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TRIGONOMETRIC INTERPOLATION FOR BIVARIATE FUNCTIONS OF BOUNDED VARIATION

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In this note we estimate the L^p-error of trigonometric interpolation for bivariate functions of bounded variation in the sense of Hardy–Krause. Especially we consider blending interpolation and Lagrange interpolation with respect to uniform meshes. Some of these results can be given with explicit constants and can be extended in a simple manner to the multivariate case.

1. Introduction

Let BV be the set of all 2π-periodic real-valued functions $f\colon R \to R$ with bounded variation on $I = [0, 2\pi]$. By Vf we denote the total variation of f on I. Further, let

(1) $$\|f\|_p = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_I |f(x)|^p\,dx\right)^{1/p} & \text{if } 1 \leqslant p < \infty, \\ \sup\limits_{x\in I} |f(x)| & \text{if } p = \infty. \end{cases}$$

A real-valued bivariate function $f\colon R^2 \to R$, 2π-periodic with respect to both variables, is said to be of *bounded variation on $Q = I^2$ in the sense of Hardy–Krause* [1] if $f(\cdot, y), f(x, \cdot) \in BV$ for some fixed $x, y \in I$ and if

$$Hf = \sup_{\mathfrak{z}_1, \mathfrak{z}_2} \sum_{k=1}^{u-1} \sum_{m=1}^{v-1} |\Delta f(\xi_k, \eta_m)|$$

is finite, where

(2) $$\mathfrak{z}_1\colon\ 0 \leqslant \xi_1 < \ldots < \xi_u \leqslant 2\pi,$$

(3) $$\mathfrak{z}_2\colon\ 0 \leqslant \eta_1 < \ldots < \eta_v \leqslant 2\pi$$

are arbitrary decompositions of I and

$$\Delta f(\xi_k, \eta_m) = f(\xi_{k+1}, \eta_{m+1}) - f(\xi_{k+1}, \eta_m) - f(\xi_k, \eta_{m+1}) + f(\xi_k, \eta_m).$$

The class of these functions will be denoted by HBV. If $f \in HBV$, then $f(\cdot, y)$, $f(x, \cdot) \in BV$ for any fixed $x, y \in I$. The values $V_1 f(y)$ and $V_2 f(x)$ are defined as the total variations of $f(\cdot, y)$ and $f(x, \cdot)$ on I, respectively. For $f \in HBV$ we also have $V_1 f$, $V_2 f \in BV$ (see [1]). Hence

$$V'f = \sup_{y \in I} V_1 f(y), \qquad V''f = \sup_{x \in I} V_2 f(x)$$

are finite. Analogously to (1) we denote the norm of a bivariate function f by

$$\|\|f\|\|_p = \begin{cases} \left(\dfrac{1}{4\pi^2} \iint\limits_Q |f(x, y)|^p \, dx \, dy\right)^{1/p} & \text{if } 1 \leqslant p < \infty, \\ \sup\limits_{(x,y) \in Q} |f(x, y)| & \text{if } p = \infty. \end{cases} \tag{4}$$

Now we introduce interpolation operators with respect to equidistant nodes

$$x_k = y_k = 2k\pi/(2n+1), \qquad k = 0, \ldots, 2n,$$

by

$$Lf(x) = \frac{2}{2n+1} \sum_{k=0}^{2n} f(x_k) K_n(x - x_k)$$

in the univariate case and by

$$L_1 f(x, y) = \frac{2}{2n+1} \sum_{k=0}^{2n} f(x_k, y) K_n(x - x_k),$$

$$L_2 f(x, y) = \frac{2}{2n+1} \sum_{m=0}^{2n} f(x, y_m) K_n(y - y_m),$$

$$\begin{aligned} \tilde{L} f(x, y) &= L_1 L_2 f(x, y) = L_2 L_1 f(x, y) \\ &= \frac{4}{(2n+1)^2} \sum_{k=0}^{2n} \sum_{m=0}^{2n} f(x_k, y_m) K_n(x - x_k) K_n(y - y_m) \end{aligned}$$

in the bivariate case, where

$$K_n(x) = \tfrac{1}{2} + \sum_{k=1}^{n} \cos kx$$

is the nth Dirichlet kernel. Here $\tilde{L}$ is the *bivariate Lagrange interpolation operator*. The *blending interpolation operator* [2] is given by

$$B = L_1 + L_2 - \tilde{L}.$$

Further, we consider for $1 \leqslant p \leqslant \infty$ and a function f the moduli of continuity defined by

$$\omega(f, p, \delta) = \sup_{0 \leqslant h \leqslant \delta} \|f(\cdot + h) - f(\cdot)\|_p$$

in the univariate case and by

$$\tilde{\omega}(f, p, \delta) = \sup_{0 \leqslant v, w \leqslant \delta} \||f(\cdot + v, \cdot + w) - f(\cdot + v, \cdot) - f(\cdot, \cdot + w) + f(\cdot, \cdot)\||_p$$

in the bivariate case.

2. Univariate interpolation

The proofs of our theorems are based on known results of univariate interpolation, which we briefly recall here.

THEOREM 1 ([5], [6]). *If $f \in BV$ and $1 \leqslant p < \infty$, then for all n*

$$\|f - Lf\|_p \leqslant Vf(2n+1)^{-1/p} A(p, n)$$

with

$$A(p, n) \leqslant \begin{cases} 20 + 4.2 \ln n & \text{if } p = 1, \\ 8.3 + 5(p-1)^{-1} & \text{if } 1 < p < 2, \\ 7.6 & \text{if } 2 \leqslant p < \infty. \end{cases}$$

The order of the estimate is best possible. This is shown by the example

$$g(x) = \begin{cases} 1/2 & \text{if } x = 0, \\ 0 & \text{if } 0 < x < 2\pi, \\ g(x + 2\pi) & \text{if } x \in R. \end{cases}$$

A simple calculation yields

$$\|g - Lg\|_p = \frac{1}{2n+1} \|K_n\|_p \geqslant \begin{cases} c_1 n^{-1} \ln n & \text{if } p = 1, \\ c_2 n^{-1/p} & \text{if } 1 < p < \infty. \end{cases} \tag{5}$$

Further, from the method of proof in [7] we get the estimate

$$\tfrac{1}{2} \leqslant \sup_{Vf = 1} (2n+1)^{1/2} \|f - Lf\|_2 < 4$$

for all n.

THEOREM 2. *If the derivative $f^{(r)} \in BV$, $r \geqslant 1$, and $1 \leqslant p < \infty$, then for all n*

$$\|f - Lf\|_p \leqslant Vf^{(r)} n^{-r-1/p} \cdot \begin{cases} C_1 \ln n & \text{if } p = 1, \\ C_2 & \text{if } 1 < p \leqslant \infty. \end{cases} \tag{6}$$

Proof. From [3] (see also [6]) we obtain

$$\|f-Lf\|_p \leqslant n^{-r}\omega(f^{(r)}, p, 1/n)\cdot\begin{cases} C_1' \ln n & \text{if } p=1, \\ C_2' & \text{if } 1<p<\infty \end{cases} \tag{7}$$

and

$$\omega(f^{(r)}, p, 1/n) \leqslant Vf^{(r)} n^{-1/p}. \ \blacksquare$$

The order of (6) is again best possible and in the case $2 \leqslant p < \infty$ we have $C_2 \leqslant 13.75$ [4].

THEOREM 3. *If* $f^{(r)} \in BV \cap C$, $r \geqslant 0$, *and* $1 < p \leqslant \infty$, *then*

$$\|f-Lf\|_p = o(n^{-r-1/p}) \quad \text{as } n \to \infty.$$

Proof. The assertion is well known for $p=\infty$. Hence let $1<p<\infty$ and $0<t<1-1/p$. Then we use the inequalities

$$\|f-Lf\|_p \leqslant \|f-Lf\|_\infty^t \, \|f-Lf\|_{p-tp}^{1-t}$$

in the case $r=0$, and

$$\omega(f^{(r)}, p, 1/n) \leqslant \omega(f^{(r)}, \infty, 1/n)\, \omega(f^{(r)}, p-tp, 1/n)^{1-t}$$

in the case $r>0$. The assertion now follows from Theorem 1 and (7). $\blacksquare$

3. Preliminary results

LEMMA 1. *If* $f \in HBV$, $1 \leqslant p \leqslant \infty$ *and* $0 \leqslant t \leqslant 1-1/p$, *then for all* n

$$\tilde{\omega}(f, p, 1/n) \leqslant n^{-2/p}(Hf)^{1-t}\,\tilde{\omega}(f, \infty, 1/n)^t.$$

Proof. (i) The case $p=\infty$ is trivial. Because of

$$|||f|||_p \leqslant |||f|||_\infty^t \, |||f|||_{p-tp}^{1-t}$$

for $1 \leqslant p < \infty$ and $0 \leqslant t \leqslant 1-1/p$ we can assume in the following that $t=0$.

(ii) By defining $H_{a,c}^{b,d}$ similarly to Hf, but with respect to the rectangle $[a, b]\times[c, d]$ instead of Q, it is sufficient to show, for $1 \leqslant p < \infty$, that

$$\iint_Q (H_{x,y}^{x+v,y+w})^p \, dx\, dy \leqslant vw(Hf)^p.$$

This follows from

$$\iint_Q H_{x,y}^{x+v,y+w}\, dx\, dy = \iint_Q (H_{0,0}^{x+v,y+w} - H_{0,0}^{x+v,y} - H_{0,0}^{x,y+w} + H_{0,0}^{x,y})\, dx\, dy$$

$$= \left(\int_w^{2\pi+w}\int_v^{2\pi+v} - \int_0^{2\pi}\int_v^{2\pi+v} - \int_w^{2\pi+w}\int_0^{2\pi} + \int_0^{2\pi}\int_0^{2\pi}\right) H_{0,0}^{x,y}\, dx\, dy$$

$$= \Big(\int_{2\pi}^{2\pi+w} - \int_0^w\Big)\Big(\int_{2\pi}^{2\pi+v} - \int_0^v\Big) H_{0,0}^{x,y}\,dx\,dy$$

$$= \Big(\int_{2\pi}^{2\pi+w} - \int_0^w\Big) \int_{2\pi}^{2\pi+v} H_{0,0}^{2\pi,y}\,dx\,dy = \int_{2\pi}^{2\pi+w}\int_{2\pi}^{2\pi+v} H_{0,0}^{2\pi,2\pi}\,dx\,dy$$

$$= vw\,Hf. \quad \blacksquare$$

LEMMA 2. *Let $f \in HBV$. Further, let $\mathfrak{z}_1$, $\mathfrak{z}_2$ be arbitrary decompositions of I of the form* (2), (3) *respectively and let*

$$h_1(\mathfrak{z}_1, y) = \sum_{k=1}^{u-1} |f(\xi_{k+1}, y) - f(\xi_k, y)|,$$

$$h_2(\mathfrak{z}_2, x) = \sum_{m=1}^{v-1} |f(x, \eta_{m+1}) - f(x, \eta_m)|.$$

Then for $x, y \in I$

$$V_1 f(y) = \sup_{\mathfrak{z}_1} h_1(\mathfrak{z}_1, y), \qquad V_2 f(x) = \sup_{\mathfrak{z}_2} h_2(\mathfrak{z}_2, x).$$

Proof. We only consider $V_1 f(y)$ for $y \in I$. Let $\varepsilon > 0$ be arbitrary given. We set $h(y) = H_{0,0}^{2\pi,y}$. Since $V_1 f$, $h \in BV$, we can find a decomposition $\mathfrak{z}_2$ of the form (3) so that for every $y \in I$ there exists an $\eta_m \in \mathfrak{z}_2$ with

$$|V_1 f(y) - V_1 f(\eta_m)| < \varepsilon, \qquad |h(y) - h(\eta_m)| < \varepsilon.$$

Now we choose a decomposition $\mathfrak{z}_1$ so that for every $m = 1, \ldots, v$

$$V_1 f(\eta_m) \leqslant h_1(\mathfrak{z}_1, \eta_m) + \varepsilon.$$

By construction for every $y \in I$ there exists an $\eta_m \in \mathfrak{z}_2$ so that

$$|h_1(\mathfrak{z}_1, y) - h_1(\mathfrak{z}_1, \eta_m)| \leqslant |h(y) - h(\eta_m)| < \varepsilon.$$

Hence we obtain

$$h_1(\mathfrak{z}_1, y) \leqslant V_1 f(y) \leqslant V_1 f(\eta_m) + \varepsilon \leqslant h_1(\mathfrak{z}_1, \eta_m) + 2\varepsilon \leqslant h_1(\mathfrak{z}_1, y) + 3\varepsilon.$$

This completes the proof. $\square$

4. Blending interpolation

Here we give estimates for the error $|||f - Bf|||_p$ of blending interpolation. Using the representation

$$f - Bf = (E - L_1)(f - L_2 f),$$

where E denotes the identity, we derive the following results from those of Sections 2 and 3.

THEOREM 4. *If $f \in HBV$ and $1 \leqslant p < \infty$, then for all n*

$$|||f - Bf|||_p \leqslant Hf\,(2n+1)^{-2/p} A(p, n)^2$$

with $A(p, n)$ as in Theorem 1.

Note that the order of this result is again best possible. This is shown by the example (see (5))

$$g(x, y) = \begin{cases} 1/4 & \text{if } x = y = 0, \\ 0 & \text{if } (x, y) \in [0, 2\pi)^2 \setminus \{(0, 0)\}, \\ g(x+2\pi, y+2\pi) & \text{if } (x, y) \in R^2, \end{cases}$$

with

$$|||g - Bg|||_p = \frac{1}{(2n+1)^2} \|K_n\|_p^2.$$

More generally, but without explicit constants we obtain

THEOREM 5. *If the partial derivative $f^{(r,s)} \in HBV$, $r, s \geqslant 0$, and $1 \leqslant p < \infty$, then for all n*

$$|||f - Bf|||_p \leqslant (Hf^{(r,s)})\, n^{-r-s-2/p} . \begin{cases} c \ln^2 n & \text{if } p = 1, \\ c_p & \text{if } 1 < p < \infty. \end{cases}$$

For $2 \leqslant p < \infty$, $c_p \leqslant 190$.

THEOREM 6. *If $f^{(r,s)} \in HBV \cap C$, $r, s \geqslant 0$, and $1 < p \leqslant \infty$, then*

$$|||f - Bf|||_p = o(n^{-r-s-2/p}) \quad \text{as } n \to \infty.$$

Proof of Theorems 4–6. From

$$|||f - Bf|||_p \leqslant |||(E - L_1)(f - L_2 f)|||_p$$

it follows by Section 2 that

$$|||f - Bf|||_p \leqslant D_1 \|V_1 (f^{(r,0)} - L_2 f^{(r,0)})\|_p$$

with some $D_1 = D_1(p, n, r)$. We note that $(L_2 f)^{(r,0)} = L_2 f^{(r,0)}$ and that D_1 can be specified from Theorem 1 or 2. Because of Lemma 2 we have

$$\|V_1 (f^{(r,0)} - L_2 f^{(r,0)})\|_p = \sup_{\mathfrak{Z}_1} \sum_{k=1}^{u-1} \left\|(E - L_2)\left(f^{(r,0)}(\xi_{k+1}, \cdot) - f^{(r,0)}(\xi_k, \cdot)\right)\right\|_p.$$

If we apply again Theorem 1, 2 or 3 to

$$f^{(r,0)}(\xi_{k+1}, \cdot) - f^{(r,0)}(\xi_k, \cdot), \quad k = 1, \ldots, u-1,$$

we finally obtain

$$|||f - Bf|||_p \leqslant D_1 D_2 Hf^{(r,s)}$$

with some $D_2 = D_2(p, n, s)$, which can be specified from Theorem 1 or 2.

The modifications needed to verify Theorem 6 are obvious. ∎

Note that we can also prove Theorems 5 and 6 beginning with (7) and applying Lemma 1. This yields, for $f^{(r,s)} \in HBV$, $rs > 0$, $1 < p \leqslant \infty$ and $0 \leqslant t \leqslant 1-1/p$, the more general inequality

$$|||f-Bf|||_p \leqslant C_p n^{-r-s-2/p}(Hf^{(r,s)})^{1-t}\tilde{\omega}(f^{(r,s)}, \infty, 1/n)^t.$$

5. Lagrange interpolation

Now we describe two possibilities of proving error estimates for Lagrange interpolation.

A first way is to apply the inequality

$$|||f-\tilde{L}f|||_p \leqslant |||f-L_1 f|||_p+|||f-L_2 f|||_p+|||f-Bf|||_p.$$

Using Theorems 1, 2, 4 and 5 we immediately obtain the following results.

THEOREM 7. *If $f \in HBV$ and $1 \leqslant p < \infty$, then for all n*

$$|||f-\tilde{L}f|||_p \leqslant A(p, n)(2n+1)^{-1/p}(\|V_1 f\|_p+\|V_2 f\|_p+A(p, n)(2n+1)^{-1/p} Hf)$$

with $A(p, n)$ as in Theorem 1.

THEOREM 8. *If $f^{(r,s)} \in HBV$, $r, s \geqslant 0$, and $2 \leqslant p < \infty$, then for all n*

$$|||f-\tilde{L}f|||_p \leqslant \frac{13.75}{n^{1/p}}\left(\frac{\|V_1 f^{(r,0)}\|_p}{n^r}+\frac{\|V_2 f^{(0,s)}\|_p}{n^s}+\frac{13.75\, Hf^{(r,s)}}{n^{r+s+1/p}}\right).$$

An analogous result is true for $1 \leqslant p < 2$, but the corresponding constants are unknown. This approach works only for functions belonging to HBV. However, the same order of convergence can be obtained for a wider class of functions with weaker conditions on the variation. For that reason we present a second approach to error estimates for Lagrange interpolation.

Setting for $i = 1, 2$ and $1 \leqslant p < \infty$

$$M_i f = \|V_i f\|_p, \quad N_i f = \left(\frac{1}{2n+1}\sum_{k=0}^{2n} V_i f(x_k)^p\right)^{1/p}$$

we now consider functions $f\colon R^2 \to R$, 2π-periodic with respect to both variables, which satisfy the conditon

$$\min(M_1 f^{(r,0)}+N_2 f^{(0,s)}, M_2 f^{(0,s)}+N_1 f^{(r,0)}) < \infty \tag{8}$$

for some $r, s \geqslant 0$. Note that $f \in HBV$ fulfils (8) with $r = s = 0$ and $1 \leqslant p < \infty$.

On the other hand the function

$$g(x, y) = \begin{cases} 1 & \text{if } x = y \in [0, 2\pi), \\ 0 & \text{if } (x, y) \in [0, 2\pi)^2,\ x \neq y, \\ g(x+2\pi, y+2\pi) & \text{if } (x, y) \in R^2 \end{cases}$$

does not belong to HBV, but satisfies (8) with $r = s = 0$ and $1 \leqslant p < \infty$.

Now let f be a function satisfying (8). Setting

$$M = \min\left(M_1 f^{(r,0)} + B(p, n) N_2 f^{(0,s)},\ M_2 f^{(0,s)} + B(p, n) N_1 f^{(r,0)}\right)$$

with

$$B(p, n) = \begin{cases} 1.5 + (4/\pi^2) \ln n & \text{if } p = 1, \\ 17/(p-1) + 3 & \text{if } 1 < p < 2, \\ 1 & \text{if } p = 2, \\ 12p + 3 & \text{if } 2 < p < \infty \end{cases}$$

we have obviously

$$M \leqslant \frac{1 + B(p, n)}{2} (V' f^{(r,0)} + V'' f^{(0,s)}).$$

THEOREM 9. *Let f: $R^2 \to R$ be a bivariate function, 2π-periodic with respect to both variables, which satisfies* (8) *for some $r, s \geqslant 0$ and $1 \leqslant p < \infty$. Then we have as $n \to \infty$*

$$|||f - \tilde{L}f|||_p = \begin{cases} O(n^{-1/p-t} \ln^2 n) & \text{if } p = 1, \\ O(n^{-1/p-t}) & \text{if } 1 < p < \infty, \end{cases}$$

with $t = \min(r, s)$. Especially we have in the case $2 \leqslant p < \infty$

$$|||f - \tilde{L}f|||_p \leqslant 13.75\, M n^{-1/p-t}$$

for all n. In the case $r = s = 0$ and $1 \leqslant p < \infty$

$$|||f - \tilde{L}f|||_p \leqslant A(p, n) M (2n+1)^{-1/p}$$

for all n with $A(p, n)$ from Theorem 1.

Proof. We can estimate the error in the form

$$|||f - \tilde{L}f|||_p \leqslant |||f - L_1 f|||_p + |||L_1 (f - L_2 f)|||_p$$

or

$$|||f - \tilde{L}f|||_p \leqslant |||f - L_2 f|||_p + |||L_2 (f - L_1 f)|||_p.$$

Now we apply the inequality (see [8], Chap. 7 and 10)

$$\begin{aligned}|||L_1 f|||_p &\leq B(p,n)\left(\frac{1}{2\pi}\int_I \frac{1}{2n+1}\sum_{k=0}^{2n}|L_1 f(x_k,y)|^p\,dy\right)^{1/p}\\ &= B(p,n)\left(\frac{1}{2n+1}\sum_{k=0}^{2n}\|f(x_k,\cdot)\|_p^p\right)^{1/p},\end{aligned}$$

which immediately yields all the results by the corresponding univariate estimates. ∎

From the above proof a corresponding o-result follows:

THEOREM 10. *Let* $1<p<\infty$. *Suppose that* f *fulfils the assumptions of Theorem* 9 *and that* $f^{(r,0)}, f^{(0,s)}\in C$ *if* $r=t$ *or* $s=t$ *with* $t=\min(r,s)$. *Then*

$$|||f-\tilde{L}f|||_p = o(n^{-t-1/p}) \quad \text{as } n\to\infty.$$

6. Discrete Fourier coefficients

The Lagrange interpolation operator can be represented in the form

$$\tilde{L}f(x,y)=\sum_{j=-n}^{n}\sum_{l=-n}^{n} c_{jl}^{(n)} e^{i(jx+ly)}$$

with the *discrete Fourier coefficients*

$$c_{jl}^{(n)}=\frac{1}{(2n+1)^2}\sum_{k=0}^{2n}\sum_{m=0}^{2n} f_{km}e^{-ih(jk+lm)}, \tag{9}$$

where $h=2\pi/(2n+1)$ and $f_{km}=f(x_k,y_m)$. Note that the discrete Fourier coefficients may be computed very efficiently by a *FFT*-algorithm.

Now we discuss the asymptotic behaviour of discrete Fourier coefficients for a given function $f\in HBV$.

THEOREM 11. *If* $f\in HBV$, *then for* $j,l=-n,\ldots,n$

$$|c_{jl}^{(n)}|\leq\begin{cases}\frac{1}{4}|j|^{-1}V'f & \text{if } j\neq 0,\ l=0,\\ \frac{1}{4}|l|^{-1}V''f & \text{if } j=0,\ l\neq 0,\\ \frac{1}{16}|jl|^{-1}Hf & \text{if } jl\neq 0.\end{cases}$$

Proof. For short we only consider the case $jl\neq 0$. Setting

$$w_{km}=\frac{e^{-i(k+1/2)jh}}{-2i\sin j\frac{h}{2}}\cdot\frac{e^{-i(m+1/2)lh}}{-2i\sin l\frac{h}{2}}$$

for $k, m = 0, 1, \ldots,$ we deduce from

$$\text{(10)}\qquad \frac{e^{-ix/2}}{-2i\sin(x/2)} + e^{-ix} + \ldots + e^{-ikx} = \frac{e^{-i(k+1/2)x}}{-2i\sin(x/2)}, \qquad 0 < |x| < 2\pi,$$

that

$$\Delta w_{k-1,m-1} = w_{km} - w_{k-1,m} - w_{k,m-1} + w_{k-1,m-1} = e^{-ih(jk+lm)}$$

for $k, m = 1, 2, \ldots$ By using the bivariate Abelian summation it follows from (9) that

$$\begin{aligned} c_{jl}^{(n)} &= \frac{1}{(2n+1)^2} \sum_{k=1}^{2n+1} \sum_{m=1}^{2n+1} f_{km} \Delta w_{k-1,m-1} \\ &= \frac{1}{(2n+1)^2} \sum_{k=1}^{2n+1} \sum_{m=1}^{2n+1} w_{km} \Delta f_{km}. \end{aligned}$$

Now we have

$$\sum_{k=1}^{2n+1} \sum_{m=1}^{2n+1} |\Delta f_{km}| \leqslant Hf$$

and

$$|w_{km}| \leqslant \left(4 \sin|j|\frac{h}{2} \cdot \sin|l|\frac{h}{2}\right)^{-1} = \left(\frac{\pi}{2}\right)^2 |jl|^{-1} h^{-2},$$

so that the assertion follows. ∎

If we consider the Fourier coefficients

$$c_{jl} = \frac{1}{(2\pi)^2} \iint_Q f(x, y) e^{-i(jx+ly)} \, dx \, dy,$$

then we find for $f \in HBV$ that

$$|c_{jl}| \leqslant \begin{cases} \dfrac{1}{2\pi} |j|^{-1} V'f & \text{if } j \neq 0,\ l = 0, \\ \dfrac{1}{2\pi} |l|^{-1} V''f & \text{if } j = 0,\ l \neq 0, \\ \dfrac{1}{(2\pi)^2} |jl|^{-1} Hf & \text{if } jl \neq 0. \end{cases}$$

Finally, we estimate the error between the discrete Fourier coefficients $c_{jl}^{(n)}$ and the Fourier coefficients c_{jl}.

THEOREM 12. *If $f \in HBV$, then for $j, l = -n, \ldots, n$ and for all n*

$$|c_{jl}^{(n)}-c_{jl}| \leqslant \begin{cases} \dfrac{1}{2n+1}(V'f+V''f) & \text{if } j=l=0, \\ \dfrac{1}{2n+1}\left(\left(1+\dfrac{\pi}{4}\right)V'f+\dfrac{1}{4}|j|^{-1}\,Hf\right) & \text{if } j\neq 0,\ l=0, \\ \dfrac{1}{2n+1}\left(\left(1+\dfrac{\pi}{4}\right)V''f+\dfrac{1}{4}|l|^{-1}\,Hf\right) & \text{if } j=0,\ l\neq 0, \\ \dfrac{1}{2n+1}Hf\left(\dfrac{1}{2n+1}+\left(\dfrac{1}{4}+\dfrac{\pi}{16}\right)(|j|^{-1}+|l|^{-1})\right) & \text{if } jl\neq 0. \end{cases}$$

Proof. For short we only consider the case $jl \neq 0$. By writing

$$c_{jl}^{(n)}-c_{jl}=I_1+I_2+I_3+I_4$$

with

$$I_1=\frac{1}{(2\pi)^2}\sum_{k=0}^{2n}\sum_{m=0}^{2n}\iint_{Q_{km}}(f(x_k,y)-f_{km}-f(x,y)+f(x,y_m))e^{-i(jx+ly)}\,dx\,dy,$$

$$I_2=\frac{1}{(2\pi)^2}\sum_{k=0}^{2n}\sum_{m=0}^{2n}\iint_{Q_{km}}(f_{km}-f(x_k,y))e^{-i(jx+ly)}\,dx\,dy,$$

$$I_3=\frac{1}{(2\pi)^2}\sum_{k=0}^{2n}\sum_{m=0}^{2n}\iint_{Q_{km}}(f_{km}-f(x,y_m))e^{-i(jx+ly)}\,dx\,dy,$$

$$I_4=\frac{1}{(2\pi)^2}\sum_{k=0}^{2n}\sum_{m=0}^{2n}\iint_{Q_{km}}f_{km}(e^{-ih(jk+lm)}-e^{-i(jx+ly)})\,dx\,dy,$$

where $Q_{km}=[kh,(k+1)h]\times[mh,(m+1)h]$, we have to estimate these four integrals. It is easy to see that

$$|I_1|\leqslant\frac{1}{(2n+1)^2}Hf.$$

Integrating with respect to x, we obtain for I_2 the expression

$$I_2=\frac{1}{(2\pi)^2}\,\frac{e^{-ihj}-1}{-ij}\sum_{k=1}^{2n+1}f_k e^{-ihjk} \tag{11}$$

with

$$f_k=\sum_{m=0}^{2n}\int_{mh}^{(m+1)h}(f_{km}-f(x_k,y))e^{-ily}\,dy.$$

Setting

$$u_k = \frac{e^{-i(k+1/2)jh}}{-2i \sin j\frac{h}{2}}$$

we have by (10) for $k = 1, 2, \ldots$

$$\Delta u_{k-1} = u_k - u_{k-1} = e^{-ihjk}.$$

By using Abelian summation it follows from (11) that

$$I_2 = -\frac{1}{(2\pi)^2} \frac{e^{-ihj} - 1}{-ij} \sum_{k=1}^{2n+1} u_k \Delta f_k$$

and hence

$$|I_2| \leqslant \frac{1}{4(2n+1)|j|} Hf.$$

Analogously we obtain the estimate

$$|I_3| \leqslant \frac{1}{4(2n+1)|l|} Hf.$$

Computing the integral I_4 we get

$$I_4 = c_{jl}^{(n)} \left(h^2 jl + (e^{-ihj} - 1)(e^{-ihl} - 1)\right) h^{-2} (jl)^{-1}$$

and hence by Theorem 11

$$|I_4| \leqslant \frac{Hf}{16(hjl)^2} |h^2 jl + (e^{-ihj} - 1)(e^{-ihl} - 1)|.$$

Since for $x, y \in R$

$$xy + (e^{-ix} - 1)(e^{-iy} - 1) = \int_0^y \int_0^x (1 - e^{-i(s+t)})\, ds\, dt,$$

we obtain the inequality

$$|xy + (e^{-ix} - 1)(e^{-iy} - 1)| \leqslant \int_0^{|y|} \int_0^{|x|} (s+t)\, ds\, dt = \tfrac{1}{2} |xy| (|x| + |y|),$$

so that

$$|I_4| \leqslant \frac{\pi}{16(2n+1)} \left(\frac{1}{|j|} + \frac{1}{|l|}\right) Hf.$$

Summarizing, we get

$$|c_{jl}^{(n)} - c_{jl}| \leqslant \frac{1}{2n+1} \left(\frac{1}{2n+1} + \left(\frac{1}{4} + \frac{\pi}{16}\right)\left(\frac{1}{|j|} + \frac{1}{|l|}\right)\right) Hf. \quad \blacksquare$$

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