

ESTIMATES OF REARRANGEMENTS AND EMBEDDING THEOREMS

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1. Introduction

Let $I^N \equiv [0, 1]^N$ be the N -dimensional unit cube. If f is a measurable function on I^N , then the nonincreasing rearrangement of f is a function f^* on $[0, 1]$ which is nonincreasing and equimeasurable with $|f|$. f^* may be given by the formula

$$f^*(t) = \sup_{E \subset I^N, |E|=t} \operatorname{ess\,inf}_{x \in E} |f(x)| \quad (0 < t \leq 1).$$

Rearrangements have important applications in various problems of approximation theory, harmonic analysis and the theory of function spaces. In this paper they are studied basically in connection with the theory of embedding of function classes.

The sources of embedding theory are contained in the papers of Hardy and Littlewood [15], [17]. They obtained embedding theorems for the classes $\operatorname{Lip}(\alpha; p)$ with various norms, as well embedding theorems with limit exponent for the fractional integrals and analytic functions of class H^p .

The embedding theorem for the class of fractional integrals was proved by Hardy and Littlewood with the use of F. Riesz's inequality for rearrangements of three functions (see [18], Ch. X). Generalizing this inequality to radial rearrangements of functions of several variables, S. L. Sobolev [29] proved the embedding theorem with limit exponent for the classes W_p^k . Sobolev's paper initiated the theory of embedding for classes of differentiable functions in several variables (see [2], [25]).

In the late sixties, P. L. Ul'yanov [33]–[35] formulated a number of extremal problems on finding necessary and sufficient conditions for embeddings of some function classes. The methods he proposed were based on estimating nonincreasing rearrangements by moduli of continuity.

The *modulus of continuity* of $f \in L^p(I^N)$ ($1 \leq p < \infty$) is defined to be the function

$$\omega_p(f; \delta) = \sup_{\substack{|h_i| \leq \delta \\ i=1, \dots, N}} \left(\int_{I^N} |f(x) - f(x+h)|^p dx \right)^{1/p} \quad (0 \leq \delta \leq 1)$$

(for simplicity we assume that f is periodic in each variable with period 1; in the nonperiodic case, the integral on the right-hand side is over $I_h^N = \{x: 0 \leq x_i \leq 1 - h_i\}$).

Every nondecreasing continuous function ω on $[0, 1]$ which satisfies $\omega(0) = 0$, $\omega(\delta + \eta) \leq \omega(\delta) + \omega(\eta)$ for $0 \leq \delta \leq \delta + \eta \leq 1$ is called a *modulus of continuity*. If ω is a modulus of continuity and $1 \leq p < \infty$, we denote by $H_{p,N}^\omega$ the class of all $f \in L^p(I^N)$ satisfying $\omega_p(f; \delta) = O(\omega(\delta))$. We also put $\text{Lip}(\alpha; p) \equiv H_{p,N}^{\delta^\alpha}$ ($0 < \alpha \leq 1$).

Basically, the present paper deals with estimating rearrangements and some maximal operators by integral moduli of continuity. We also consider embedding theorems for $H_{p,N}^\omega$ classes which can be proved by using these estimates.

2. Estimates for rearrangements of L^p functions

In [34] P. L. Ul'yanov established the following inequalities: if $f \in L^p[0, 1]$ then

$$(1) \quad f^{**}(t) - f^*(t) \leq ct^{-1/p} \omega_p(f; t) \quad (1 < p < \infty),$$

$$(2) \quad f^*(t) \leq c[\omega_1(f; t)/t + \|f\|_1] \quad (p = 1),$$

where

$$f^{**}(t) = t^{-1} \int_0^t f^*(s) ds.$$

Later E. A. Storozhenko [30] proved that for $f \in L^p[0, 1]$ ($1 < p < \infty$)

$$(3) \quad f^*(t) \leq 2 \left(\int_t^1 u^{-1-1/p} \omega_p(f; u) du + \|f\|_p \right).$$

An inequality of the same type was obtained by A. M. Garsia [10] who used it in some problems of the theory of Fourier series. Garsia's proof is based on the following remarkable inequality.

THEOREM 1. For every even function φ nondecreasing on $[0, \infty)$ and for every measurable a.e. finite function f on $[0, 1]$

$$(4) \quad \iint_{E_\delta} \varphi(f^*(x) - f^*(y)) dx dy \leq \iint_{E_\delta} \varphi(f(x) - f(y)) dx dy$$

for $\delta \in [0, 1]$, where $E_\delta = \{(x, y) \in [0, 1]^2: |x - y| \leq \delta\}$.

Inequality (4) was obtained independently by A. M. Garsia and E. Rodemich [11], P. Oswald [26] and I. Wik [36]; its proofs use combinatorial methods and are rather complicated. Using this inequality, P. Oswald [26] and independently I. Wik [36] proved that for all $f \in L^p[0, 1]$ ($1 \leq p < \infty$)

$$(5) \quad \omega_p(f^*; \delta) \leq 2\omega_p(f; \delta) \quad (0 \leq \delta \leq 1/2)$$

(the moduli of continuity here are the nonperiodic ones). For $p = 1$, such an inequality was obtained by P. L. Ul'yanov [34]; he also conjectured its validity for all $p \geq 1$. Yu. A. Brudnyi [4] also proved (5) by other methods (with a less precise constant).

Let us point out that inequality (5) has applications in embedding theory (see [21], [27]).

In the sequel we consider periodic functions only.

Inequalities (1) and (3) are "weak type estimates" — they give bounds for the individual values of f^{**} and f^* . These bounds are not exact if f is not smooth enough. The author [20] obtained a more precise estimate: for $f \in L^p(I^N)$ ($1 \leq p < \infty$, $N \geq 1$)

$$(6) \quad \int_0^\tau [f^*(t) - f^*(\tau)]^p dt \leq 2\omega_p^p(f; \tau^{1/N}) \quad (0 < \tau \leq 1/2).$$

For $N = 1$ this follows immediately from (5); however, as we shall see, the proof for $N \geq 1$ can be obtained by very simple considerations.

Note that for $N \geq 2$ the estimate (6) is not sharp if the order of $\omega_p(f; \delta)$ is close to the limit order $O(\delta)$. Investigating this case, P. Oswald [27] proved that for all $f \in L(I^2)$

$$\int_{\delta^2}^1 t^{-1/2} f^*(t) dt \leq c[\omega_1(f; \delta)/\delta + \|f\|_1].$$

His method of proof makes essential use of the specific character of the case $p = 1$. By other means, the author [23] established the corresponding estimates of f^* by $\omega_p(f; \delta)$ for all p , $N \geq 1$.

THEOREM 2. For $f \in L^p(I^N)$ ($1 \leq p < \infty$) and $s = 1, 2, \dots$

$$(7) \quad \sum_{n=1}^s 2^{n(p-N)} (\Delta f^*(2^{-nN}))^p \leq c 2^{sp} \omega_p^p(f; 2^{-s}),$$

where c is a constant depending on p and N only ⁽¹⁾.

Note that the estimates (6) and (7) are complementary — the first is exact whenever $\omega_p(f; \delta)$ decreases not too fast as $\delta \rightarrow 0$; if, however, the order of $\omega_p(f; \delta)$ is close to $O(\delta)$, then (7) is an essential strengthening of (6).

⁽¹⁾ Here and in the sequel, $\Delta A_n = A_{n+1} - A_n$.

In the case $1 \leq p < N$, inequality (7) can be transformed to

$$(8) \quad \int_{\delta^N}^1 t^{-p/N} (f - f_{I^N})^{*p}(t) dt \leq c\omega_p^p(f; \delta) \delta^{-p},$$

where $f_{I^N} = \int_{I^N} f(x) dx$.

Let $0 < p, q < \infty$. A measurable function f on I^N belongs to the Lorentz space $L^{q,p}(I^N)$ if

$$\int_0^1 (t^{1/q} f^*(t))^p \frac{dt}{t} < \infty.$$

Clearly, $L^{q,p} \subset L^q$ for $q \geq p$.

From (8) we immediately obtain the embedding

$$(9) \quad \text{Lip}(1; p) \subset L^{q^*,p} \quad (1 \leq p < N, q^* = Np/(N-p)).$$

Since the Sobolev space W_p^1 is contained in $\text{Lip}(1; p)$ ($1 \leq p < \infty$) (see [25], p. 166), (9) in turn yields the Sobolev embedding theorem [29]

$$W_p^1(I^N) \subset L^{q^*}(I^N) \quad (1 \leq p < N, q^* = Np/(N-p))$$

(for $p = 1$ this was proved by Gagliardo and Nirenberg; see [2], p. 143).

In Section 5 we cite some embedding theorems which can be established with the aid of the estimates (6) and (7); now, we briefly discuss their proofs.

Denote by $J(x, h)$ ($h > 0$) the cube centered at x with edge length $2h$.

Let $f \in L^p(I^N)$ ($1 \leq p < \infty$); without loss of generality we can assume that $f \geq 0$. Further, let $\tau \in (0, 1/2]$. It can easily be seen that there is a set $E \subset I^N$ with $|E| = \tau$ such that $f(x) \geq f^*(\tau)$ for all $x \in E$ and

$$\int_0^\tau [f^*(t) - f^*(\tau)]^p dt = \int_E [f(x) - f^*(\tau)]^p dx.$$

Let $x \in E$ and put $h = 2^{1/N-1} \tau^{1/N}$. Since $|J(x, h)| = 2\tau$, we have

$$|\{y \in J(x, h): f(y) \leq f^*(\tau)\}| \geq \tau.$$

Consequently,

$$\begin{aligned} \int_E [f(x) - f^*(\tau)]^p dx &\leq \tau^{-1} \int_E dx \int_{J(x,h)} |f(x) - f(y)|^p dy \\ &\leq \tau^{-1} \int_{J(0,h)} du \int_{I^N} |f(x) - f(x+u)|^p dx \\ &\leq 2\omega_p^p(f; h), \end{aligned}$$

and this proves inequality (6).

In the proof of (7) we use approximation by Steklov means. For $f \in L(I^N)$ and $h > 0$ define

$$f_h(x) = (2h)^{-N} \int_{J(x,h)} F(y) dy, \quad \bar{f}_h(x) = (f_h)_h(x).$$

If $f \in L^p(I^N)$ ($1 \leq p < \infty$), then for $h \in (0, 1]$

$$(10) \quad \|f_h\|_p \leq \|f\|_p, \quad \|f - f_h\|_p \leq \omega_p(f; h),$$

$$(11) \quad \|\partial f_h / \partial x_j\|_p \leq \omega_p(f; h)/h \quad (j = 1, \dots, N).$$

These inequalities are well known.

Let first $p > 1$. Set $h = 2^{-s}$ ($s \geq 1$) and $J_n(x) = J(x, 2^{-n})$. For $x \in I^N$ and $n = 1, 2, \dots$

$$|\{y \in J_n(x): f(y) \leq f^*(2^{-nN})\}| \geq \frac{1}{2} |J_n(x)|.$$

Therefore

$$f(x) - f^*(2^{-nN}) \leq \frac{2}{|J_n(x)|} \int_{J_n(x)} |f(x) - f(y)| dy \equiv 2\delta_n(x).$$

Hence

$$f^*(2^{-(n+1)N}) - f^*(2^{-nN}) \leq 2\delta_n^*(2^{-(n+1)N}).$$

Furthermore,

$$\delta_n(x) \leq \tilde{\delta}_n(x) + |f(x) - \bar{f}_h(x)| + 2^{nN/p} \|f - \bar{f}_h\|_p,$$

where

$$\tilde{\delta}_n(x) = 2^{(n-1)N} \int_{J_n(x)} |\bar{f}_h(x) - \bar{f}_h(y)| dy.$$

Consequently (see (10)),

$$\begin{aligned} \delta_n^*(2^{-(n+1)N}) &\leq \tilde{\delta}_n^*(2^{-(n+2)N}) + (f - \bar{f}_h)^*(2^{-(n+2)N}) + 2^{nN/p} \|f - \bar{f}_h\|_p \\ &\leq \tilde{\delta}_n^*(2^{-(n+2)N}) + c2^{nN/p} \omega_p(f; h). \end{aligned}$$

We now estimate $\tilde{\delta}_n^*(2^{-(n+2)N})$. It can easily be seen that for $y \in J_n(x)$

$$|\bar{f}_h(x) - \bar{f}_h(y)| \leq \sqrt{N} 2^{-n} \int_0^1 |g(x+t(y-x))| dt,$$

where $g(z) = |\text{grad } \bar{f}_h(z)|$. Consequently,

$$\tilde{\delta}_n(x) \leq c2^{(N-1)n} \int_0^1 dt \int_{J_n(0)} g(x+tz) dz.$$

Since for every set $E \subset I^N$ with $|E| = 2^{-(n+2)N}$ and fixed t, z we have

$$|E|^{-1} \int_E g(x+tz) dx \leq g^{**}(2^{-(n+2)N}),$$

it follows that

$$\tilde{\delta}_n^*(2^{-(n+2)N}) \leq c2^{-n} g^{**}(2^{-(n+2)N}).$$

Thus for $n = 1, 2, \dots$

$$\Delta f^*(2^{-nN}) \leq c [2^{-n} g^{**}(2^{-(n+2)N}) + 2^{nN/p} \omega_p(f; h)].$$

Hence (recall that $h = 2^{-s}$)

$$\sum_{n=1}^s 2^{n(p-N)} (\Delta f^*(2^{-nN}))^p \leq c [\|g^{**}\|_p^p + 2^{sp} \omega_p^p(f; 2^{-s})].$$

Since $p > 1$, the Hardy inequality (see [18], p. 289) and (11) yield

$$\|g^{**}\|_p \leq c_p \|g\|_p \leq c'_{p,N} 2^{sp} \omega_p^p(f; 2^{-s}).$$

This finishes the proof of (7) for $p > 1$.

Let now $p = 1$, $N \geq 2$. We will prove inequality (8) (which implies (7)). Write $g(x) = |f(x) - f_{1N}|$. For $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, put

$$\hat{x}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N), \quad x = (x_i, \hat{x}_i) \quad (i = 1, \dots, N).$$

For fixed $h \in (0, 1]$, define for $i = 1, \dots, N$

$$g_i(x) = \frac{1}{2h} \int_{x_i-h}^{x_i+h} g(t, \hat{x}_i) dt, \quad \bar{g}_i(\hat{x}_i) = \sup_{x_i \in [0,1]} g_i(x).$$

It is not difficult to show that

$$(12) \quad \|g - g_i\|_1 \leq \omega_1(f; h),$$

$$(13) \quad \int_{I^{N-1}} \bar{g}_i(y) dy \leq c \omega_1(f; h)/h.$$

Let $0 < t \leq 1$; we estimate $g^*(t)$. It is easily seen that there is a measurable set $E \subset I^N$ with measure t such that

$$g^*(t) = \operatorname{ess\,inf}_{x \in E} g(x).$$

Furthermore, there are measurable sets E_1, \dots, E_N such that

$$E \equiv E_0 \supset E_1 \supset \dots \supset E_N, \quad |E_i| = \frac{1}{2} |E_{i-1}| = 2^{-i} t,$$

and for $i = 1, \dots, N$

$$(14) \quad g_i(x) \leq \operatorname{ess\,inf}_{y \in E_i} g_i(y) \quad \text{for } x \in E_{i-1} - E_i.$$

Let $Q \subset E_N$ be a reduced ⁽²⁾ F_σ -set with measure $2^{-N} t$. Denote by $Q^{(i)}$ the orthogonal projection of Q onto the hyperplane $x_i = 0$. By the Loomis-Whitney inequality (see [14], p. 227), there is j such that the $(N-1)$ -

⁽²⁾ A measurable set Q is called *reduced* if $|Q \cap I| > 0$ for every cube I such that $Q \cap I \neq \emptyset$.

dimensional measure of $Q^{(j)}$ satisfies

$$(15) \quad m' Q^{(j)} \geq 2^{1-N} t^{(N-1)/N}.$$

By (14), $g(x) \leq |g(x) - g_j(x)| + \text{ess inf}_{y \in E_j} g_j(y)$ for $x \in E^* \equiv E_{j-1} - E_j$. Hence (see (12))

$$g^*(t) \leq \text{ess inf}_{x \in E^*} g(x) \leq \frac{2^N}{t} \omega_1(f; h) + \text{ess inf}_{y \in Q} g_j(y).$$

Since Q is a reduced set, using (15) we obtain

$$\text{ess inf}_{y \in Q} g_j(y) \leq \text{ess inf}_{z \in Q^{(j)}} \bar{g}_j(z) \leq \bar{g}_j^*(2^{-N} t^{(N-1)/N}).$$

Consequently, (13) yields

$$\int_{h^N}^1 t^{-1/N} g^*(t) dt \leq c [\omega_1(f; h)/h + \|\bar{g}_j\|_1] \leq c' \omega_1(f; h)/h,$$

which completes the proof of inequality (8) for $p = 1$, $N \geq 2$.

As already noted, for $p = 1$, $N = 2$ inequality (8) was proved by P. Oswald [27] by other methods (in particular, he used inequality (5)).

3. Estimates for rearrangements of maximal functions

In the sequel we denote by Q an N -dimensional cube with sides parallel to the coordinate axes.

Let $f \in L(I^N)$. The *Hardy-Littlewood maximal function* is defined by

$$Mf(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(t)| dt,$$

where the supremum is taken over all cubes Q containing x with $|Q| \leq 1$.

We have the inequalities ([19], [1])

$$(16) \quad 3^{-N} f^{**}(t) \leq (Mf)^*(t) \leq f^{**}(2^{-N} t).$$

Let us observe that the idea of estimating $(Mf)^*$ from above by f^{**} goes back to Hardy and Littlewood [16].

Using (16), it is not difficult to deduce from (6) and (7) the analogous estimates for the maximal function Mf (for $1 < p < \infty$).

In connection with some interpolation problems, Fefferman and Stein [9] introduced the maximal function f^* , measuring local mean oscillations:

$$f^*(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(t) - f_Q| dt, \quad f_Q = |Q|^{-1} \int_Q f(y) dy.$$

Bennett and Sharpley [1] proved that Peetre's K -functional satisfies

$$K(f; t; L^1, BMO) \sim t(f^*)^*(t).$$

Using this relation, R. De Vore [7] obtained the following result.

THEOREM 3. *Let $f \in L^p[0, 1]$ ($1 \leq p < \infty$). Then*

$$(17) \quad (f^*)^*(t) \leq c \sup_{t \leq \delta \leq 1} \delta^{-1/p} \omega_p(f; \delta) \quad (0 < t \leq 1).$$

The author [22] obtained an analogous inequality for functions of several variables (taking into account the different behaviour of the function in the direction of different coordinate axes). However, as remarked earlier, the estimates of the type (17) are not sharp in the limit cases. The following more exact estimates were established by the author [23] in terms of the "isotropic" modulus of continuity $\omega_p(f; \delta)$.

THEOREM 4. *Let $f \in L^p(I^N)$ ($1 < p < \infty$, $N \geq 1$) and*

$$\sigma = \{n: \Delta(f^*)^*(2^{-(n-1)N}) > 0\}.$$

Then for $s = 1, 2, \dots$

$$(18) \quad \sum_{n \leq s, n \in \sigma} 2^{n(p-N)} A_n^p \leq c 2^{sp} \omega_p^p(f; 2^{-s}),$$

$$(19) \quad \sum_{n \geq s, n \in \sigma} 2^{-nN} A_n^p \leq c \omega_p^p(f; 2^{-s}),$$

where $A_n = (f^*)^*(2^{-nN})$.

The estimates (18) and (19) reflect special structural properties of f^* . In particular, the following result follows from (18): if $f \in L^p(I^N)$ ($N < p < \infty$) and $\liminf_{\delta \rightarrow +0} \delta^{-N/p} \omega_p(f; \delta) = 0$, then $(f^*)^*$ is constant on some subinterval of each interval $[0, \tau]$, $0 < \tau \leq 1$.

If $1 < p < N$, then the assumption $n \in \sigma$ on the left-hand sides of (18) and (19) may be dropped. (18) remains valid also for $p = 1$, $N \geq 2$.

In the sequel, K denotes the class of all positive concave nondecreasing functions on $(0, 1]$.

Let $f \in L(I^N)$ and $\eta \in K$. Put

$$f_{\eta}^{\#}(x) = \sup_{x \in Q} \frac{1}{\eta(l_Q)|Q|} \int_Q |f(t) - f_Q| dt,$$

$$f_{\eta}^{\wedge}(x) = \sup_{x \in Q} \frac{1}{\eta(l_Q)|Q|} \int_Q |f(t) - f(x)| dt,$$

where the supremum is taken over all cubes Q containing x with $|Q| \leq 1$, and l_Q is the length of the edge of Q .

Observe that $f_{\eta}^{\#}(x) \leq 2 f_{\eta}^{\wedge}(x)$, and if $\eta(t)t^{-\alpha}$ is increasing for some $\alpha > 0$, then $f_{\eta}^{\wedge}(x) \leq c f_{\eta}^{\#}(x)$ a.e.

For $\eta(t) = t^{\alpha}$ ($0 < \alpha \leq 1$) such maximal functions were studied in a

number of papers of several authors, beginning from a paper by A. Calderón [5] (see [1], [3], [6], [8]).

Owing to the arbitrariness of $\eta \in K$, the maximal functions $f_{\eta}^{\#}$ and \dot{f}_{η} yield flexible characteristics of various metric properties of f . On the one hand, $f_{\eta}^{\#}$ measures the local mean oscillations, and \dot{f}_{η} the local smoothness of f : for $x, y \in I^N$

$$(20) \quad |f(x) - f(y)| \leq [\dot{f}_{\eta}(x) + \dot{f}_{\eta}(y)] \eta(|x - y|).$$

On the other hand, using $f_{\eta}^{\#}$ we can also estimate the maximal function Mf .

THEOREM 5. *Let $f \in L(I^N)$ and $\eta \in K$. Then*

$$(Mf)^*(t) - (Mf)^*(2t) \leq c\eta(t^{1/N})(f_{\eta}^{\#})^*(t) \quad (0 < t \leq 1/2).$$

This theorem was proved by Bennett and Sharpley [1] for $\eta(t) = t^{\alpha}$ ($0 \leq \alpha \leq 1$); in the case of a general $\eta \in K$ the proof is similar.

K. I. Oskolkov [28] investigated the question of quantitative estimates related to Lusin's C -property:

$$|f(x) - f(y)| \leq (C_f(x) + C_f(y)) \eta(|x - y|),$$

where C_f is an a.e. finite function and $\eta(t) \rightarrow 0$ as $t \rightarrow 0$. By (20), one can take $C_f(x) = \dot{f}_{\eta}(x)$. The corresponding result of K. I. Oskolkov [28] may essentially be interpreted to be a necessary and sufficient condition for $\eta \in K$ to satisfy $\dot{f}_{\eta}(x) < \infty$ a.e. for all $f \in H_{p,N}^{\omega}$. This condition may be formulated in the form (see [23])

$$(21) \quad \sum_{n=1}^{\infty} [\omega(2^{-n})/\eta(2^{-n})]^p \gamma_n < \infty,$$

where

$$(22) \quad \gamma_n = \frac{1}{\omega_n} \min(\omega_n - \omega_{n+1}, \omega_n - \omega_{n-1}), \quad \omega_n = \omega(2^{-n}).$$

The author [23] established the estimates of the rearrangements of the maximal functions \dot{f}_{η} and $f_{\eta}^{\#}$ by $\omega_p(f; \delta)$.

Denote by Φ the collection of all even nonnegative functions φ nondecreasing on $[0, \infty)$. If $\varphi \in \Phi$ and $\varphi(2t) \leq c\varphi(t)$ ($0 \leq t < \infty$), then φ is said to satisfy the Δ_2 condition.

Further, we introduce the following definition. Let ω be a modulus of continuity and $1 \leq p < \infty$. We denote by $E_p(\omega)$ the class of all nonnegative sequences $\varepsilon = \{\varepsilon_n\}$ such that for $s = 1, 2, \dots$

$$\sum_{n=1}^s 2^{np} \varepsilon_n^p \leq 2^{sp} \omega^p(2^{-s}), \quad \sum_{n=s}^{\infty} \varepsilon_n^p \leq \omega^p(2^{-s}).$$

THEOREM 6. Let $f \in L^p(I^N)$ ($1 < p < \infty$), $\omega(\delta) = \omega_p(f; \delta)$, $\eta \in K$, and suppose

$$(23) \quad \sup_{\varepsilon \in E_p(\omega)} \sum_{n=1}^{\infty} \varepsilon_n^p / \eta^p (2^{-n}) < \infty.$$

Then $V_n f \in L^p(I^N)$, and there is a sequence $\{\varepsilon_n\} \in E_p(\omega)$ such that for any $\varphi \in \Phi$ satisfying the Δ_2 condition with $\varphi(t)t^{-p}$ increasing on $(0, \infty)$,

$$(24) \quad \sum_{n=s}^{\infty} \varphi(\Delta(V_n f)^*(2^{-nN})) 2^{-nN} \\ \leq c(p, N, \varphi) \sum_{n=s}^{\infty} \varphi(2^{nN/p} \varepsilon_n / \eta (2^{-n})) 2^{-nN} \quad (s = 1, 2, \dots).$$

For $p = 1$ the theorem is no longer true. Under the same assumptions, an analogous inequality is also valid for $f_n^\#$.

One important point has to be emphasized: the sequence $\{\varepsilon_n\}$ in Theorem 6 depends not only on $\omega_p(f; \delta)$, but also on other structural properties of f . If ω is a modulus of continuity, then in general it is not possible to choose a sequence $\varepsilon \in E_p(\omega)$ in such a way that (24) holds for all f with $\omega_p(f; \delta) \leq \omega(\delta)$.

Note, moreover, that conditions (21) and (23) are equivalent.

4. Rearrangements of the modulus of an analytic function

A function f analytic in the unit disc $D_0 = \{z: |z| < 1\}$ belongs to the Hardy space H^p ($0 < p < \infty$) if

$$\|f\|_p \equiv \sup_{0 < r < 1} \left\{ \int_0^{2\pi} |f(re^{i\varphi})|^p d\varphi \right\}^{1/p} < \infty.$$

If $f \in H^p$, then according to a theorem of F. Riesz the limit

$$\lim_{r \rightarrow 1-0} f(re^{i\varphi}) \equiv f(e^{i\varphi})$$

exists for a.e. $\varphi \in [0, 2\pi]$ and represents an L^p function.

Hardy and Littlewood [17] proved

THEOREM 7. Suppose f is an analytic function in the disc D_0 and $f' \in H^p$ for some p with $0 < p < 1$. Then $f \in H^{q^*}$, $q^* = p/(1-p)$.

The problem arises of obtaining exact estimates for the rearrangement of the modulus of the boundary function which would in particular imply Theorem 7.

Let $f \in H^p$ ($0 < p < \infty$). The L^p modulus of continuity of the boundary function $f(e^{i\varphi})$ is defined by

$$\omega_p(f; \delta) = \sup_{0 \leq h \leq \delta} \left\{ \int_0^{2\pi} |f(e^{i(\varphi+h)}) - f(e^{i\varphi})|^p d\varphi \right\}^{1/p}.$$

If $f \in H^p$ ($0 < p < \infty$), $f \neq \text{const}$, then as shown by E. A. Storozhenko,

$$\liminf_{\delta \rightarrow +0} \omega_p(f; \delta)/\delta > 0.$$

For $0 < p < 1$ this is caused by the analyticity of $f(z)$ and is not true for real L^p functions.

If f is analytic in the unit disc, then $f'(z)$ is in H^p ($0 < p < \infty$) if and only if $f \in H^p$ and

$$(25) \quad \omega_p(f; \delta) = O(\delta)$$

(this was established by Hardy and Littlewood [17] for $1 \leq p < \infty$ and by A. E. Gwilliam [13] for $0 < p < 1$).

The author obtained estimates of the rearrangement of $|f(e^{i\varphi})|$ by $\omega_p(f; \delta)$, analogous to (6) and (7).

THEOREM 8. *Let $f \in H^p$ ($0 < p < 1$) and let f^* be the nondecreasing rearrangement of $|f(e^{i\varphi})|$. Then for $s = 1, 2, \dots$*

$$(26) \quad \sum_{n=1}^s 2^{n(p-1)} (\Delta f^*(2^{-nN}))^p \leq c_p 2^{sp} \omega_p^p(f; 2^{-s}),$$

$$(27) \quad \sum_{n=s}^{\infty} 2^{-nN} (\Delta f^*(2^{-nN}))^p \leq c_p \omega_p^p(f; 2^{-s}).$$

Using (26) and the above-mentioned result of A. E. Gwilliam we obtain

COROLLARY. *Suppose f is an analytic function in the disc D_0 and $f' \in H^p$ ($0 < p < 1$). Then the boundary function $f(e^{i\varphi})$ is in the Lorentz class L^{q^*} , $q^* = p/(1-p)$.*

This assertion is a strengthening of Theorem 7.

Note that for real functions in $L^p[0, 2\pi]$ with $0 < p < 1$ the estimate (26) does not hold, and (25) does not ensure that f belongs to L^{q^*} with the limit exponent $q^* = p/(1-p)$.

On the other hand, inequality (27) is not related to analyticity. It holds for any $f \in L^p[0, 2\pi]$ ($0 < p < \infty$), and the proof is exactly the same as that of (6) for $p \geq 1$.

We now sketch the proof of the main estimate (26).

First, we use the following inequalities due to E. A. Storozhenko [31].

[32]: if $f \in H^p$ ($0 < p < 1$), then for all $h \in (0, 1]$

$$(28) \quad \int_0^{2\pi} |f(e^{i\varphi}) - f((1-h)e^{i\varphi})|^p d\varphi \leq c_p \omega_p^p(f; h),$$

$$(29) \quad \int_0^{2\pi} |f'((1-h)e^{i\varphi})|^p d\varphi \leq c_p \omega_p^p(f; h) h^{-p}.$$

For $f \in H^p$ we denote by $\cdot // f$ the nontangential maximal function

$$\cdot // f(\varphi) = \sup_{z \in \Omega_\varphi} |f(z)|, \quad \varphi \in [0, 2\pi],$$

where Ω_φ is the region bounded by the two tangents to the circle $|z| = 1/\sqrt{2}$ drawn from the point $e^{i\varphi}$ and by the longer of the two arcs of the circle with ends at the points of tangency. By the Hardy–Littlewood theorem [16]

$$(30) \quad \|\cdot // f\|_p \leq A_p \|f\|_p \quad (0 < p < \infty).$$

With the above results, the proof of the inequality (26) is very easy.

Let $h = 2^{-s}$ ($s \geq 1$). Define

$$F_h(\varphi) = |f(e^{i\varphi}) - f((1-h)e^{i\varphi})|,$$

$$g(re^{i\varphi}) = f((1-h)re^{i\varphi}), \quad 0 \leq r < 1.$$

Fix $\varphi \in [0, 2\pi]$. For every $n = 1, 2, \dots$ there is $\theta \equiv \theta(\varphi, n)$ such that

$$|\varphi - \theta| \leq 2^{-n+2} \quad \text{and} \quad |f'(e^{i\theta})| \leq f^*(2^{-n}),$$

$$F_h(\theta) \leq F_h^*(2^{-n}), \quad (\cdot // g')(\theta) \leq (\cdot // g')^*(2^{-n}).$$

We have

$$|f(e^{i\varphi})| - f^*(2^{-n}) \leq |f(e^{i\varphi})| - |f(e^{i\theta})|$$

$$\leq F_h(\varphi) + F_h(\theta) + |g(e^{i\varphi}) - g(e^{i\theta})|$$

$$\leq F_h(\varphi) + F_h^*(2^{-n}) + |g(e^{i\varphi}) - g(e^{i\theta})|.$$

Let z_0 be the intersection point of the tangents forming part of the boundaries of Ω_φ and Ω_θ respectively. Then

$$|z_0 - e^{i\varphi}| = |z_0 - e^{i\theta}| \leq c2^{-n}.$$

Consequently,

$$|g(e^{i\varphi}) - g(e^{i\theta})| \leq |g(e^{i\varphi}) - g(z_0)| + |g(z_0) - g(e^{i\theta})|$$

$$\leq c[(\cdot // g')(\varphi) + (\cdot // g')(\theta)]2^{-n}.$$

Thus

$$|f(e^{i\varphi})| - f^*(2^{-n}) \leq F_h(\varphi) + c2^{-n}(\cdot // g')(\varphi) + F_h^*(2^{-n}) + c2^{-n}(\cdot // g')^*(2^{-n}).$$

Hence

$$f^*(2^{-n-1}) - f^*(2^{-n}) \leq 2F_h^*(2^{-n-2}) + c' 2^{-n} (f/g)^*(2^{-n-2}),$$

and we obtain (see (28)–(30))

$$\begin{aligned} \sum_{n=1}^s 2^{n(p-1)} [f^*(2^{-n-1}) - f^*(2^{-n})]^p &\leq c_1 (2^{sp} \|F_h\|_p^p + \|Mg'\|_p^p) \\ &\leq c_2 (2^{sp} \omega_p^p(f; 2^{-s}) + \|g'\|_p^p) \leq c 2^{sp} \omega_p^p(f; 2^{-s}). \end{aligned}$$

The proof of inequality (26) is complete.

5. Some embedding theorems

As already noted in Section 1, P. L. Ulyanov was the first to propose methods for investigating extremal problems of embedding theory, based on estimates of nonincreasing rearrangements. Using inequalities (1) and (2), P. L. Ulyanov proved the following

THEOREM 9. *Let $1 \leq p < q < \infty$ and let ω be a modulus of continuity. Then $H_{p,1}^\omega \subset L^q[0, 1]$ if and only if*

$$\int_0^1 x^{-q/p} \omega^q(x) dx < \infty.$$

For $N \geq 2$ the problem of embedding $H_{p,N}^\omega \subset L^q(I^N)$ has been considered in a number of papers of various authors (for the references see [21], where necessary and sufficient conditions for the anisotropic classes $H_p^{\omega_1, \dots, \omega_N}$ to embed in L^q are obtained).

Let $\varphi \in \Phi$ (see §3). We denote by $\varphi(L)$ the class of all measurable functions f on I^N with $\varphi \circ f \in L(I^N)$.

P. L. Ulyanov posed the problem of finding necessary and sufficient conditions in order that

$$(31) \quad H_{p,N}^\omega \subset \varphi(L),$$

where $\varphi \in \Phi$ satisfies the Δ_2 condition.

In the papers of P. L. Ulyanov [34], [35], L. Leindler [24] and E. A. Storozhenko [30] necessary and sufficient conditions were obtained to have the embedding

$$H_{p,1}^\omega \subset L^q \psi(L) \quad (1 \leq p \leq q < \infty, N = 1)$$

under additional assumptions on the function ψ ; in particular, it was assumed that $\psi(t^2) = O(\psi(t))$ as $t \rightarrow \infty$.

The author [20] proved that under minimal additional assumptions on

φ the embedding problem (31) may be solved with the use of inequality (6) together with a decomposition of the sequence $\{\omega(2^{-n})\}$ according to a decreasing geometrical progression.

THEOREM 10. *Suppose $N \leq p < \infty$, $\varphi \in \Phi$ satisfies the Δ_2 condition and $\varphi(t)t^{-p}$ is increasing on $(0, \infty)$. Then the embedding (31) holds if and only if*

$$\sum_{n=1}^{\infty} \varphi(2^{nN/p} \omega(2^{-n})) 2^{-nN} \alpha_n < \infty,$$

where $\alpha_n = [\omega(2^{-n}) - \omega(2^{-n-1})]/\omega(2^{-n})$.

As already noted, in general inequality (6) is not exact if the smoothness of $\omega_p(f; \delta)$ is close to the limit one $O(\delta)$; however, this degree of smoothness is not required to have the embedding (31) for $p \geq N$. For $N \geq 2$ and $1 \leq p < N$ the situation is different (e.g. $H_{p,N}^\omega \subset L^{q^*}(I^N)$, $q^* = Np/(N-p)$, if and only if $\omega(\delta) = O(\delta)$). In this case, for solving the embedding problem (31) one uses both estimates (6) and (7) together with simultaneous decomposition of the sequences $\{\omega(2^{-n})\}$ and $\{2^n \omega(2^{-n})\}$ (3).

We obtain the following result.

THEOREM 11. *Suppose $1 \leq p < N$, $q^* = Np/(N-p)$, and $\varphi \in \Phi$ is such that $\varphi(t)t^{-p}$ increases and $\varphi(t)t^{-q^*}$ decreases on $(0, \infty)$. Then the condition*

$$\sum_{n=1}^{\infty} \varphi(2^{nN/p} \omega(2^{-n})) 2^{-nN} \gamma_n < \infty$$

is necessary and sufficient for the embedding (31) to hold (4).

Note that the assumption of $\varphi(t)t^{-q^*}$ being decreasing does not restrict the growth order: we show that $\text{Lip}(1; p) \subset \varphi(L)$ if and only if $\varphi(t) = O(t^{q^*})$ ($1 \leq p < N$).

The main difficulties in Theorem 10 and 11 arise in the cases where the order of growth of φ is close to $|t|^p$ or $|t|^{q^*}$ (the latter case is most complicated). By excluding these cases, we obtain

COROLLARY. *Suppose $1 \leq p < \infty$ and $\varphi \in \Phi$ has the following properties:*

- 1) φ satisfies the Δ_2 condition.
 - 2) For some $\varepsilon > 0$, $\varphi(t)t^{-p-\varepsilon}$ is increasing on $(0, \infty)$.
 - 3) If $1 \leq p < N$, then for some $\varepsilon > 0$, $\varphi(t)t^{-q^*-\varepsilon}$ is decreasing on $(0, \infty)$.
- Then the condition

$$(32) \quad \sum_{n=1}^{\infty} \varphi(2^{nN/p} \omega(2^{-n})) 2^{-nN} < \infty$$

is necessary and sufficient for the embedding (31) to hold.

(3) Such a simultaneous decomposition was first used by K. I. Oskolkov [28].

(4) The sequence $\{\gamma_n\}$ is defined in (22).

Observe that the sufficiency of (32) is easily established with the use of the inequality (see (1), (6))

$$f^*(t) - f^*(2t) \leq 2^{1+1/p} t^{-1/p} \omega_p(f; t^{1/N}),$$

and remains valid for any $\varphi \in \Phi$ satisfying the Δ_2 condition (for $N = 1$ this was shown by L. Leindler [24]).

We also point out that M. L. Gol'dman obtained the assertion of the above corollary for moduli of continuity of any order.

Using the estimates (26) and (27) we establish the analogue of Theorem 11 for subclasses of the Hardy space H^p .

Let $0 < p < \infty$ and let ω be a modulus of continuity. We denote by $H^p(\omega)$ the class of all functions in H^p for which the modulus of continuity of the boundary function satisfies $\omega_p(f; \delta) = O(\omega(\delta))$.

Put

$$M_p(r; f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\varphi})|^p d\varphi \right)^{1/p}, \quad 0 < r < 1.$$

E. A. Storozhenko [32] proved

THEOREM 12. *Let $f \in H^p$, $0 < p < \infty$. Then for all $q \in (p, \infty)$*

$$M_q(r; f) \leq c_{p,q} [|f(0)| + I(r)],$$

where

$$I(r) = \left(\int_{1-r}^1 x^{-q/p} \omega_p^q(f; x) dx \right)^{1/q}.$$

This theorem gives an exact condition for the embedding $H^p(\omega) \subset H^q$ ($0 < p < q < \infty$) to hold, except the limit case $0 < p < 1$, $q = p/(1-p)$, $\omega(\delta) = O(\delta)$, covered by Theorem 7.

Let $\varphi \in \Phi$. We denote by $\varphi(H)$ the class of all functions $f(re^{i\varphi})$ analytic in the unit disc with

$$\sup_{0 < r < 1} \int_0^{2\pi} \varphi(|f(re^{i\theta})|) d\theta < \infty.$$

Applying Theorem 8, we obtain the following result (cf. (22)):

THEOREM 13. *Suppose $0 < p < 1$, $q^* = p/(1-p)$, and $\varphi \in \Phi$ is such that $\varphi(t)t^{-p}$ increases and $\varphi(t)t^{-q^*}$ decreases on $(0, \infty)$. Let ω be a modulus of continuity. Then the condition*

$$\sum_{n=1}^{\infty} \varphi(2^{n/p} \omega(2^{-n})) 2^{-n} \gamma_n < \infty$$

is necessary and sufficient for the embedding $H^p(\omega) \subset \varphi(H)$ to hold.

Finally, we note that using Theorem 6 it is possible to obtain a necessary and sufficient condition for the embedding

$$H_{p,N}^\omega \subset C_\varphi^\eta \quad (1 < p < \infty, \eta \in K, \varphi \in \Phi)$$

to hold, where

$$C_\varphi^\eta = \{f \in L(I^N): \dots, f \in \varphi(L)\}$$

(in case $\varphi(t) = |t|^r$ ($1 \leq r < \infty$) we write $C_\varphi^\eta \equiv C_r^\eta$). In particular, the following theorem holds [23]:

THEOREM 14. *The embedding $H_{p,N}^\omega \subset C_q^\eta$ ($1 < p \leq q < \infty$) holds if and only if*

$$(33) \quad \sum_{n=1}^{\infty} 2^{nN(q/p-1)} [\omega(2^{-n})/\bar{\eta}(2^{-n})]^q \gamma_n < \infty,$$

where $\bar{\eta}(t) = t^{1-\beta} \inf_{0 < \tau \leq t} \tau^{\beta-1} \eta(\tau)$, $\beta = N(1/p - 1/q) < 1$.

For $p = q$, Theorem 14 can be deduced from the results of K. I. Oskolkov [28]. For $\eta(t) = t^\alpha$, the classes C_p^η are studied in detail in the monograph [8]. If $\eta(t) = t^\alpha$ with $0 < \alpha < 1 - N(1/p - 1/q)$, then the sequence $\{\gamma_n\}$ in condition (33) may be dropped; in this case Theorem 14 is contained in [8].

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*Translated from the Russian
by Jerzy Trzeciak*

*Presented to the Semester
Approximation and Function Spaces
February 27–May 27, 1986*
