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EQUIVALENT NORMS IN SPACES OF BESOV-TRIEBEL-LIZORKIN TYPE DEFINED BY PSEUDODIFFERENTIAL OPERATORS

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In function spaces of Besov–Triebel–Lizorkin type $B^s_{p,q}(R^n)$ and $F^s_{p,q}(R^n)$ norms can be defined by a resolution of unity $\{\psi_k(\xi)\}_{k=0}^{\infty}$ in R^n_ξ which is connected with the symbol $|\xi|^2$ of the Laplacian $-\Delta$.

In this paper we consider decompositions of $R^n_x \times R^n_\xi$ which are induced by symbols $a(x, \xi)$ of appropriate pseudodifferential operators. This means that we may have different resolutions $\{\varphi_j(x, \xi)\}_{j=0}^{\infty}$ in R^n_ξ for different $x \in R^n_x$.

In the special case that the decomposition is induced by an elliptic pseudodifferential operator we will prove that such systems $\{\varphi_j(x, \xi)\}_{j=0}^{\infty}$ of symbols of pseudodifferential operators define equivalent norms in the usual function spaces $B^s_{p,q}(R^n)$ and $F^s_{p,q}(R^n)$.

In Section 1 we recall some facts about classical pseudodifferential operators (see also [3], [6] or [7]). We use the notation of [3].

Section 2 deals with the proof of a vector-valued multiplier theorem in $L^\Omega_p(l_q)$ for pseudodifferential operators.

In the last section we define systems of pseudodifferential operators $\{\varphi_j(x, \xi)\}_{j=0}^{\infty}$ belonging to an elliptic pseudodifferential operator and prove that the norms defined by these systems are equivalent norms in the usual function spaces.

1. Basic properties of pseudodifferential operators

Let $p(x, \xi)$ be a polynomially bounded complex-valued function defined on $R^n_x \times R^n_\xi$. The pseudodifferential operator $P(x, D_x)$ with symbol $p(x, \xi)$ is defined by

$$P(x, D_x)u(x) = (2\pi)^{-n} \int e^{ix\xi} p(x, \xi)(Fu)(\xi)\,d\xi \qquad \text{for } u \in S(R^n)$$

where $S(R^n)$ denotes the Schwartz class and $(Fu)(\xi) = \int e^{-iy\xi} u(y)\,dy$ denotes the Fourier transform of u.

A function $p(x, \xi)$ belongs to the class $S^m_{\varrho,\delta}$ $(-\infty < m < \infty$, $0 \leqslant \delta \leqslant \varrho \leqslant 1$, $\delta < 1)$ if for any multi-indices α, β there exists a constant $c_{\alpha\beta}$ such that

$$|p^{(\alpha)}_{(\beta)}(x, \xi)| \leqslant c_{\alpha\beta} \langle\xi\rangle^{m-\varrho|\alpha|+\delta|\beta|} \quad \text{for } (x, \xi) \in R^n_x \times R^n_\xi,$$

where $p^{(\alpha)}_{(\beta)}(x, \xi) = \partial^\alpha_\xi D^\beta_x p(x, \xi)$, $\partial^\alpha_\xi = \partial^{|\alpha|}/\partial\xi_1^{\alpha_1} \dots \partial\xi_n^{\alpha_n}$, $D^\beta_x = (-i)^{|\beta|}\partial^\beta_x$ and $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$.

We set $S^{-\infty} = \bigcap_m S^m_{\varrho,\delta}$. It is easy to see that $\bigcap_m S^m_{\varrho,\delta} = \bigcap_m S^m_{1,0}$ for any ϱ and δ.

The pseudodifferential operator $P(x, D_x)$ with symbol $p \in S^m_{\varrho,\delta}$ maps $S(R^n)$ continuously into itself and can be extended to a continuous operator from $S'(R^n)$ into $S'(R^n)$, the space of all tempered distributions on R^n. The mapping $p(x, \xi) \to P(x, D_x)$ is a bijection.

For $p \in S^m_{\varrho,\delta}$ we define the seminorms $|p|^{(m)}_{(l,k)}$ by

$$(1) \qquad |p|^{(m)}_{(l,k)} = \max_{|\alpha| \leqslant l, |\beta| \leqslant k} \sup_{(x,\xi)} \{|p^{(\alpha)}_{(\beta)}(x, \xi)| \langle\xi\rangle^{-m+\varrho|\alpha|-\delta|\beta|}\}.$$

THEOREM 1. *Assume that* $0 \leqslant \delta < \varrho \leqslant 1$. *Let* $P_1(x, D_x) \in S^{m_1}_{\varrho,\delta}$ *and* $P_2(x, D_x) \in S^{m_2}_{\varrho,\delta}$. *Then* $P(x, D_x) = P_1(x, D_x) P_2(x, D_x)$ *belongs to* $S^{m_1+m_2}_{\varrho,\delta}$.

For the symbol $p(x, \xi)$ of $P(x, D_x)$ and for any N we have the expansion formula

$$(2) \quad p(x, \xi) = \sum_{|\alpha| < N} \frac{1}{\alpha!} p_1^{(\alpha)}(x, \xi) p_{2(\alpha)}(x, \xi) + N \sum_{|\gamma| = N} \int_0^1 \frac{(1-\vartheta)^{N-1}}{\gamma!} r_{\gamma,\vartheta}(x, \xi)\,d\vartheta,$$

where

$$(3) \qquad r_{\gamma,\vartheta}(x, \xi) = \text{Os-}(2\pi)^{-n} \int\int e^{-iy\eta} p_1^{(\gamma)}(x, \xi+\vartheta\eta) p_{2(\gamma)}(x+y, \xi)\,dy\,d\eta.$$

$\{r_{\gamma,\vartheta}\}_{|\vartheta| \leqslant 1}$ is a bounded subset of $S^{m_1+m_2-|\gamma|(\varrho-\delta)}_{\varrho,\delta}$. Furthermore, for any integers (l, k) there exist constants c, c' and integers l', k' independet of ϑ such that

$$(4) \qquad |p_1^{(\alpha)} p_{2(\alpha)}|^{(m_1+m_2-|\alpha|(\varrho-\delta))}_{(l,k)} \leqslant c\,|p_1|^{(m_1)}_{(l+|\alpha|,k)} |p_2|^{(m_2)}_{(l,k+|\alpha|)},$$

$$(5) \qquad |r_{\gamma,\vartheta}|^{(m_1+m_2-|\gamma|(\varrho-\delta))}_{(l,k)} \leqslant c'\,|p_1|^{(m_1)}_{(l',k)} |p_2|^{(m_2)}_{(l,k')}.$$

The theorem gives an estimate of each term of the sum (2) which is obtained by the composition of two pseudodifferential operators. Especially the estimate of the remainder term will often be useful. The proof is a direct consequence of the definition of the seminorms and of [3] (Section 2); see also there for details.

THEOREM 2. *Let* $P(x, D_x) \in S^0_{1,\delta}$ *and* $\delta < 1$. *Then for all* p *with* $1 < p < \infty$ *there exist integers* (l, k) *and a constant* c, *all independent of* $P(x, D_x)$, *such* *that*

$$\|P(x, D_x)u \mid L_p\| \leqslant c\,|p|^{(0)}_{(l,k)}\,\|u \mid L_p\| \tag{6}$$

for all $u \in L_p(R^n)$.

This was proved first by Illner [2] in 1975. Later for example Bourdaud [1] and Nagase [5] considered non-regular symbols and got weaker conditions on $p(x, \xi)$.

COROLLARY 1. *Let* $P(x, D_x) \in S^m_{1,\delta}$, $\delta < 1$, $1 < p < \infty$ *and* $-\infty < t, m < \infty$. *Then there exist integers* (l, k) *and a constant* c *such that for* $u \in H^{t+m}_p(R^n)$

$$\|P(x, D_x)u \mid H^t_p\| \leqslant c\,|p|^{(m)}_{(l,k)}\,\|u \mid H^{t+m}_p\|. \tag{7}$$

Again the constants are independent of $P(x, D_x)$ *and* u. $H^t_p(R^n)$ *denotes the Bessel potential spaces.*

Let $A(x, D_x) \in S^m_{1,\delta}$ and $\delta < 1$. We say that A is *elliptic of order m* if there exist constants $c' > 0$, $c > 0$ and $R \geqslant 0$ such that

$$c' \langle\xi\rangle^m \leqslant |a(x, \xi)| \leqslant c\,\langle\xi\rangle^m \quad \text{for } |\xi| \geqslant R. \tag{8}$$

DEFINITION 1. Let N be an integer. A symbol $a(x, \xi)$ induces a *variable covering* $\{\Omega^{N,a}_j\}^\infty_{j=0}$ of $R^n_x \times R^n_\xi$ by

$$\Omega^{N,a}_j = \begin{cases} \{(x, \xi)\colon |a(x, \xi)| < 2^{J+N+j}\} & \text{if } j = 0, 1, \ldots, N-1, \\ \{(x, \xi)\colon 2^{J-N+j} < |a(x, \xi)| < 2^{J+N+j}\} & \text{if } j = N, N+1, \ldots \end{cases} \tag{9}$$

J is a constant which is fixed in such a way that $|\xi| \leqslant R$ always implies $(x, \xi) \in \Omega^{N,a}_0$.

In view of (8) we can find such a J for every elliptic pseudodifferential operator.

Variable covering means that at different points $x \in R^n_x$ we may have different coverings of R^n_ξ, in contrast to the classical dyadic coverings of R^n_ξ which are the basis for the definition of the spaces $B^s_{p,q}(R^n)$ and $F^s_{p,q}(R^n)$.

The definition is a special case of Definition 2.2 in [4] where variable coverings of $R^n_x \times R^n_\xi$ were considered also for hypoelliptic symbols of the class $S(m, m'; \delta)$. $S(m, m'; \delta)$ denotes the class of all symbols $a(x, \xi)$ which fulfil Condition (H) in [3], p. 83, with the parameters $\varrho = 1$, $\delta < 1$ and $0 < m' \leqslant m$.

2. A vector-valued multiplier theorem

Let $0 < p < \infty$ and $0 < q \leqslant \infty$. If $\Omega = \{\Omega_k\}^\infty_{k=0}$ is a sequence of compact subsets of R^n, then we denote by

$$L^\Omega_p(l_q) = \{u\colon u = \{u_k\}^\infty_{k=0} \subset S'(R^n), \operatorname{supp} Fu_k \subset \Omega_k \text{ if } k = 0, 1, 2, \ldots \text{ and } \|u_k \mid L_p(l_q)\| < \infty\},$$

where

$$\|u_k \mid L_p(l_q)\| = \Big(\int_{R^n} \Big(\sum_{k=0}^{\infty} |u_k(x)|^q \Big)^{p/q} dx \Big)^{1/p} \quad \text{(modified if } q = \infty\text{)},$$

an $L_p(l_q)$-space of analytic functions.

The following theorem generalizes a Fourier multiplier theorem of Triebel ([8], 1.6.3) to the case of pseudodifferential operators and will be useful in the next section.

THEOREM 3. *Let* $0 < p < \infty$, $0 < q \leqslant \infty$, *let* $\{d_k\}_{k=0}^{\infty}$ *be a sequence of positive numbers and* $\{\Omega_k\}_{k=0}^{\infty}$ *a sequence of compact subsets of* R^n *with* $\Omega_k \subset \{\xi\colon |\xi| \leqslant d_k\}$. *Let* $\{m_k(x, \xi)\}_{k=0}^{\infty} \subset S^{-\infty}$ *be a sequence of symbols with*

$$\operatorname{supp} m_k \subset \{(x, \xi)\colon (x, \xi) \in R_x^n \times R_\xi^n \text{ and } \varepsilon d_k \leqslant |\xi| \leqslant d_k\} \tag{10}$$

where ε *is a fixed real number, independent of* k.

If l *is an even natural number with* $l > n + n/\min(p, q)$, *then there exists a constant* c *such that*

$$\|M_k(x, D_x) u_k \mid L_p(l_q)\| \leqslant c\varepsilon^{-l} \sup_j |m_j|_{(l,0)}^{(0)} \|u_k \mid L_p(l_q)\| \tag{11}$$

for all systems $\{u_k\}_{k=0}^{\infty} \in L_p^{\Omega}(l_q)$. *Here* $|m_j|_{(l,0)}^{(0)}$ *are the seminorms in* $S_{1,\delta}^0$ *and* $M_j(x, D_x)$ *are the corresponding pseudodifferential operators.*

Proof. Because the symbol $m_k(x, \xi)$ belongs to $S^{-\infty}$ it is easy to verify that the corresponding pseudodifferential operator $M_k(x, D_x)$ has the C^∞-kernel representation

$$M_k(x, D_x) u(x) = \int K_k(x, x-y) u(y)\, dy \quad \text{for } u \in L_\infty(R^n)$$

where

$$K_k(x, w) = (2\pi)^{-n} \int e^{iw\xi} m_k(x, \xi)\, d\xi, \tag{12}$$

and for any multi-indices β, γ and natural numbers l' the kernel $K_k(x, w)$ satisfies

$$\sup_{(x,w)} \{\langle w \rangle^{l'} |\partial_x^\beta \partial_w^\gamma K_k(x, w)|\} < \infty.$$

On the other hand, we have $\{u_k\} \in L_p^{\Omega}(l_q)$ which implies $u_k \in L_\infty(R^n)$ for all k.

Now in complete analogy to the proof of Theorem 1.6.3 in [8] we get for $0 < r < \min(p, q)$

$$\begin{aligned} &|(M_k(x, D_x) u_k)(x-z)| \\ &\qquad \leqslant c \sup_w \frac{|u_k(w)|}{1 + |d_k(x-w)|^{n/r}} \int |K_k(x-z, x-z-y)| \big(1 + |d_k(x-y)|^{n/r}\big)\, dy \end{aligned}$$

and this leads to

$$(13)\quad |M_k(x, D_x)u_k(x)| \leqslant c' \sup_w \frac{|u_k(w)|}{1+|d_k(x-w)|^{n/r}} \times \sup_z \int |K_k(x-z, x-z-y)|\left(1+|d_k(x-y-z)|^{n/r}\right)dy$$

where the constant c' is independent of x, k, d_k and of the functions u_k and m_k.

It suffices to estimate the second factor in (13) uniformly for arbitrary $x \in R^n_x$. Then the first factor will be estimated by a maximal inequality.

Let l be an even natural number with $l > n+n/r$. So we obtain in view of (12)

$$(14)\quad \sup_{(x,z)} \int |K_k(x-z, x-z-y)|\left(1+|d_k(x-y-z)|^{n/r}\right)dy \leqslant c \int \sup_{(x,z)} \left| \int e^{iw\eta} m_k(x-z, d_k\eta)(1+|w|^l)\,d\eta \right| (1+|w|^{n/r})(1+|w|^l)^{-1}\,dw.$$

Furthermore, if γ is an arbitrary multi-index with $|\gamma| = l$, we get by (12) and (10) the estimate

$$(15)\quad \left|\int e^{iw\eta} m_k(x-z, d_k\eta)\,w^\gamma\,d\eta\right| \leqslant \left|\int e^{iw\eta} d_k^l (D_\xi^\gamma m_k)(x-z, d_k\eta)\,d\eta\right| \leqslant c\varepsilon^{-l} |m_k|^{(0)}_{(l,0)}$$

where the constant c is independent of x, z, w and m_k. Since $l > n+n/r$ (11) is now an immediate consequence of (13), (14), (15) and the maximal inequality in [8], Theorem 1.6.2.

By a small modification in the estimate (14) we can prove, in complete analogy to [8], 1.6.3, the following assertion.

COROLLARY 2.

$$\|M_k(x, D_x)u_k \mid L_p(l_q)\| \leqslant c \sup_j \|m_j(x, d_j\xi) \mid H^\mu_2(R^n_\xi) \mid L_\infty(R^n_x)\| \, \|u_k \mid L_p(l_q)\|$$

for all systems $\{u_k\}_{k=0}^\infty \in L_p^\Omega(l_q)$, *all sequences of symbols* $\{m_k(x, \xi)\}_{k=0}^\infty \subset S^{-\infty}$ *with* $\|m_k(x, \xi) \mid H^\mu_2(R^n_\xi) \mid L_\infty(R^n_x)\| < \infty$ *and* $\mu > n/2 + n/\min(p, q)$.

3. Equivalent norms in $F^s_{p,q}(R^n)$

Let $a(x, \xi) \in S^m_{1,\delta}$ be the symbol of an elliptic pseudodifferential operator of order m and $\delta < 1$.

DEFINITION 2. We say that a *system* $\{\varphi_j(x, \xi)\}_{j=0}^\infty$ *belongs to the symbol* $a(x, \xi)$ if for an arbitrary fixed N and all $j = 0, 1, 2, \dots$ the following conditions are satisfied:

(i) $\varphi_j \in C^\infty(R^n_x \times R^n_\xi)$ and $\varphi_j(x, \xi) \geqslant 0$.

(ii) $\operatorname{supp} \varphi_j \subset \Omega_j^{N,a}$.

(iii) $|\varphi_{j(\beta)}^{(\alpha)}(x, \xi)| \leqslant c_{\alpha\beta} \langle \xi \rangle^{-|\alpha| + \delta|\beta|}$ for any multi-indices α and β, where the constants $c_{\alpha\beta}$ are independent of j.

(iv) $\sum_{j=0}^{\infty} \varphi_j(x, \xi) = c^\varphi > 0$.

It is easily seen that there always exist non-trivial systems $\{\varphi_j(x, \xi)\}_{j=0}^\infty$ (non-trivial means that the symbol $\varphi_j(x, \xi)$ does depend on x).

By assumption (ii) and (iii) we get $\varphi_j \in S^{-\infty}$. The following estimates for the seminorms of φ_j are simple consequences of (ii), (iii) and (9):

$$|\varphi_j|_{(l,k)}^{(\varkappa)} \leqslant c_{lk\varkappa} 2^{-j\varkappa/m} \tag{16}$$

for all real numbers $\varkappa$ and with constants $c_{lk\varkappa}$ independent of j.

Also, from (iii), respectively (16), it follows that the seminorms of the φ_j are uniformly bounded in $S^0_{1,\delta}$. Together with (ii) and (iv) we get in this way

$$\sum_{j=0}^{J} \varphi_j(x, \xi) \to c^\varphi \quad \text{in } S^0_{1,\delta} \text{ weakly as } J \to \infty.$$

The weak convergence in $S^0_{1,\delta}$ and Corollary 1 imply that for every $v \in H^s_p(R^n)$

$$\sum_{j=0}^{J} \varphi_j(x, D_x) v \to c^\varphi v \quad \text{in } H^s_p(R^n) \text{ as } J \to \infty$$

(see also [3], Ch. 3, §7, where this fact was proved for $L_2(R^n)$). But in view of Corollary 1 there is no difficulty in carrying over the proof to the case $1 < p < \infty$ and the Bessel potential spaces for arbitrary real s (see [4], Lemma 2.1).

DEFINITION 3. Let $1 < p < \infty$, $0 < q \leqslant \infty$, $-\infty < s < \infty$ and let $\{\varphi_j(x, \xi)\}_{j=0}^\infty$ be a system belonging to $a(x, \xi)$. We define

$$B^{s,a}_{p,q}(R^n) = \{u\colon u \in S'(R^n) \text{ and } \|u \mid B^{s,a}_{p,q}\|^\varphi = \|2^{sj} \varphi_j(x, D_x) u \mid l_q(L_p(R^n))\| < \infty\},$$

$$F^{s,a}_{p,q}(R^n) = \{u\colon u \in S'(R^n) \text{ and } \|u \mid F^{s,a}_{p,q}\|^\varphi = \|2^{sj} \varphi_j(x, D_x) u \mid L_p(R^n, l_q)\| < \infty\}.$$

Remark. Let $a(x, \xi) = \langle \xi \rangle$, so that $\{\Omega_j^{N,a}\}_{j=0}^\infty$ is a fixed covering of $R^n_x \times R^n_\xi$ which is independent of R^n_x. Furthermore, suppose that the system $\{\varphi_j(x, \xi)\}_{j=0}^\infty$ belonging to $\langle \xi \rangle$ is also independent of x. Then $\varphi_j(\xi) = \varphi_j(x, \xi)$ can be interpreted as a Fourier multiplier and Definition 3 is the usual definition of Besov–Triebel–Lizorkin spaces. As far the notation used is concerned we refer to [8], 2.3.

Remark. The definition of $B^{s,a}_{p,q}(R^n)$ is a special case of Definition 3.1 in [4] where we introduced and studied in detail Besov spaces of variable order of differentiation based on more general hypoelliptic symbols $a(x, \xi) \in S(m, m'; \delta)$. In the case of elliptic symbols $a(x, \xi)$ of order m it turns out that the spaces $B^{s,a}_{p,q}(R^n)$ coincide with the usual Besov spaces $B^{sm}_{p,q}(R^n)$ ([4], Corollary 3.3).

Here we have also defined spaces of Triebel–Lizorkin type for elliptic symbols and in view of Theorems 2 and 3 we will prove for the spaces $F^{s,a}_{p,q}(R^n)$ a similar result to the one obtained for the Besov spaces $B^{s,a}_{p,q}(R^n)$. But in order to prove this we need a lemma. This lemma contains a first, not necessarily sharp embedding of $F^{s,a}_{p,q}(R^n)$ in the scale of Bessel potential spaces and will be useful in the last step of the proof of Theorem 4.

LEMMA. *Let* $1 < p < \infty$, $0 < q \leqslant \infty$ *and* $-\infty < s < \infty$. *Then*

$$F^{s,a}_{p,q}(R^n) \hookrightarrow H^{\varkappa}_p(R^n)$$

if $\varkappa < sm$.

Proof. Step 1. By the monotonicity of the l_q-spaces and by a simple calculation we get the first elementary embedding

$$F^{s,a}_{p,q}(R^n) \hookrightarrow B^{s',a}_{p,1}(R^n) \tag{17}$$

if $1 < p < \infty$, $0 < q \leqslant \infty$ and $-\infty < s' < s < \infty$.

Step 2. Let $\{\varphi_j(x, \xi)\}_{j=0}^{\infty}$ be a system belonging to the elliptic symbol $a(x, \xi)$ with $c^{\varphi} = 1$.

We introduce a second function system $\{\varphi_j^*(\xi)\}_{j=0}^{\infty} \subset S(R^n)$ with:

$$\varphi_j^*(\xi) = 1 \quad \text{on } \operatorname{supp} \varphi_j,$$

$$|\varphi_j^{*(\alpha)}(\xi)| \leqslant c_\alpha \langle \xi \rangle^{-|\alpha|} \quad \text{for all } \alpha \text{ and } c_\alpha \text{ independent of } j,$$

$$\tag{18} \begin{cases} \operatorname{supp} \varphi_j^* \subset \Omega_j^{N,a,*} = \{(x, \xi)\colon \langle \xi \rangle < 2^{d+(j+N)/M}\} \\ \qquad\qquad\qquad\qquad \text{if } j = 0, 1, \ldots, N-1, \\ \operatorname{supp} \varphi_j^* \subset \Omega_j^{N,a,*} = \{(x, \xi)\colon 2^{-d'+(j-N)/m} < \langle \xi \rangle < 2^{d+(j+N)/m}\} \\ \qquad\qquad\qquad\qquad \text{if } j = N, N+1, \ldots \end{cases}$$

In view of (8) there always exist constants d' and d, both independent of j, such that $\Omega_j^{N,a} \subset \Omega_j^{N,a,*}$ for all j. Then by Theorem 1 we obtain

$$\tag{19} \begin{aligned} \varphi_j^*(D_x)\varphi_j(x, D_x) &= \varphi_j(x, D_x) + R_j(x, D_x), \\ \varphi_j(x, D_x)\varphi_j^*(D_x) &= \varphi_j(x, D_x) \end{aligned}$$

and

$$|r_j|^{(\mu)}_{(l,k)} \leqslant c_{lk\mu\nu} 2^{-j\nu} \tag{20}$$

for arbitrary real numbers μ and ν where the constants $c_{lk\mu\nu}$ are independent of j.

This yields for $J = 0, 1, 2, \ldots$

$$u = \sum_{j=0}^{J-1} \varphi_j(x, D_x)u + \sum_{j=J}^{\infty} \varphi_j^*(D_x)\varphi_j(x, D_x)u - R^J(x, D_x)u \tag{21}$$

with

$$R^J(x, D_x) = \sum_{j=J}^{\infty} R_j(x, D_x). \tag{22}$$

By (20) we get the convergence of the infinite sum (22) in $S^{\mu}_{1,\delta}$ for arbitrary μ. The seminorms of $R^J(x, D_x)$ can be estimated by

$$|r^J|^{(\mu)}_{(l,k)} \leqslant c_{lk\mu} 2^{-J} \tag{23}$$

where the constants $c_{lk\mu}$ are independent of J.

Step 3. Let $s \leqslant 0$ and let $\varkappa < sm$ be fixed. Then by Corollary 1 and (23) with $\mu = 0$ we have

$$\|R^J(x, D_x) \,|\, L(H^{\varkappa}_p, H^{\varkappa}_p)\| \leqslant c_{\varkappa p} 2^{-J}.$$

This implies that the inverse operator of $I + R^J(x, D_x)$ exists and it also belongs to $L(H^{\varkappa}_p, H^{\varkappa}_p)$ if $J \geqslant J_0(\varkappa, p)$. I stands for the identity.

Step 4. Let $s' = \varkappa/m$, $u \in B^{s';a}_{p,1}(R^n)$ and

$$v_j = \begin{cases} \varphi_j(x, D_x)u & \text{if } j = 0, 1, \ldots, J_0, \\ \varphi_j^*(D_x)\varphi_j(x, D_x)u & \text{if } j = J_0+1, J_0+2, \ldots \end{cases}$$

As a consequence of Corollary 1 and (18) we see that

$$\begin{aligned} \|\varphi_j^*(D_x)\varphi_j(x, D_x)u \,|\, H^{\varkappa}_p\| &\leqslant c_{\varkappa p} |\varphi_j^*|^{(-\varkappa)}_{(l,k)} \|\varphi_j(x, D_x)u \,|L_p\| \\ &\leqslant c 2^{js'} \|\varphi_j(x, D_x)u \,|\, L_p\| \end{aligned}$$

if $j = J_0 + 1, J_0 + 2, \ldots$

A trivial estimate gives

$$\|\varphi_j(x, D_x)u \,|\, H^{\varkappa}_p\| \leqslant c' 2^{-J_0 s'} 2^{js'} \|\varphi_j(x, D_x)u \,|\, L_p\|$$

if $j = 0, 1, \ldots, J_0$. c and c' are independent of j. We obtain from these two estimates

$$\sum_{j=0}^{\infty} \|v_j \,|\, H^{\varkappa}_p\| \leqslant \max(c, c' 2^{-J_0 s'}) \|u \,|\, B^{s';a}_{p,1}\|^{\varphi}.$$

Together with (21) this implies that

$$v = \sum_{j=0}^{\infty} v_j = u + R^J(x, D_x)u$$

belongs to $H^{\varkappa}_p(R^n)$.

By step 3, the same must be true for u and we get

$$\|u \mid H_p^\varkappa\| \leqslant c'' \|(I+R^J)^{-1} \mid L(H_p^\varkappa, H_p^\varkappa)\| \, \|u \mid B_{p,1}^{s';a}\|^\varphi$$

if $s \leqslant 0$, $\varkappa < sm$ and $s' = \varkappa/m$.

Now the assertion of the lemma follows in view of (17).

Step 5. In the case $s > 0$ the proof is simpler. Let $0 < \varkappa < sm$ and $s' = \varkappa/m$. If $u \in B_{p,1}^{s';a}(R^n)$ then it is straightforward to see that

$$\sum_{j=0}^{\infty} \|\varphi_j(x, D_x)u \mid L_p\| \leqslant c\|u \mid B_{p,1}^{s';a}\|^\varphi$$

is absolutely convergent in $L_p(R^n)$. In this way we get $u \in L_p(R^n)$ and

$$\|u \mid L_p\| \leqslant c\|u \mid B_{p,1}^{s';a}\|^\varphi .$$

On the other hand, it follows from (19), (20) and Corollary 1 that

$$\begin{aligned} &\sum_{j=0}^{\infty} \|\varphi_j(x, D_x)u \mid H_p^\varkappa\| \\ &\qquad \leqslant c_{\varkappa p}\Big(\sum_{j=0}^{\infty} |\varphi_j^*|_{(l,k)}^{(-\varkappa)} \|\varphi_j(x, D_x)u \mid L_p\| + \sum_{j=0}^{\infty} |r_j|_{(l,k)}^{(-\varkappa)} \|u \mid L_p\|\Big) \\ &\qquad \leqslant c \sum_{j=0}^{\infty} 2^{js'} \|\varphi_j(x, D_x)u \mid L_p\| + c'\|u \mid L_p\| \\ &\qquad \leqslant c''\|u \mid B_{p,1}^{s';a}\|^\varphi . \end{aligned}$$

We get $u \in H_p^\varkappa(R^n)$ with

$$\|u \mid H_p^\varkappa\| \leqslant c''\|u \mid B_{p,1}^{s';a}\|^\varphi$$

and again the assertion of the lemma follows in view of (17).

THEOREM 4. *Let $\{\varphi_j(x, \xi)\}_{j=0}^{\infty}$ be a system belonging to an elliptic symbol $a(x, \xi)$ and let $1 < p < \infty$, $0 < q \leqslant \infty$, $-\infty < s < \infty$.*

Then the spaces $F_{p,q}^{s,a}(R^n)$ and $F_{p,q}^{sm}(R^n)$ coincide and there exist constants $c' > 0$ and $c > 0$ such that for all $u \in F_{p,q}^{sm}(R^n)$

$$c'\|u \mid F_{p,q}^{sm}\| \leqslant \|u \mid F_{p,q}^{s,a}\|^\varphi \leqslant c\|u \mid F_{p,q}^{sm}\| .$$

Proof. Step 1. We assume that $\{\varphi_j(x, \xi)\}_{j=0}^{\infty}$ is a system with $c^\varphi = 1$ and $\{\varphi_j^*(\xi)\}_{j=0}^{\infty}$ is a function system as described in the last proof. Hence (18)–(23) hold.

Let $\{\psi_k(\xi)\}_{k=0}^{\infty} \subset S(R^n)$ be a classical dyadic resolution of unity, i.e.

$$\operatorname{supp}\psi_k \subset \{\xi\colon 2^{k-1} < \langle\xi\rangle < 2^{k+1}\} \qquad \text{for } k = 1, 2, \ldots,$$

$$2^{k|\alpha|}|\psi_k^{(\alpha)}(\xi)| \leqslant c_\alpha \quad \text{and} \quad \sum_{k=0}^{\infty} \psi_k(\xi) = 1 .$$

$\mathscr{J}(k)$ and $\mathscr{K}(j)$ denote the index sets

$$\mathscr{J}(k) = \{j\colon \max(0, (k-1-d)m-N) \leqslant j < (k+1+d')m+N\},$$
$$\mathscr{K}(j) = \{k\colon \max(0, (j-N)/m-d'-1) \leqslant k < (j+N)/m+d+1\}.$$

Then for any fixed k we obtain

$$\operatorname{supp}\psi_k \cap \operatorname{supp}\varphi_j^* = \emptyset \quad \text{if } j \notin \mathscr{J}(k), \tag{24}$$

and for any fixed j we have

$$\operatorname{supp}\psi_k \cap \operatorname{supp}\varphi_j^* = \emptyset \quad \text{if } k \notin \mathscr{K}(j). \tag{25}$$

Step 2. Let $u \in F^{sm}_{p,q}(R^n)$. As in the classical case we get

$$\| \{2^{js}\varphi_j(x, D_x)u\}_{j=0}^{\infty} \mid L_p(l_q)\| \leqslant c \Big\| \Big\{ \sum_{j\in \mathscr{J}(k)} |\varphi_j(x, D_x) 2^{ksm}\psi_k(D_x)u| \Big\}_{k=0}^{\infty} \mid L_p(l_q)\Big\|. \tag{26}$$

Let $u_k = 2^{ksm}\psi_k(D_x)u$, $d_k = 2^{k+1+d'}(c'^{-1}2^N)^{1/m}$ and let $\{m_k^i\}_{k=0}^{\infty}$ be a finite set of systems. For fixed k and $i = 1, 2, \ldots, [2N+(2+d+d')m]$ we have

$$m_k^i(x, \xi) = \varphi_{j(k,i)}(x, \xi)$$

where $j(k, i)$ runs over the index set $\mathscr{J}(k)$.

By Definition 2, $\{\varphi_{j(k,i)}(x, \xi)\}_{k=0}^{\infty}$ fulfils (10) and the seminorms $|\varphi_{j(k,i)}|^{(0)}_{(l,0)}$ may be estimated uniformly for all k and $j \in \mathscr{J}(k)$. Now Theorem 3 applied to (26) shows that

$$\|u \mid F^{s,a}_{p,q}\|^{\varphi} \leqslant c\,\|u \mid F^{sm}_{p,q}\|.$$

Step 3. Let $u \in F^{s,a}_{p,q}(R^n)$. Then we get by (19), (21) and (25)

$$\begin{aligned} \| \{2^{ksm}\psi_k(D_x)u\}_{k=0}^{\infty} \mid L_p(l_q)\| &\leqslant c \Big\| \Big\{ \sum_{k\in \mathscr{K}(j)} |\psi_k(D_x) 2^{js}\varphi_j^*(D_x)\varphi_j(x, D_x)u| \Big\}_{k=0}^{\infty} \mid L_p(l_q)\Big\| \\ &\quad + c'\,\| \{2^{ks}\psi_k(D_x)R^0(x, D_x)u\}_{k=0}^{\infty} \mid L_p(l_q)\|. \end{aligned} \tag{27}$$

We can apply Theorem 3 only to the first term on the right-hand side of (27). For the second term the assumption that $\{R^0(x, D_x)u\}_{k=0}^{\infty}$ belongs to a space $L_p^{\Omega}(l_q)$ of analytic functions is not fulfilled.

It is then obvious that as in the previous step we get from (27), by Theorem 3,

$$\|u \mid F^{sm}_{p,q}\| \leqslant c\,\|2^{js}\varphi_j^*(D_x)\varphi_j(x, D_x)u \mid L_p(l_q)\| + c'\,\|2^{ks}\psi_k(D_x)R^0(x, D_x)u \mid L_p(l_q)\|.$$

Now by (19) we obtain

$$(28)\qquad \|u \mid F^{sm}_{p,q}\| \leqslant c\left(\|u \mid F^{s,a}_{p,q}\|^{\varphi} + \|2^{js} R_j(x, D_x) u \mid L_p(l_q)\| + \right.$$
$$\left. + \|2^{ks} \psi_k(D_x) R^0(x, D_x) u \mid L_p(l_q)\|\right).$$

Step 4. The proof will be completed by showing that the remainder terms in (28) can also be estimated by the norm of u in $F^{s,a}_{p,q}$.

By a simple calculation, Corollary 1 and (20) with $\mu = sm-\varepsilon$, $\nu = s+2\varepsilon$ and fixed $\varepsilon > 0$ we obtain

$$\|2^{js} R_j(x, D_x) u \mid L_p(l_q)\| \leqslant c\,\|u \mid H^{sm-\varepsilon}_p\|.$$

Now from the embedding proved in the lemma it follows that

$$\|2^{js} R_j(x, D_x) u \mid L_p(l_q)\| \leqslant c'\,\|u \mid F^{s,a}_{p,q}\|^{\varphi}.$$

The third term in (28) may be estimated in the same way and so the proof is complete.

Remark. We cannot carry over this proof to the case $0 \not< p \leqslant 1$. Since the remainder terms in (28) do not fulfil the assumptions of Theorem 3 it is not possible to estimate them by this theorem. So we need Corollary 1 and Theorem 2 in the last step of the proof. But in the presented form they are only true in the case $1 < p < \infty$.

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