COMBINATORICS AND GRAPH THEORY
BANACH CENTER PUBLICATIONS, VOLUME 25
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1989

ON THE MINIMAL BASIS OF THE SPINDLE-SURFACE $\mathfrak{S}_1$

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If Γ denotes the set of finite undirected graphs $G = (V, E)$ without loops and multiple edges we can define five elementary relations R_i on Γ, $i = 0, 1, 2, 3, 4$, in the following way: Let G and H be two graphs of Γ. Then we define:

(0) $(G, H) \in R_0$ iff the graph H arises from G by deleting an edge e of G or an isolated vertex v of G with $H = G - e$ or $H = G \dot{-} v$.

(1) $(G, H) \in R_1$ iff H arises from G by contracting the edge $e = \{v_1 v_2\}$ of G with $1 \leqslant \gamma(v_i, G\} \leqslant 2$ for some $i = 1, 2$, where $\gamma(v_i, G)$ is the degree of v_i in G.

(2) $(G, H) \in R_2$ iff we obtain H from G by contracting the edge $e = \{v_1 v_2\}$ of G with $\gamma(v_i, G) \geqslant 3$, $i = 1, 2$.

(3) $(G, H) \in R_3$ iff H arises from G by substituting the trihedral $v * \{v_1, v_2, v_3\}$ of G by the triangle (v_1, v_2, v_3, v_1).

(4) $(G, H) \in R_4$ iff H arises from G by substituting the double trihedral $\{u_1, u_2\} * \{u\} * \{v\} * \{v_1, v_2\}$ of G by the double triangle $K_2 * \{w\} * K_2'$ with $K_2 = \{u_1\} * \{u_2\}$ and $K_2' = \{v_1\} * \{v_2\}$.

If $(G, H) \in R_i$, $i = 0, 1, 2, 3, 4$, we can also write $G\dot{R}_i H$ or $H = R_i(G)$. Figure 1 shows $R_3(2*3) = K_4$ and $R_4(K_{3,3}) = K_5$ where K_4 is the complete graph of order 4 and where $K_{3,3}$ and K_5 are the two Kuratowski graphs. By means of these five elementary relations, we are now able to introduce the following five partial orderings $>_i$, $i = 0, 1, 2, 3, 4$; If G, H are two graphs in Γ then we define: $(G, H) \in >_i$ or $G >_i H$ iff either $G = H$ or there is a sequence of graphs $G_1, \ldots, G_n$, $n \geqslant 2$, in Γ such that $G_1 = G$, $G_n = H$ and $(G_\nu, G_{\nu+1}) \in R_{j_\nu}$ for each $\nu = 1, \ldots, n-1$ with $j_\nu \in \{0, 1, \ldots, i\}$.

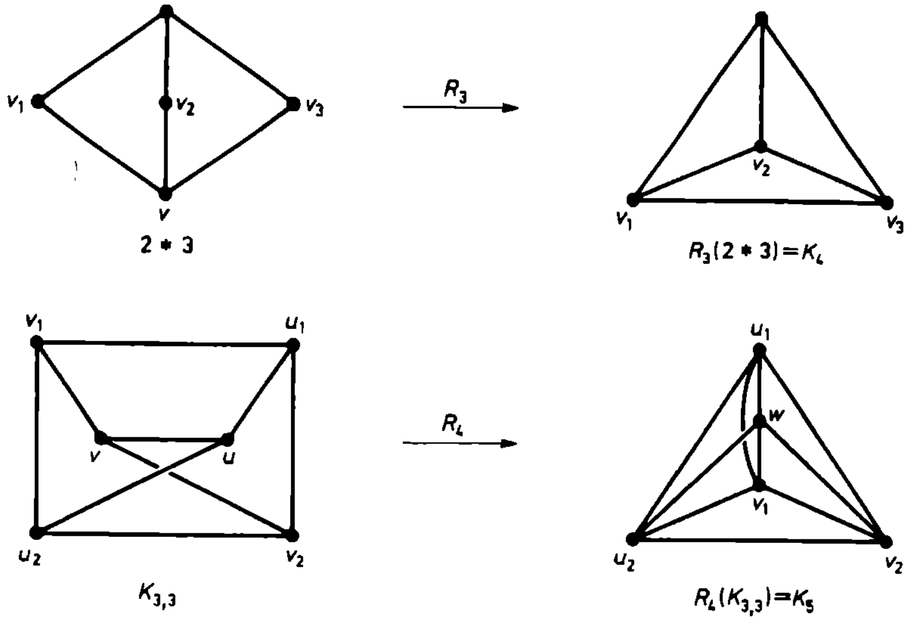

Fig. 1

It is obvious that $>_0$ and $>_1$ are the well-known partial orderings called subgraph relation and subdivision relation respectively where $G >_0 H$ means $G \supseteq H$ and $G >_1 H$ means $G >_s H$. Furthermore, it is clear that $G >_2 H$ means that G is subcontractable to H. $>_3$ and $>_4$ are two new partial orderings which are very useful as we shall demonstrate below.

If $M_i(\Gamma)$, $i = 0, 1, 2, 3, 4$, denotes the minimal basis of Γ with respect to $>_i$ with $M_i(\Gamma) = \{G \in \Gamma \mid G \text{ is } >_i\text{-minimal}\}$, we obviously get the two inclusion chains:

(i) $>_4 \supseteq >_3 \supseteq >_2 \supseteq >_1 \supseteq >_0$ and
(ii) $M_4(\Gamma) \subseteq M_3(\Gamma) \subseteq M_2(\Gamma) \subseteq M_1(\Gamma) \subseteq M_0(\Gamma)$.

Using these notions and denoting the set of all nonplanar graphs in Γ by Γ_0, we obtain two very short and elegant statements each equivalent to Kuratowski's theorem:

(iii) $M_1(\Gamma_0) = M_2(\Gamma_0) = M_3(\Gamma_0) = \{K_5, K_{3,3}\}$ and
(iv) $M_4(\Gamma_0) = \{K_5\}$.

If $\mathfrak{F}$ denotes an orientable surface $\mathfrak{F}_p$ of genus $p \in N_0$, or a nonorientable surface $\tilde{\mathfrak{F}}_q$ of genus $q \in N$, or the spindle-surface $\mathfrak{S}_1$ obtained from a torus or a Klein bottle by contracting a meridian to a single point, and if $\Gamma(\mathfrak{F}) = \{G \in \Gamma \mid G \text{ is not embeddable in } \mathfrak{F}\}$, we can generalize König's question by asking: Is it possible to obtain a Kuratowski type theorem for every surface $\mathfrak{F}$?

In the case of the projective plane $\tilde{\mathfrak{F}}_1$, we know the answer. While C. S. Wang [9] proved that

(v) The minimal basis $M_1(\Gamma(\mathfrak{F}_1))$ consists of exactly 103 $>_1$-minimal graphs,

R. Bodendiek, H. Schumacher and K. Wagner [2] were able to show that

(vi) $M_4(\Gamma(\mathfrak{F}_1))$ consists of exactly 12 $>_4$-minimal graphs.

In the case of the torus $\mathfrak{F}_1$ H. Glover *et al.* conjecture that the minimal basis $M_1(\Gamma(\mathfrak{F}_1))$ consists of more than 1000 graphs. R. Bodendiek and K. Wagner were able to prove that the minimal basis $M_4(\Gamma(\mathfrak{F}_1))$ has at least 23 graphs. They conjecture that $M_4(\Gamma(\mathfrak{F}_1))$ consists of less than 30 graphs.

On the basis of these facts it is obvious that the partial ordering $>_4$ eases the problem of determination of the minimal basis $M(\Gamma(\mathfrak{F}))$ for every $\mathfrak{F}$. Therefore we shall use $>_4$ for constructing a theorem in the style of Kuratowski's theorem.

For the spindle surface $\mathfrak{S}_1$, the problem of determination of the minimal basis $M_4(\Gamma(\mathfrak{S}_1))$ is much easier than the determination of $M_4(\Gamma(\mathfrak{F}_1))$. Therefore we restrict ourselves to the investigation of $M_4(\Gamma(\mathfrak{S}_1))$. Before we can start to investigate if a graph $G = (V, E)$ is embeddable in $\mathfrak{S}_1$, we have to explain the meaning of the singular point s of $\mathfrak{S}_1$. Figure 2 shows a planar model of $\mathfrak{S}_1$. In order to get useful embeddings of graphs in $\mathfrak{S}_1$, we have to postulate that the singular point s of $\mathfrak{S}_1$ is at most an element of the vertex set V of $G = (V, E)$ or, in other words, that s cannot be an inner point of any edge in embeddings of G.

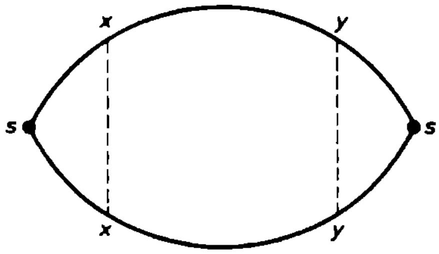

Fig. 2

Furthermore, the authors proved in [1] the following theorem:

(a) A graph $G = (V, E)$ belongs to $M_4(\Gamma(\mathfrak{S}_1))$ iff G is not embeddable in $\mathfrak{S}_1$ and $R_i(G)$ is embeddable in $\mathfrak{S}_1$ for each $i = 0, 1, 2, 3, 4$.

(b) A graph $G = (V, E)$ is embeddable in $\mathfrak{S}_1$ iff there is at least one vertex $v \in V$ such that the following condition holds: The graph $G \dot{-} v$, arising from G by deleting v and all edges incident to v, is so embeddable in the plane (or in the sphere) that exactly two countries of $G \dot{-} v$ contain all the vertices of G which are adjacent to v on their boundaries.

Now, let $G = (V, E)$ be a graph in $M_4(\Gamma(\mathfrak{S}_1))$. Then we know that G is not embeddable in $\mathfrak{S}_1$. According to the definition of $\mathfrak{S}_1$, G is not embeddable in the plane. Hence, G contains a subdivision graph $S(K_5)$ or a subdivision graph $S(K_{3,3})$. So it is obvious that the investigations are simplified if we know all possibilities of embeddings of $S(K_5)$ and $S(K_{3,3})$. Figure 3 shows the two topologically different embeddings of $S(K_{3,3})$ in $\mathfrak{S}_1$ and the three topologically different embeddings of $S(K_5)$ in $\mathfrak{S}_1$. By means of Fig. 3 and of the above theorems (a) and (b) we are able to prove the following theorem (cf. [1] and [5]):

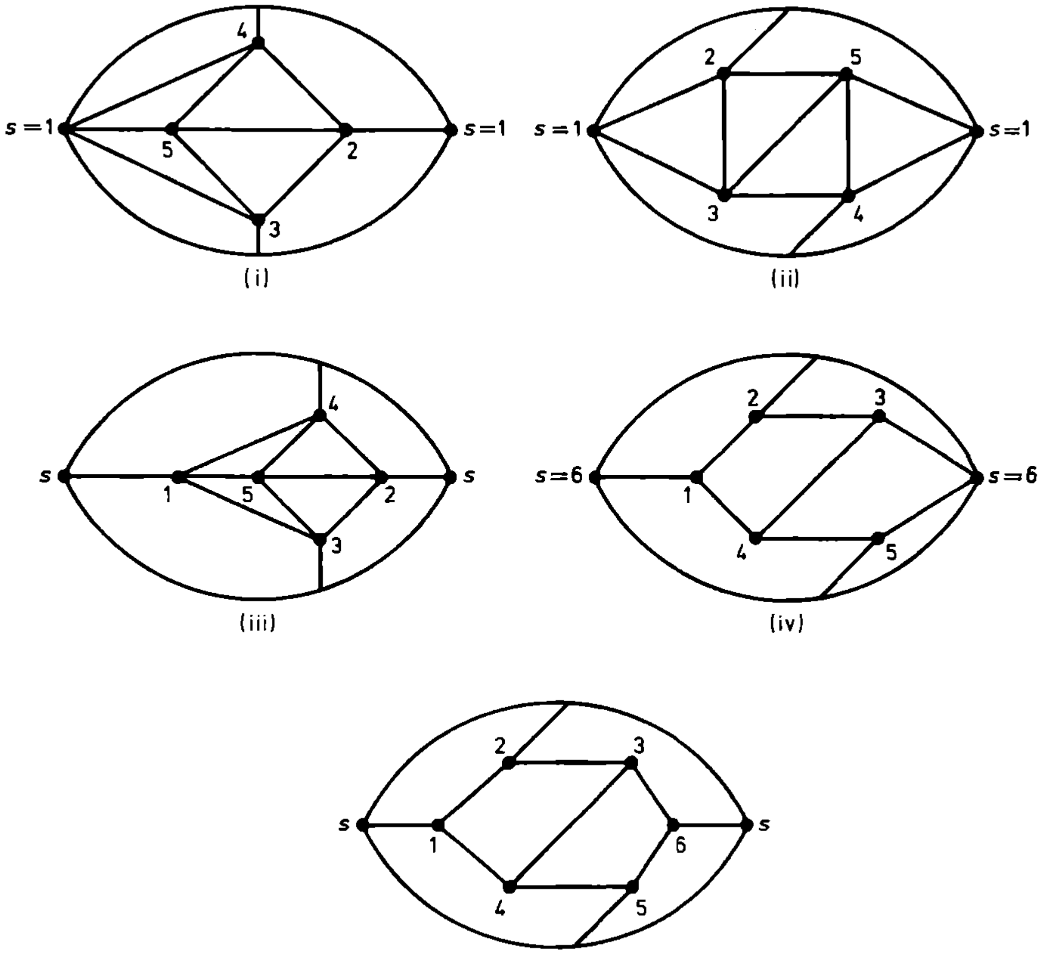

Fig. 3

Theorem 1. *The twelve graphs* $G_1 = K_5 \dot{\cup} K_5$ *(the union of two disjoint* K_5*),* $G_2 = K_4 * 1 * K_4$ *(two disjoint* K_5 *are stitched in one vertex),* $G_3 = K_3 * K_2 * K_3$ *(two disjoint* K_5 *are stitched along an edge),* $G_4 = 1 * K_{3,3}$, $G_5 = 2 * 3 * 3$, $G_6 = K_6$, *and* $G_7, G_8, \ldots, G_{12}$, *shown in Fig. 4, belong to* $M_4(\Gamma(\mathfrak{S}_1))$.

Remarks. More precisely, we can say according to [5] that there are exactly three graphs G in $M_4(\Gamma(\mathfrak{S}_1))$ with $G \supseteq S(K_5)$ and $G \not\supseteq S(K_{3,3})$.

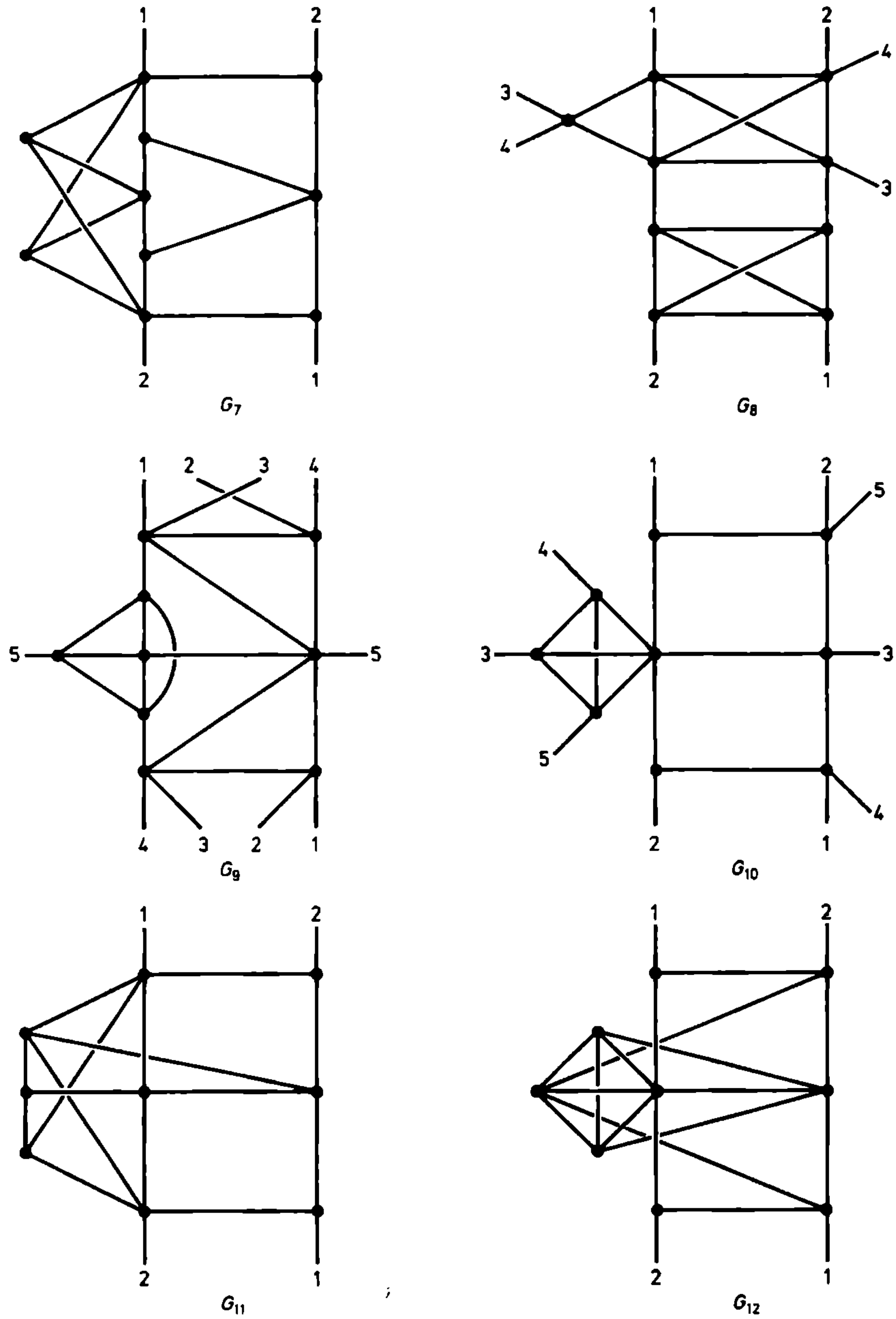

Fig. 4

Furthermore, it has been shown in [5] that every graph $G \in M_4(\Gamma(\mathfrak{S}_1))$ with $G \neq G_1, G_2, G_3$ is 3-connected and contains a subdivision graph $S(K_{3,3})$. Therefore, it remains to determine the set of $>_4$-minimal graphs G in $\Gamma(\mathfrak{S}_1)$ with $G \supseteq S(K_{3,3})$.

In order to describe the method of determining this set, we introduce some new notions and notations. Let $G=(V, E) \neq G_1, G_2, G_3$ be a 3-connected graph in $M_4(\Gamma(\mathfrak{S}_1))$ with $G \supseteq S(K_{3,3})$. As $S(K_{3,3})$ is embeddable in $\mathfrak{S}_1$, the graph G contains edges or vertices that do not belong to $S(K_{3,3})$. If such an edge e joins two vertices u and v of $S(K_{3,3})$, then $e=\{u, v\}$ is said to be a *chord* of $S(K_{3,3})$. For example, the $>_4$-minimal graph $G=G_6=K_6$ of Theorem 1 arises from $S(K_{3,3})=K_{3,3}$ by adding six chords. Let G^* be the induced subgraph with vertex set $V^*=\{v \in V | v$ does not belong to $S(K_{3,3})\}$ and with $G^*=X_1 \cup \ldots \cup X_n$, $n \in N_0$, where $X_1, \ldots, X_n$ are the components of G^*. As G is 3-connected, each component X_i, $i=1, \ldots, n$, is joined to vertices of $S(K_{3,3})$ by edges which are said to be the *bridges* of X_i. These bridges with their endpoints form a bipartite graph, denoted by Y_i. The union graph $Q_i=X_i \cup Y_i$, $i=1, \ldots, n$, is said to be a *relative component* of G with respect to $S(K_{3,3})$.

The graph $X_i=X(Q_i)$ is called the *center* of Q_i. The order of X_i is called the order of Q_i. A relative component Q_i of order 1 is called a *star*, analogously Q_i of order 2 is called a *double star*. The vertices of Y_i belonging to $S(K_{3,3})$ are called the *basis points* of Q_i. As a consequence, we know that G consists of $S(K_{3,3})$, m $(\in N_0)$ chords $e_1, \ldots, e_m$ of $S(K_{3,3})$ and n $(\in N_0)$ relative components $Q_1, \ldots, Q_n$ of G with respect to $S(K_{3,3})$, and that G is the following union graph:

$$(*) \qquad G=S(K_{3,3}) \cup Q_1 \cup \ldots \cup Q_n \cup \{e_1, \ldots, e_m\}, \qquad m+n \in N.$$

For example, $G=G_4=1*K_{3,3}$ is the union graph $G_4=K_{3,3} \cup Q$, where Q, being a star, is the only relative component of G with respect to $K_{3,3}$ with six basis points. Analogously, G_5 is the union graph $G=K_{3,3} \cup Q_1 \cup Q_2$, where Q_1, Q_2 are two stars with the same three basis points.

The equation $(*)$ expresses the fact that we know all $>_4$-minimal graphs of $M_4(\Gamma(\mathfrak{S}_1)) \backslash \{G_1, G_2, G_3\}$ iff we are able to answer the following three questions:

(1) Which chords $e_1, \ldots, e_m$ of $S(K_{3,3})$ might be added to $S(K_{3,3})$?

(2) What can we say about the center $X_i=X(Q_i)$ of the relative component $Q_i=X_i \cup Y_i$, $i=1, \ldots, n$, of G with respect to $S(K_{3,3})$?

(3) What can we prove about the basis points of $Q_i=X_i \cup Y_i$, $i=1, \ldots, n$?

In order to answer these questions it is useful to define two more notions.

The *trace* $T(Q)$ of a relative component Q of G with respect to $S(K_{3,3})$ is the union of the set of all edges of $K_{3,3}$ on which at least one basis point lies which is an inserted vertex of Q, and of the set of all vertices of $K_{3,3}$ which are basis points of Q and are incident to no edge of $T(Q)$.

Furthermore, a relative component Q of G with respect to $S(K_{3,3})$ is said to be of *type q* iff all the basis points of Q lie on a (subdivided) quadrilateral q of $S(K_{3,3})$. If the basis points of Q do not lie on a (subdivided) quadrilateral q of $S(K_{3,3})$ we call Q of *type* $\bar{q}$.

In this paper, we shall investigate some properties of relative components $Q = X \cup Y$ of $G \in M_4(\Gamma(\mathfrak{S}_1))\setminus\{G_1, G_2, G_3\}$ of order $|X(Q)| \geqslant 2$ and prove some statements on the structure of Q. We start by repeating a theorem proved in an earlier paper:

THEOREM 2. *If a graph* $G \in M_4(\Gamma(\mathfrak{S}_1))\setminus\{G_1, G_2, G_3\}$ *contains exactly one relative component* Q *with respect to* $S(K_{3,3})$ *with* $G = S(K_{3,3}) \cup Q$, *then* G *is of type* $\bar{q}$. *If* Q *is a star, then necessarily* $G = G_4 = 1 * K_{3,3}$.

Proof. The second part of the statement is clear. The proof of the first part is a little more difficult. We suppose that all basis points of Q lie on a quadrilateral q of $S(K_{3,3})$ with $q = (1, 2, 3, 4, 1)$. As Q is the only relative component of G with respect to $S(K_{3,3})$, all main paths incident to the main vertices 5 or 6 of $S(K_{3,3})$ have length 1. If we apply the elementary relation R_4 to the vertex 5 or 6, then we know that the graph $R_4(G)$ is embeddable in $\mathfrak{S}_1$ and that the graph $q \cup Q$ is planar. Hence G is embeddable in $\mathfrak{S}_1$. This contradiction proves the first part of Theorem 2.

A similar theorem is

THEOREM 3. *If a* $>_4$*-minimal graph* $G \in M_4(\Gamma(\mathfrak{S}_1))\setminus\{G_1, G_2, G_3\}$ *has a relative component* $Q = X \cup Y$ *of order* $|X(Q)| \geqslant 2$ *with respect to* $S(K_{3,3})$, *whose trace* $T(Q)$ *contains the four edges of a quadrilateral* q *of* $K_{3,3}$, *then* Q *is of type* $\bar{q}$.

Proof. Suppose that $G \in M_4(\Gamma(\mathfrak{S}_1))\setminus\{G_1, G_1, G_3\}$ is a graph with a relative component Q of order $\geqslant 2$, whose trace $T(Q)$ contains the four edges of a quadrilateral q of $K_{3,3}$ and which is of type q. Then we consider the two graphs G' and G^*, where $G' = S(K_{3,3}) \cup Q$ and G^* arises from G by substituting Q by the star Q^* with the same basis points, or, in other words, by contracting the center of $X(Q)$ to a single point. While G^* is embeddable in $\mathfrak{S}_1$ because of the minimality of G, the embeddability of G' in $\mathfrak{S}_1$ follows directly from Theorem 2. Because of the assumption that the trace $T(Q)$ contains the quadrilateral q there are two possibilities (α') and (α'') of embedding the relative component Q of $G' = S(K_{3,3}) \cup Q$ in $\mathfrak{S}_1$, illustrated in Fig. 5. In the embedding of G^*, we may not substitute Q^*, lying in q, by Q for G is not embeddable in $\mathfrak{S}_1$. But therefore we know that Q of $G' = S(K_{3,3}) \cup Q$ is in (α') and in (α'') normally embeddable for a cleavable embedding of Q might be possible at most in (α'') of Fig. 5. But in this case, we can identify the two vertices 4 of Fig. 5 so that Q is even embeddable in (α''). Therefore we are able to substitute Q^* by Q in each embedding of G^* in $\mathfrak{S}_1$. Hence G would be embeddable in $\mathfrak{S}_1$. This is a contradiction to the hypothesis.

Before we prove further propositions concerning the structure of relative components Q of $G \in M_4(\Gamma(\mathfrak{S}_1))\setminus\{G_1, G_2, G_3\}$ we shall try to answer the question how many relative components are possible in the representation of G. The first statement is

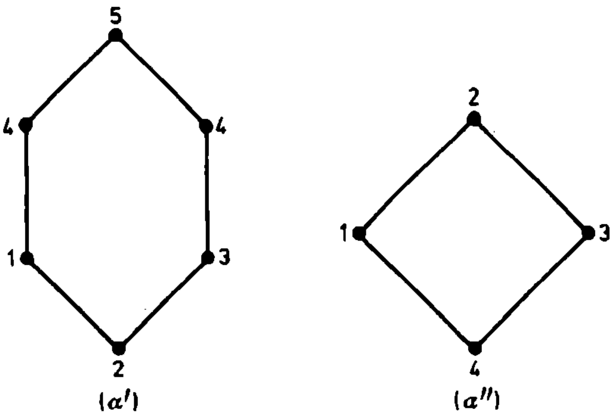

Fig. 5

THEOREM 4. *For every graph* $G \in M_4(\Gamma(\mathfrak{S}_1)) \setminus \{G_1, G_2, G_3\}$ *the following holds: For every* $S(K_{3,3})$ *with* $G \supseteq S(K_{3,3})$, G *contains at most two relative components* Q_1, Q_2 *of order* $\geqslant 1$ *and of type* $\bar{q}$.

Proof. Suppose that G is a graph in $M_4(\Gamma(\mathfrak{S}_1)) \setminus \{G_1, G_2, G_3\}$ which contains three relative components Q_1, Q_2, Q_3 of order $\geqslant 1$ and of type $\bar{q}$ with respect to $S(K_{3,3})$. As G is 3-connected G contains at least three basis points b_1, b_2, b_3 so that for at least one of the two triples $\{2, 4, 6\}, \{1, 3, 5\}$ of main vertices of $K_{3,3}$—without loss of generality, we can choose the triple $\{2, 4, 6\}$—the following holds: For each $i = 1, 2, 3$, b_i is either equal to the vertex $2i$ or equal to an inserted vertex of one of the three main paths of $S(K_{3,3})$ (i.e. the subdivided edges of $K_{3,3}$). By applying R_2 to G several times if necessary we get

$$G >_4 S(K_{3,3}) \cup Q_1 \cup Q_2 \cup Q_3 >_4 K_{3,3} \cup Q_1^* \cup Q_2^* \cup Q_3^*,$$

where Q_1^*, Q_2^*, Q_3^* are stars whose basis points are either 2, 4, 6 or 1, 3, 5. Hence at least two of these three stars have the same basis points. We can assume without loss of generality that Q_1^*, Q_2^* have the basis points 2, 4, 6 so that

$$G >_4 K_{3,3} \cup Q_1^* \cup Q_2^* >_4 2 * 3 * 3 = G_5.$$

That is a contradiction to the minimality of G, which completes the proof.

Now, we turn to the question of whether there are also $>_4$-minimal graphs $G \in M_4(\Gamma(\mathfrak{S}_1)) \setminus \{G_1, G_2, G_3\}$ containing relative components Q of type q with respect to $S(K_{3,3})$. The first statement is given in:

THEOREM 5. *If* Q *is a relative component of a graph* $G \in M_4(\Gamma(\mathfrak{S}_1)) \setminus \{G_1, G_2, G_3\}$ *of type* q *and of order* $\geqslant 2$ *with respect to* $S(K_{3,3}) \subseteq G$, *then the trace* $T(Q)$ *is necessarily equal to one of the five cases* (i)–(v) *illustrated in Fig.* 6.

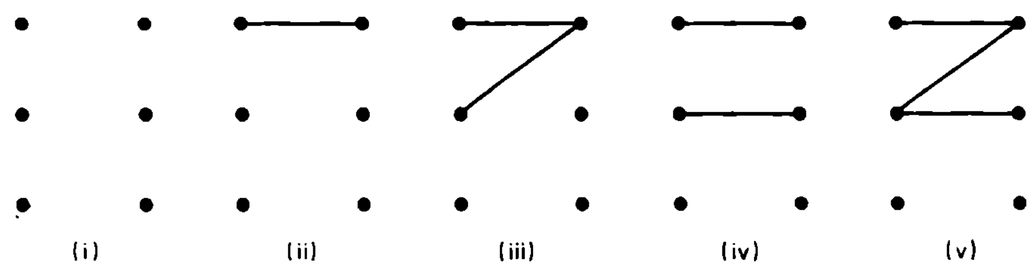

Fig. 6

Proof by verifying the cases (i)–(v) to be the only possible ones for the trace $T(Q)$.

The next theorem is a refinement of Theorem 5:

THEOREM 6. *If Q is a relative component of a graph $G \in M_4(\Gamma(\mathfrak{S}_1)) \backslash \{G_1, G_2, G_3\}$ of type q and of order $\geqslant 2$ with respect to $S(K_{3,3}) \subseteq G$, then Q has one of the following five properties* (1)–(5):

(1) *Q contains three or four basis points which are main vertices of $S(K_{3,3})$ (i.e. vertices of $K_{3,3}$) and which lie on a quadrilateral of $S(K_{3,3})$.*

(2) *Exactly one main path P (i.e. a (subdivided) edge of $K_{3,3}$) of the quadrilateral q—we can choose the main path $P = 1\ldots2$ without loss of generality—contains an inserted vertex of P. All other basis points of Q lie on P or are main vertices of q.*

(3) *Each of exactly two adjacent main paths P_1 and P_2—we can choose $P_1 = 1\ldots2$ and $P_2 = 2\ldots3$ without loss of generality—contains at least one basis point of Q as an inserted vertex. Each other basis point of Q lies on $P_1 \cup P_2$ or is equal to the fourth vertex 4 of q.*

(4) *Each of exactly two disjoint main paths P_1, P_2 of q—we can choose $P_1 = 1\ldots2$ and $P_2 = 3\ldots4$ without loss of generality—contains at least one inserted vertex as basis point of Q. All other basis points of Q lie on $P_1 \cup P_2$.*

(5) *Each of exactly three main paths P_1, P_2, P_3—we can choose $P_1 = 1\ldots2$, $P_2 = 2\ldots3$, $P_3 = 3\ldots4$ without loss of generality—contains at least one inserted vertex as basis point of Q. All other basis points of Q lie on $P_1 \cup P_2 \cup P_3$.*

The proof follows from Theorem 5.

In the following, we investigate the centers of the relative components Q of $G \in M_4(\Gamma(\mathfrak{S}_1)) \backslash \{G_1, G_2, G_3\}$ with respect to $S(K_{3,3}) \subseteq G$. Due to [4] we know that the center $X(Q)$ of every relative component Q of G is a Husimi tree. Now, we want to try to improve this result.

In order to do this, let Q be a relative component of order $\geqslant 2$ with respect to $S(K_{3,3}) \subseteq G$. Furthermore, let G^* be the graph arising from G by contracting $X(Q)$ to a single point or, in other words, by substituting Q by the star Q^*

having the same basis points as Q. Because of the minimality of G, the graph G^* is embeddable in $\mathfrak{S}_1$. For every embedding of G^* in $\mathfrak{S}_1$ the star Q^* lies either inside a (subdivided) quadrilateral or inside a (subdivided) hexagon of $S(K_{3,3})$. If we denote this n-gon, $n = 4$ or 6, of $S(K_{3,3})$ by Z, we can say that all basis points of Q^* lie on Z. If we substitute Q^* by Q, then the graph $Z \cup Q$ is not planar because otherwise G would be embeddable in $\mathfrak{S}_1$. By means of Theorem 1 of [8] we are now able to characterize the graph $Z \cup Q$. Before we carry out this in an elegant way we make the following definition. Let Q, Q' be two relative components of G with respect to $S(K_{3,3})$. Then Q *contains a subdivision* $S(Q')$ iff the basis point set of Q' is a subset of the basis point set of Q with $S(Q') \subseteq Q$.

Now it is possible to state a theorem in the style of Theorem 1 of [8]:

THEOREM 7. *If Q is a relative component of $G \in M_4(\Gamma(\mathfrak{S}_1)) \backslash \{G_1, G_2, G_3\}$ of order $\geqslant 2$ with respect to $S(K_{3,3}) \subseteq G$, then Q necessarily contains a subdivision of one of the four relative components Q'_1, Q'_2, Q'_3, Q'_4 illustrated in Fig.* 7 (*where Z has the meaning introduced above*).

Proof. As Theorem 7 is a generalization of Theorem 1 of [8] it will do for

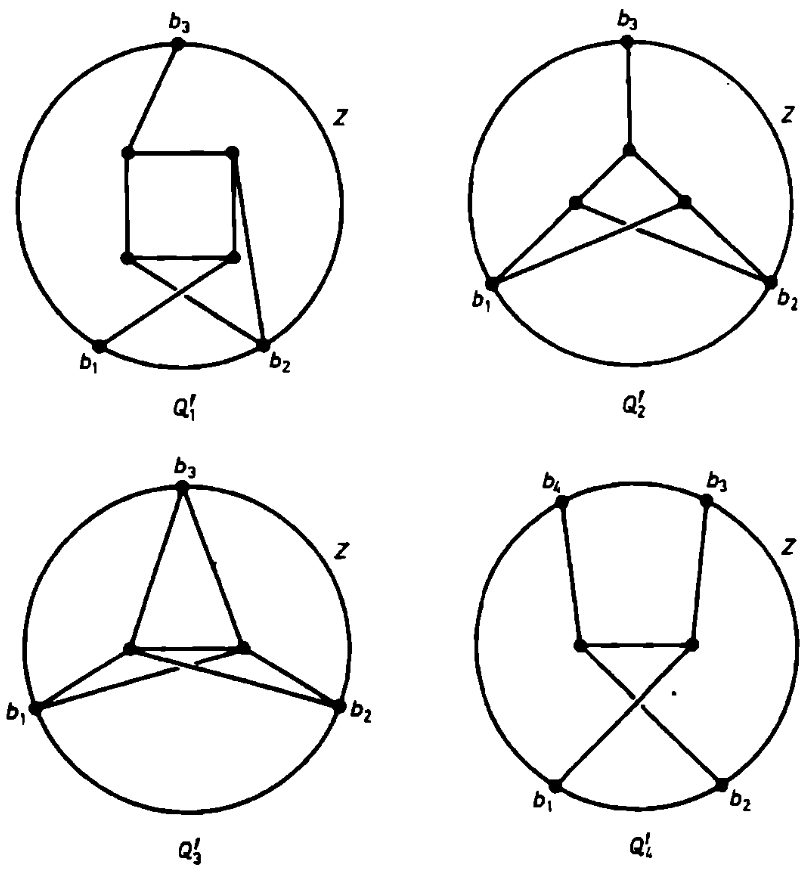

Fig. 7

our purpose to say that the proof of Theorem 7 follows from the proof of Theorem 1 of [8].

Now, we are leaving the centers for a moment and prove another relevant theorem concerning the vertices v of $G \in M_4(\Gamma(\mathfrak{S}_1)) \backslash \{G_1, G_2, G_3\}$ of degree $\gamma(v, G) = 3$.

THEOREM 8. *Let v be a vertex of a $>_4$-minimal graph $G \in M_4(\Gamma(\mathfrak{S}_1)) \backslash \{G_1, G_2, G_3\}$ of degree $\gamma(v, G) = 3$, and let v_1, v_2, v_3 be the three vertices of G adjacent to v. Then the set $\{v_1, v_2, v_3\}$ is independent, i.e. no two vertices in $\{v_1, v_2, v_3\}$ are adjacent.*

Proof. Assume that v_1 and v_2 are adjacent in G and that the edge $e = \{v_1, v_2\}$ joins v_1 and v_2 in G. Because of the minimality of G, the graph $G-e$ is embeddable in $\mathfrak{S}_1$. If there is an embedding of $G-e$ in $\mathfrak{S}_1$ such that v is not equal to the singular point s of $\mathfrak{S}_1$, then it follows from $\gamma(v, G-e) = 3$ that we can embed the edge e in $\mathfrak{S}_1$ without crossing any edge of $G-e$. Hence G is embeddable in $\mathfrak{S}_1$. Therefore it follows from this contradiction that there exists an embedding of $G-e$ in $\mathfrak{S}_1$ such that v is equal to the singular point s of $\mathfrak{S}_1$. Because of $\gamma(v, G-e) = 3$ we can reduce this case to the above one by finding a new embedding in which v_1 or v_2 is equal to the singular point s of $\mathfrak{S}_1$. Similarly to the first case, we can add the edge e without crossing other edges of $G-e$. So we get a contradiction again. That proves Theorem 8.

Now we are interested in the main vertices of degree 3 of $S(K_{3,3}) \subseteq G \in M_4(\Gamma(\mathfrak{S}_1)) \backslash \{G_1, G_2, G_3\}$.

THEOREM 9. *If $\{1, 3, 5\}$ and $\{2, 4, 6\}$ are the two triples of main vertices of $G \in M_4(\Gamma(\mathfrak{S}_1)) \backslash \{G_1, G_2, G_3\}$ and if $\gamma(1, G) = 3$, then the vertices 2, 4, 6 are adjacent to 1.*

Proof. The proof is quite involved. Therefore we have to omit it.

THEOREM 10. *The set of all graphs $G \in M_4(\Gamma(\mathfrak{S}_1)) \backslash \{G_1, G_2, G_3\}$ with $G \supseteq S(K_{3,3})$ and $G \nsupseteq S(K_{3,4})$ and with the property that G contains only one relative component Q with respect to at least one $S(K_{3,3}) \subseteq G$ is empty.*

Proof. Assume that there is a graph $G \in M_4(\Gamma(\mathfrak{S}_1)) \backslash \{G_1, G_2, G_3\}$ with $G = S(K_{3,3}) \cup Q$ and $G \nsupseteq S(K_{3,4})$. According to Theorem 2 we know that Q is of type $\bar{q}$. If $\{1, 3, 5\}$ and $\{2, 4, 6\}$ are the two triples of main vertices of $S(K_{3,3})$ then at most two vertices in $\{1, 3, 5\}$ are basis points of Q. Analogously, at most two vertices in $\{2, 4, 6\}$ are basis points of Q. Therefore there are at least two main vertices of $S(K_{3,3})$—we can choose the vertices 1 and 2 without loss of generality—of degree $\gamma(1, G) = \gamma(2, G) = 3$. As Q is of type $\bar{q}$, at least one of the five main paths which are incident to 1 or to 2 contains an inserted vertex. We can choose the main path $1 \ldots 2i$, $i = 1, 2, 3$, without loss of generality. If we apply R_3 to 1, then we know because of the

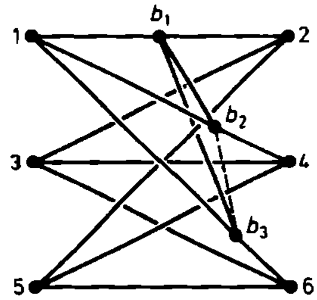

Fig. 8

minimality of G that the graph $R_3(G)$ is embeddable in $\mathfrak{S}_1$. If we denote the three vertices adjacent to 1 and lying on the main paths $1\ldots2$, $1\ldots4$, $1\ldots6$ by b_1, b_2, b_3 then it follows from $b_1 \neq 2$ that $R_3(G)$ consists of the new $S'(K_{3,3})$ (Fig. 8) (instead of $S(K_{3,3})$) with the two triples of main vertices $\{b_1, 3, 5\}$ and $\{2, 4, 6\}$, the three main paths $b_1\ldots2$, $b_1\ldots b_2\ldots4$ and $b_1\ldots b_3\ldots6$, the same Q and the additional edge $\{b_2, b_3\}$ with $R_3(G) = S'(K_{3,3}) \cup Q \cup \{b_2, b_3\}$. Then it follows from this representation of $R_3(G)$ that the graph $G = S(K_{3,3}) \cup Q$ is embeddable in $\mathfrak{S}_1$. This contradiction proves Theorem 10.

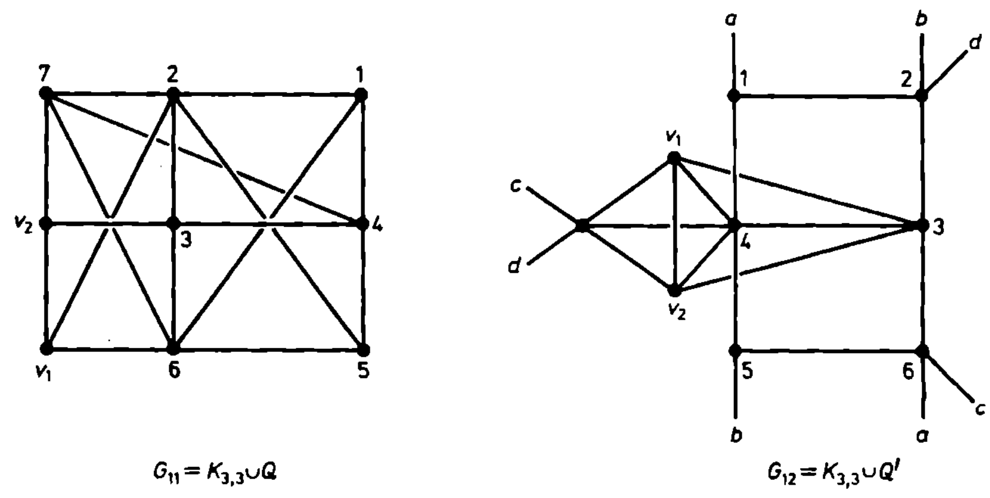

Fig. 9

Remark. It follows from Theorem 10 that every graph $G \in M_4(\Gamma(\mathfrak{S}_1)) \setminus \{G_1, G_2, G_3\}$ that contains only one relative component Q with respect to one $S(K_{3,3}) \subseteq G$ contains an $S(K_{3,4})$ as a subgraph. Figure 9 illustrates this remark. It exhibits the two $>_4$-minimal graphs $G_{11} = K_{3,3} \cup Q$ and $G_{12} = K_{3,3} \cup Q'$.

We finish this paper by formulating the following conjecture: $M_4(\Gamma(\mathfrak{S}_1))$ consists of exactly 12 graphs.

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Presented to the Semester
Combinatorics and Graph Theory
September 14 – December 17, 1987