

ON HAMILTONIAN CONSECUTIVE- d DIGRAPHS*

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The concept of a consecutive- d digraph was defined by Du, Hsu and Hwang. It generalizes many interconnection networks of computer and multiprocessor systems. The problem of characterizing which consecutive- d digraphs are Hamiltonian has been solved for $d = 1$ and $d \geq 5$. In this paper, we settle the case $d = 2$ and also give some results for $d = 3, 4$.

I. Introduction

A consecutive- d digraph $G(d, n, q, r)$ studied in [4] is a digraph with n nodes labeled by the residues modulo n , Z_n , and such that a link from node i to node j exists iff $j \equiv qi + r, \dots, qi + r + d - 1 \pmod{n}$ for some given $r \in Z_n$ and nonzero $q \in Z_n$. It generalizes the class of generalized deBruijn digraphs ($q = d, r = 0$) [2, 9, 12] and the class of Imase-Itoh digraphs ($q = r = n - d$) [10]. A digraph is said to be *Hamiltonian* if it contains a spanning circuit as subgraph. The Hamiltonian property is important in applications since a ring structure facilitates implementation of certain protocols and algorithms [11]. The characterization of Hamiltonian generalized deBruijn digraphs and Hamiltonian Imase-Itoh digraphs has been completely settled [2, 3, 7]. However, the story for the consecutive- d digraph is a little different. Hwang [8] studied the case $d = 1$. Du, Hsu and Hwang [4] showed that the consecutive- d digraph for $d > 5$ is Hamiltonian iff $\gcd(n, q) \leq d$. In this paper, we settle the case $d = 2$, the case $0 < q \leq d$ and the case $0 < n - q \leq d$.

* This work was supported in part by NSF of China. Part of the work was done when the first author visited the Stefan Banach International Mathematical Center, Warsaw, Poland.

Du, Hsu and Hwang [4] showed that if $\gcd(n, q) > 1$, then $G(d, n, q, r)$ is Hamiltonian iff $d \geq \gcd(n, q)$. Thus, without loss of generality, we can assume $\gcd(n, q) = 1$ throughout this paper.

II. $d = 2$

For a digraph $G = (V, E)$, G^* denotes the bipartite graph with node set $V \cup V^*$ where $V^* = \{v^* | v \in V\}$, and with edges from each node u to node v^* if link $(u, v) \in E$. Let φ denote the map from link (u, v) of G to edge (u, v^*) of G^* . Obviously, φ is one-to-one and onto. It is a useful fact that a subset H of E forms a 1-factor of G iff $\varphi(H)$ forms a perfect matching of G^* .

THEOREM 1. $G(2, n, q, r)$ is Hamiltonian iff either $G(1, n, q, r)$ or $G(1, n, q, r+1)$ is Hamiltonian.

Proof. The “if” part is trivial. Next we show the “only if” part. It suffices to prove that $G^*(2, n, q, r)$ is a cycle using edges of $G^*(1, n, q, r)$ and $G^*(1, n, q, r+1)$, alternately. Since $\gcd(n, q) = 1$, for every node j , we can find a node i such that $j \equiv qi + r \pmod{n}$, i.e., (i, j^*) and $(i, (j+1)^*)$ are two edges of $G^*(2, n, q, r)$. It follows that $G^*(2, n, q, r)$ is a cycle passing through $0^*, 1^*, \dots, (n-1)^*$ consecutively and edges of $G^*(1, n, q, r)$ and $G^*(1, n, q, r)$ alternately. ■

Hwang [8] showed that $G(1, n, q, r)$ is Hamiltonian iff one of the following conditions holds.

- (1) $q = 1$ and $\gcd(n, r) = 1$.
- (2) $\gcd(n, r) | \gcd(n, q-1)$ and q belongs to the exponent $n \pmod{tn}$ where $t = \gcd(n, q-1)/\gcd(n, r)$.

Thus, $G(2, n, q, r)$ is Hamiltonian iff either condition (1) or condition (2) holds for $G(1, n, q, r)$ or $G(1, n, q, r+1)$.

From the proof of Theorem 1, we can see that $G(2, n, q, r)$ has exactly two 1-factors. Clearly, more 1-factors will give more possibilities for a graph being Hamiltonian. Du, Hsu and Peck [5] improved $G(d, n, q, r)$ by replacing all loops by a circuit. The resulting digraph is denoted by $D(d, n, q, r)$. They showed that the connectivity can be enlarged by this simple improvement. The next theorem shows that the number of 1-factors can also be enlarged by that improvement.

THEOREM 2. Suppose that $G(2, n, q, r)$ has loops. Then $D(2, n, q, r)$ has at most 2^s 1-factors where $s = \gcd(n, q-1)$. Moreover, the number of 1-factors of $D(2, n, q, r)$ can reach 2^s if all loops of $G(2, n, q, r)$ are replaced by an appropriately chosen circuit.

Proof. If $s = 1$, then $G(2, n, q, r)$ has exactly two loops at nodes i and j which are solutions of the following equations, respectively:

$$i \equiv qi + r \pmod{n}, \quad j \equiv qj + r + 1 \pmod{n}.$$

$D^*(2, n, q, r)$ can be obtained from $G^*(2, n, q, r)$ by replacing two edges (i, i^*) and (j, j^*) by (i, j^*) and (j, i^*) . Note that the replacement does not break the cycle. Thus, $D(2, n, q, r)$ has exactly two 1-factors.

If $s > 1$, then $G(2, n, q, r)$ has loops iff $s|r$ or $s|(r+1)$. We first consider the case $s|r$. In this case, $s \nmid (r+1)$. Thus, $G(2, n, q, r)$ has exactly s loops, respectively, at s nodes which are solutions of $i \equiv qi + r \pmod{n}$, $0 \leq i_1 < \dots < i_s \leq n-1$. Note that each node of $D^* = D^*(2, n, q, r)$ is of degree 2. Hence, D^* is a 1-factor. Moreover, D^* can be constructed from $G^* = G^*(2, n, q, r)$ by replacing s edges (i_k, i_k^*) by other s edges. However, removing s edges from G^* would break it into s pieces. Thus, D^* consists of at most s cycles. Therefore, D^* has at most 2^s perfect matchings. Now, we choose the way that connects i_1, \dots, i_s into a circuit $i_1 \rightarrow \dots \rightarrow i_s \rightarrow i_1$. Then, the corresponding D^* consists of exactly s cycles. Note that each piece obtained by removing edges (i_k, i_k^*) from G^* is of odd length (the number of edges). Thus, each cycle of D^* is of even length. It follows that D^* has exactly 2^s perfect matchings, i.e., the corresponding $D(2, n, q, r)$ has exactly 2^s 1-factors.

In the case $s|(r+1)$, the argument is similar except that to reach the maximum number of 1-factors, we should connect nodes with a loop into a circuit in decreasing direction. ■

III. $d = 3, 4$

First of all, we describe a general approach.

Since $\gcd(n, q) = 1$, $C = G(1, n, q, r+1)$ is a 1-factor. Let C_1, \dots, C_m be the set of disjoint circuits of C . If $m = 1$, we are done. If $m > 1$ then we want to merge the m circuits into one circuit. Assume that nodes i and $i+1$ are on two different circuits C_j and C_k . Let x (y) be the node preceding i ($i+1$) on C_j (C_k). Then we can replace the two links (x, i) and $(y, i+1)$ by the two links $(x, i+1)$ and (y, i) . C_j and C_k then merge into one circuit. We call this replacement the *interchange* of $(i, i+1)$. Note that the two new links are not in C but in $G(1, n, q, r+2)$ and $G(1, n, q, r)$, respectively.

Now, we state a procedure for merging the m circuits in C into one circuit. Let S be a graph with nodes $0, 1, \dots, n-1$ and edges $(i_1, i_1+1), \dots, (i_s, i_s+1)$ such that no chain in S contains more than $d-2$ edges. Let C' be the undirected version of C . Suppose that $C' \cup S$ is connected. We use $m-1$ iterations. Initially, set $P = S$ and $Q = C$. At each iteration, choose $(i, i+1)$ from P such that i and $i+1$ are on the different circuits C_x and C_y of Q and that either $(i-1, i) \notin P$ or $i-1$ and i are on the same circuit of Q . The connectivity of $C' \cup S$ guarantees the existence of such $(i, i+1)$. Make interchange of $(i, i+1)$, merge C_x and C_y to give the updated Q and reset $P := P \setminus \{(i, i+1)\}$.

Note that according to the procedure, no interchange of $(i, i+1)$ follows

the interchange of $(i-1, i)$. Moreover, by the property of S , the maximal chain of interchange-performed edges contains at most $d-2$ edges. These two facts guarantee that the new edges used in the procedure all belong to $G(d, n, q, r)$. Thus to show $G(d, n, q, r)$ being Hamiltonian, it suffices to find the expected S .

Divide $0, 1, \dots, n-1$ into n' groups $g(0), g(1), \dots, g(n'-1)$. Each group contains at most $d-1$ consecutive numbers modulo n . Choose S such that $(i, i+1) \in S$ iff i and $i+1$ belong to the same group. Define a graph H to be with nodes $g(0), g(1), \dots, g(n'-1)$ and such that an edge between $g(i)$ and $g(j)$ exists iff C' has an edge between a member of $g(i)$ and a member of $g(j)$. Clearly, $C' \cup S$ is connected iff H is connected. In the proofs of the following theorems, we will specify the group $g(i)$.

THEOREM 3. *If $1 < \gcd(n, q+1) < d$ and r is even, then $G(d, n, q, r)$ is Hamiltonian.*

Proof. Let $p = \gcd(n, q+1)$. Define $g(i) = \{r/2 + pi, r/2 + pi + 1, \dots, r/2 + pi + p - 1\}$. Thus, $n' = n/p$. Note that

$$q(r/2 + pi) + r + 1 = r/2 + p(qi + r(q+1)/(2p)) + 1,$$

$$q(r/2 + pi + 1) + r + 1 = r/2 + p(qi + r(q+1)/(2p) + (q+1)/p).$$

Thus, $g(qi + r(q+1)/(2p))$ and $g(qi + r(q+1)/(2p) + (q+1)/p)$ are connected through $g(i)$. Since $\gcd(n', q) = 1$, $qi + r(q+1)/(2p)$ runs over $0, 1, \dots, n'-1$ as i runs through $0, 1, \dots, n'-1$ modulo n' . Therefore, for every j , $g(j)$ and $g(j + (q+1)/p)$ are connected. Moreover, $\gcd(n', (q+1)/p) = 1$. Hence, H is connected. ■

THEOREM 4. *If $1 < \gcd(n, q-1, r+1) < d$ then $G(d, n, q, r)$ is Hamiltonian.*

Proof. Let $p = \gcd(n, q-1, r+1)$. Define $g(i) = \{pi, pi + 1, \dots, pi + p - 1\}$. Then $n' = n/p$. Note that

$$q(pi) + r + 1 = p(qi + (r+1)/p),$$

$$q(pi + 1) + r + 1 = p(qi + (r+1)/p + (q-1)/p) + 1.$$

Thus, $g(qi + (r+1)/p)$ and $g(qi + (r+1)/p + (q-1)/p)$ are connected through $g(i)$. Since $\gcd(n', q) = 1$, $qi + (r+1)/p$ runs over $0, 1, \dots, n'-1$ as i runs through $0, 1, \dots, n'-1$ modulo n' . Hence, for any j , $g(j)$ and $g(j + (q-1)/p)$ are connected. It follows that if $i = j \pmod{t}$ where $t = \gcd(n', (q-1)/p)$, then $g(i)$ and $g(j)$ are connected.

For $k = 0, 1, \dots, t-1$, denote by $H(k)$ the subgraph of H induced by $\{g(i) | i = k \pmod{t}\}$. Clearly, $H(k)$ is connected. Since

$$qk + (r+1)/p = k + (r+1)/p \pmod{t},$$

for any k , $H(k)$ and $H(k + (r+1)/p)$ are connected by H . Moreover,

$\gcd((r+1)/p, t) = \gcd((r+1)/p, n', (q-1)/p) = 1$. Therefore, all $H(k)$ are connected by H . Hence, H is connected. ■

Before presenting the next theorem, we show a lemma.

LEMMA 1. *If $r \equiv r' \pmod{h}$ where $h = \gcd(n, q-1)$ then $G(d, n, q, r)$ and $G(d, n, q, r')$ are isomorphic.*

Proof. Write $r = hx + r'$. Let y be a solution of the equation $(q-1)y = hx \pmod{n}$. It is easy to verify that the map $f: i \rightarrow i + y$ gives an isomorphism from $G(d, n, q, r)$ onto $G(d, n, q, r')$. ■

The next theorem is a generalization of the result in [3].

THEOREM 5. *If $d \geq 3$ and $0 < q \leq d$ (or $0 < n - q \leq d$), then $G(d, n, q, r)$ is Hamiltonian.*

Proof. We consider six cases.

Case 1: $q = 1$. If $r \equiv n-1 \pmod{n}$ then $\gcd(n, r) = 1$, so $G(1, n, 1, r)$ is a Hamiltonian circuit. If $\gcd(n, r+1) = 1$, then $G(1, n, 1, r+1)$ is a Hamiltonian circuit. Thus, without loss of generality, we can assume that $r \not\equiv n-1 \pmod{n}$ and $\gcd(n, r+1) = m > 1$. Clearly, C consists of m disjoint circuits containing nodes $0, 1, \dots, m-1$, respectively. Choose S consisting of $m-1$ disjoint edges $(0, 1), (r+2, r+3), (2, 3), (r+4, r+5), \dots$ which interconnect the m circuits of C . Then S meets our requirement.

Case 2: $q = 2$. Since $G(d, n, 2, r)$ has the subgraph $G(3, n, 2, r)$ which is isomorphic to $G(3, n, 2, 0)$ by Lemma 1, it suffices to show that $G(3, n, 2, 0)$ is Hamiltonian. Define $g(i) = \{2i+1, 2i+2\}$ for $i = 0, 1, \dots, \lfloor n/2 \rfloor - 1$, and $g(\lfloor n/2 \rfloor - 1) = \{0\}$ if n is odd. We prove by induction on k that $g(0), g(1), \dots, g(k)$ are connected in H . For $k = 0$, this is trivial. For $k > 0$, since $(k, 2k+1)$ is a link of C and k belongs to $g(i)$ for some $i = 0, 1, \dots, k-1$, $g(k)$ is connected to such $g(i)$. However, by the induction hypothesis, $g(0), g(1), \dots, g(k-1)$ are connected. Thus, $g(0), g(1), \dots, g(k)$ are connected. Hence, H is connected.

Case 3: $3 \leq q \leq d$. By Lemma 1, it suffices to show that $G(q, n, q, r)$ for $q \geq 3$ and $0 \leq r < q-1$ is Hamiltonian. We first consider the subcase $r < q-2$. Set $n' = \lceil n/(q-1) \rceil$. Define $g(i) = \{i(q-1)+1, \dots, (i+1)(q-1)\}$ for $i = 0, 1, \dots, n'-2$ and $g(n'-1) = \{(n'-1)(q-1)+1, \dots, n-1, 0\}$. Since $(0, r+1)$ is a link of C , $g(n'-1)$ and $g(0)$ are connected in H .

Next, we show by induction on k that

$$(*) \quad g(n'-1), g(0), \dots, g(k) \text{ are connected in } H.$$

Consider $k > 0$. Since $\gcd(n, q) = 1$, for every i , there exists i^* such that $qi^* + r + 1 \equiv i \pmod{n}$. If $(i^*-1, i^*) \in S$ for some $i \in g(k)$, then $g(k)$ will be

connected to either $g(k-1)$ or $g(k-2)$ since $i-q$ and i are connected through edges $(i^*-1, i-q)$, (i^*-1, i^*) and (i^*, i) of $C' \cup S$. By the induction hypothesis, we can conclude (*). If such an $i \in g(k)$ does not exist, then for every $i \in g(k)$, $i^* = w(q-1)+1$ for some $w = 0, 1, \dots, n'-1$. Since $i^* = w(q-1)+1$ for some $i \in g(k)$ and $w = n'-1, 0, 1, \dots, k-1$ implies that $g(k)$ is connected to one of $g(n'-1), g(0), \dots, g(k-1)$, without loss of generality, we can furthermore assume that for every $i \in g(k)$, $i^* = w(q-1)+1$ for some $w = k, k+1, \dots, n'-2$. As w runs from k to $n'-2$, $q(w(q-1)+1)+r+1$ will cover the total distance of $(n'-2-k)q(q-1) \leq nq - (k+1)q(q-1)$. It covers at most q rounds of the n -cycle even if the first round is defined to be from $kq(q-1)+q+r+1$ to $n-1$.

If $(k(q-1)+q)^* = w(q-1)+1$ for some $w = k, k+1, \dots, n'-2$, then each round has to contribute one to $k(q-1)+j$, $j = 1, \dots, q$. Let v_i be the node contributed by the i th round. Then $v_i - v_{i+1}$ must be a constant which clearly should be ± 1 . Therefore, v_1 must be either $k(q-1)+1$ or $k(q-1)+q$. Note that $0 < v_1 = q(w(q-1)+1)+r+1 < n-1$. Thus, $v_1 = k(q-1)+1$ implies that $k(q-1) = (wq+1)(q-1)+r+1$, contradicting $0 \leq r < q-2$, and $v_1 = k(q-1)+q$ implies that $k(q-1) = wq(q-1)+r+1$, also contradicting $0 \leq r < q-2$.

Hence, we have either $(k(q-1)+q)^*$ not in the form $w(q-1)+1$, $0 \leq w < n'$, or $(k(q-1)+1)^* = w(q-1)+1$ for some $w = n'-1, 0, \dots, k-1$. It follows that $g(k+1)$ is connected to $g(n'-1), g(0), \dots, g(k-1)$. Since $(k(q-1)+1)^*$ is in the form $w(q-1)+1$, $k \leq w \leq n'-2$, we get $((k(q-1)+1)^*, (k(q-1)+1)^*+1) \in S$. It follows that $g(k)$ and $g(k+1)$ are connected in H . Therefore, (*) holds.

In the subcase $r = q-2$, we have $\gcd(n, q-1, r+1) = \gcd(n, q-1) < q$. If $\gcd(n, q-1) > 1$, then by Theorem 4, $G(q, n, q, r)$ is Hamiltonian. If $\gcd(n, q-1) = 1$, then by Lemma 1, $G(q, n, q, r)$ is isomorphic to $G(q, n, q, 0)$ which is Hamiltonian.

Case 4: $q = n-1$. Note that $\gcd(n, q-1) = \gcd(n, n-2) = 1$ or 2 . By Lemma 1, $G(d, n, n-1, r)$ is isomorphic to $G(d, n, n-1, r')$ for $r' = 0$ or $r' = 1$. For $G(d, n, n-1, 0)$, we choose S consisting of disjoint edges $(0, 1), (n-1, n-2), (2, 3), (n-3, n-4), \dots$. For $G(d, n, n-1, 1)$, we choose S consisting of disjoint edges $(1, 2), (n-1, n-2), (3, 4), (n-3, n-4), \dots$. It is easy to verify that the chosen S meets our requirement.

Case 5: $q = n-2$. Note that $\gcd(n, n-3) = 1$ or 3 . By Lemma 1, $G(d, n, n-2, r)$ has a subgraph isomorphic to $G(3, n, n-2, r')$ for $r' = 0, 1, 2$. Set $n' = \lceil n/2 \rceil$.

For $G(3, n, n-2, 0)$, define $g(i) = \{2i+1, 2i+2\}$ for $0 \leq i < n'-1$ and $g(n'-1) = \{0\}$ if n is odd and $\{n-1, 0\}$ if n is even. Clearly, $g(n'-1)$ and $g(0)$ are connected in H . Furthermore, we prove by induction on k that $g(n'-k-1), \dots, g(0), \dots, g(k)$ are connected in H where $0 \leq k \leq \lceil n'/2 \rceil$. Consider $k \geq 1$. Let $j = 0$ if n is odd, and $j = 1$ if n is even. Observe that

$(k+j, 2(n'-k-1)+2-j)$ is a link of C and $k+j \leq 2k$. Thus, $g(n'-k-1)$ is connected to one of $g(0), \dots, g(k-1)$. Moreover, note that $(n-k, 2k+1)$ is a link of C and $2(n'-k-1)+1 \leq n-2k \leq n-k$, so $g(k)$ is connected to one of $g(n'-k-1), \dots, g(n'-1), g(0)$. By the induction hypothesis, $g(n'-k-1), \dots, g(0), \dots, g(k)$ are connected in H .

For $G(3, n, n-2, r')$, $r' = 1, 2$, define $g(i) = \{2i, 2i+1\}$ for $0 \leq i \leq n'-2$ and $g(n'-1) = \{n-1\}$ if n is odd and $\{n-2, n-1\}$ if n is even. By an argument similar to the above, we can show that H is connected.

Case 6: $3 \leq n-q \leq d$. By Lemma 1, $G(d, n, q, r)$ has a subgraph isomorphic to $G(n-q, n, q, r')$ for some $0 \leq r' < \gcd(n, q-1)$. Thus, it suffices to show that $G(d, n, q, r)$ for $d = n-q \geq 3$ and $0 \leq r < \gcd(n, d+1)$ is Hamiltonian. We divide the proof into three subcases.

Subcase 6.1: $0 \leq r \leq d-2$. Suppose $(d-1) \nmid (n+r)$. Set $n' = \lceil n/(d-1) \rceil$. Define

$$(\$) \quad g(i) = \begin{cases} \{i(d-1)+1, \dots, (i+1)(d-1)\} & \text{if } 0 \leq i \leq n'-2, \\ \{(n'-1)(d-1), \dots, 0\} & \text{if } i = n'-1. \end{cases}$$

Clearly, $g(n'-1)$ and $g(0)$ are connected in H . Furthermore, we prove by induction on k that

$$(*) \quad g(n'-1), g(0), \dots, g(k) \text{ are connected in } H.$$

Since $\gcd(n, q) = 1$, for any i , there exists a unique $i^* \in Z_n$ such that $qi^* + r + 1 \equiv i \pmod{n}$. If $(i^*, i^* + 1) \in S$ for some $i \in g(k)$, then $g(k)$ is connected to either $g(k-1)$ or $g(k-2)$ ($g(-1) = g(n'-1)$). By the induction hypothesis, we can conclude $(*)$. If such an i does not exist, then for every $i \in g(k)$, $i^* = w(d-1)$ for $1 \leq w \leq n'-1$. (Note that here we consider $k > 0$, so $i^* \neq 0$ for $i \in g(k)$.) It follows that $g(k)$ and $g(k+1)$ are connected in H . If $g(k+1)$ is connected to $g(k-1)$, then by the induction hypothesis, $(*)$ holds. If $g(k+1)$ is not connected to $g(k-1)$, then we also have $(k(d-1)+d)^* = w(d-1)$ for some $w = 1, \dots, n'-1$. As w runs from 1 to $n'-1$, $-wd(d-1)+r+1$ will run from $-d(d-1)+r+1$ to $-(n'-1)d(d-1)+r+1 \geq -nd+r+1$ which covers at most d rounds of the n -cycle. Thus, each round has to contribute one to $k(d-1)+j$, $j = 1, \dots, d$. Let v_i denote the node contributed by the i th round. Then $v_i - v_{i+1}$ must be a constant which clearly must be ± 1 . Hence, $v_1 = k(d-1)+1$ or $k(d-1)+d$, i.e., $n-wd(d-1)+r+1 = k(d-1)+1$ or $k(d-1)+d$ for some w , contradicting $(d-1) \nmid (n+r)$. This completes our induction. Therefore, if $(d-1) \nmid (n+r)$ then $G(d, n, q, r)$ is Hamiltonian.

Similarly, we can show that if $(d-1) \nmid (n+r+2)$ then $G(d, n, q, r+d+1)$ is Hamiltonian, by defining

$$(\$) \quad g(i) = \begin{cases} \{i(d-1)+2, \dots, (i+1)(d-1)+1\} & \text{if } 0 \leq i \leq n'-2, \\ \{(n'-1)(d-1)+2, \dots, 0, 1\} & \text{if } i = n'-1, \end{cases}$$

and by noting that $(1, r+2)$ is a link of C . By Lemma 1, $G(d, n, q, r)$ and $G(d, n, q, r+d+1)$ are isomorphic.

Thus, we can assume $(d-1)|(n+r)$ and $(d-1)|(n+r+2)$. This implies $(d-1)|2$. Since $d \geq 3$, we have $d=3$. If $r=0$, then $\gcd(n, d-1)=2$, so by Theorem 3, $G(d, n, q, r)$ is Hamiltonian. If $r>0$, then $\gcd(n, d+1) = \gcd(n, 4) > r > 0$. Thus, n is even and hence r is even. By Theorem 3, $G(d, n, q, r)$ is Hamiltonian.

Subcase 6.2: $r = d-1$. By Lemma 1, $G(d, n, q, d-1)$ is isomorphic to $G(d, n, q, 2d)$. If $(d-1) \nmid n$, then we can prove that $G(d, n, q, d-1)$ is Hamiltonian, by defining (§) and noting that $(1, 0)$ is a link of C . If $(d-1) \mid (n+2)$, then we can prove that $G(d, n, q, 2d)$ is Hamiltonian, by defining (§) and noting that $(2, 1)$ is a link of C . Thus, we can assume $(d-1) \mid n$ and $(d-1) \mid (n+2)$. This implies $r = d-1 = 2$. By Theorem 3, $G(d, n, q, r)$ is Hamiltonian.

Subcase 6.3: $r = d$. First, we consider $d = 4$. If $3 \nmid (n+4)$, then we can show that $G(4, n, n-4, 4)$ is Hamiltonian, by defining

$$g(i) = \begin{cases} \{-3i+9, -3i+8, -3i+7\} & \text{if } 0 \leq i \leq n'-2, \\ \{-3(n'-1)+9, \dots, 10\} & \text{if } i = n'-1, \end{cases}$$

and by noting that $g(0)$ and $g(1)$ are connected through $g(3)$ since $(0, 5)$ and $(-1, 9)$ are two links of C . If $3 \mid (n+4)$ then $3 \nmid (n-1)$ and we can show that $G(4, n, n-4, -1)$ is Hamiltonian, by defining

$$g(i) = \begin{cases} \{-3i+8, -3i+7, -3i+6\} & \text{if } 0 \leq i \leq n'-2, \\ \{-3(n'-1)+8, \dots, 9\} & \text{if } i = n'-1, \end{cases}$$

and by noting that $g(0)$ and $g(1)$ are connected through $g(3)$ since $(-1, 4)$ and $(-2, 8)$ are two links of C . By Lemma 1, $G(4, n, n-4, 4)$ is isomorphic to $G(4, n, n-4, -1)$ and hence Hamiltonian.

Finally, we consider $d = 3$. Since $\gcd(n, 4) > 3$, we have $4 \mid n$. We consider $G(3, n, n-3, -1)$ instead of $G(3, n, n-3, 3)$. Define $g(i) = \{2i, 2i+1\}$. Note that $-3(2i) = 2(-3i)$ and $-3(2i+1) = 2(-3i-2)+1$. Thus, $g(-3i)$ and $g(-3i-2)$ are connected through $g(i)$. Since $\gcd(n/2, 3) = 1$, $-3i$ will run over $0, 1, \dots, n/2-1$ as i runs through $0, 1, \dots, n/2-1$ modulo $n/2$. Therefore, for any i , $g(i)$ and $g(i-2)$ are connected in H .

We first look at the cycle $R = \{g(0), g(2), \dots, g(n/2-2)\}$. Note that $g(4)$ and $g(6)$ are connected through $g(n/2-2)$. Now, we break $g(n/2-2)$ into two groups $\{n-4\}$ and $\{n-3\}$. Then, R will be broken into two connected chains $\{g(0), g(2), g(4)\}$ and $\{g(6), g(8), \dots, g(n/2-4)\}$. However, we have the following facts.

- (1) $g(2)$ and $g(n/2-6)$ are connected through the link $(4, n-12)$ of C .
- (2) $\{n-4\}$ and $g(6)$ are connected through the link $(n-4, 12)$ of C .
- (3) $\{n-3\}$ and $g(4)$ are connected through the link $(n-3, 9)$ of C .

Therefore, $g(0), g(2), \dots, g(n/2-4), \{n-4\}, \{n-3\}$ are connected in the graph H corresponding to the new groups. Similarly, we can break $g(n/2-1)$ into two groups $\{n-2\}$ and $\{n-1\}$ such that $g(1), g(3), \dots, g(n/2-3), \{n-2\}, \{n-1\}$ are still connected in the graph H corresponding to the new groups. Now, we reset

$$\begin{aligned} g(i) &= \{2i, 2i+1\} \quad \text{for } i = 0, 1, \dots, n/2-3, \\ g(n/2-2) &= \{n-4\}, \\ g(n/2-1) &= \{n-3, n-2\}, \\ g(n/2) &= \{n-1\}. \end{aligned}$$

It is easy to see that the corresponding H is connected. Therefore, $G(3, n, n-3, -1)$ is Hamiltonian. ■

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*Presented to the Semester
Combinatorics and Graph Theory
September 14 — December 17, 1987*
