mu)$. We start with the following statement.

THEOREM 9. *Let (X, d) be a compact metric space, μ a positive Radon measure on X with $\mu(X) \leqslant 1$, and $K(x, y)$ a continuous kernel on $X \times X$ such that $K(\cdot, y) \in C^\alpha(X)$ for all $y \in X$ and $\|K\|_{\alpha,\infty} < \infty$.*

Then the Kolmogorov numbers of $T_{K,\mu} \in \mathfrak{L}(L_2(X, \mu), L_s(X, \mu))$, $2 \leqslant s < \infty$, satisfy the inequality

$$d_{2n-1}(T_{K,\mu}) \leqslant C \cdot s^{1/2} \|K\|_{\alpha,\infty} n^{-1/2} \varepsilon_n^\alpha(X),$$

and the Gelfand numbers of $T_{K,\mu} \in \mathfrak{L}(L_r(X, \mu), L_2(X, \mu))$, $1 < r \leqslant 2$, satisfy the inequality

$$c_{2n-1}(T_{K,\mu}) \leqslant C \cdot r'^{1/2} \|K\|_{\alpha,\infty} n^{-1/2} \varepsilon_n^\alpha(X),$$

$n = 1, 2, \ldots$, where C is a universal constant and r' is defined by $1/r' + 1/r = 1$.

We briefly sketch the proof. The key to the first estimate is the composition diagram

$$\begin{tikzcd}
L_2(X,\mu) \arrow[rr, "T_{K,\mu}"] \arrow[d, "I_{2,1}"'] & & L_s(X,\mu) \\
L_1(X,\mu) \arrow[dr, "T^{(\alpha)}_{K,\mu}"'] & & C(X) \arrow[u, "I_{\infty,s}"'] \\
& C^{\alpha}(X) \arrow[ur, "I_{\alpha}"'] &
\end{tikzcd}$$

where $I_{2,1}$ denotes the embedding map of $L_2(X, \mu)$ into $L_1(X, \mu)$ and $I_{\infty,s}$ the embedding map of $C(X)$ into $L_s(X, \mu)$. The dual of $I_{2,1}$ is absolutely 2-summing with $\pi_2^{\text{dual}}(I_{2,1}) := \pi_2(I'_{2,1}) \leqslant 1$. Accordingly, the dual of the operator $T_{K,\mu} = I_{\infty,s} I_\alpha T^{(\alpha)}_{K,\mu} I_{2,1}$ turns out to be absolutely 2-summing so that the generalized approximation numbers $a_k(T_{K,\mu};\, \pi_2^{\text{dual}})$ with respect to the dual absolutely 2-summing norm make sense:

$$a_k(T_{K,\mu};\, \pi_2^{\text{dual}}) := \inf\{\pi_2^{\text{dual}}(T_{K,\mu} - L):\ \operatorname{rank}(L) < k\}.$$

They satisfy the estimate

$$a_k(T_{K,\mu};\, \pi_2^{\text{dual}}) \leqslant \|I_{\infty,s}\|\, a_k(I_\alpha)\, \|T^{(\alpha)}_{K,\mu}\|\, \pi_2^{\text{dual}}(I_{2,1}).$$

Since $\pi_2^{\text{dual}}(I_{2,1}) \leqslant 1$, $\|T^{(\alpha)}_{K,\mu}\| \leqslant \|K\|_{\alpha,\infty}$, $\|I_{\infty,s}\| \leqslant 1$, and $a_k(I) \leqslant \varrho\varepsilon_n^\alpha(X)$, we obtain

$$a_k(T_{K,\mu};\, \pi_2^{\text{dual}}) \leqslant \varrho\, \|K\|_{\alpha,\infty}\, \varepsilon_n^\alpha(X).$$

The reiteration theorem (cf. [C–S1]) together with the dual version of Theorem 7 (i) yield by a little calculation

$$d_{2n-1}(T_{K,\mu}) \leqslant Cs^{1/2} n^{-1/2} a_n(T_{K,\mu};\, \pi_2^{\text{dual}}) \leqslant \varrho C s^{1/2} \|K\|_{\alpha,\infty}\, n^{-1/2} \varepsilon_n^\alpha(X).$$

The second assertion of the theorem may be treated similarly.

From Theorem 9, Theorem 1 and the method employed in [H–K] one may derive the following statement.

THEOREM 10. *Let (X, d) be a connected compact metric space. Then, for $T_{K,\mu} \in \mathfrak{L}(L_r(X, \mu), L_r(X, \mu))$, we have*

$$\sup_{\substack{\|K\|_{\alpha,\infty} \leqslant 1 \\ \|\mu\| \leqslant 1}} d_n(T_{K,\mu}) \asymp \sup_{\substack{\|K\|_{\alpha,\infty} \leqslant 1 \\ \|\mu\| \leqslant 1}} e_n(T_{K,\mu}) \asymp n^{-1/2} \varepsilon_n^\alpha(X)$$

provided that $2 \leqslant r < \infty$, *and*

$$\sup_{\substack{\|K\|_{\alpha,\infty} \leqslant 1 \\ \|\mu\| \leqslant 1}} c_n(T_{K,\mu}) \asymp \sup_{\substack{\|K\|_{\alpha,\infty} \leqslant 1 \\ \|\mu\| \leqslant 1}} e_n(T_{K,\mu}) \asymp n^{-1/2} \varepsilon_n^\alpha(X),$$

provided that $1 < r \leqslant 2$.

Very recently the behaviour of the Kolmogorov numbers in case $T_{K,\mu} \in \mathfrak{L}(C(X), C(X))$ has been treated in [C–H–K] (cf. [C–S2]). Besides, there one can also find connections with the metric dimension of the underlying compact metric space (X, d). Moreover, the behaviour of the Kolmogorov numbers of positive-definite kernels over metric compacta satisfying a Hölder condition was also determined.

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