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THE CONNECTEDNESS OF DEGENERACY LOCI

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A corollary of the Lefschetz hyperplane theorem is that every hypersurface in the complex projective space $\mathbf{P}^n$ is connected, provided $n \geqslant 2$. Sections 1 through 5 are an exposition of some recent work that generalizes this connectedness result. In Section 6 the connectedness theorems for degeneracy loci are extended to n-ample vector bundles. In the Appendix (see the next paper), we show that a symmetric degeneracy locus of odd rank r associated to a symmetric bundle map $u\colon E \otimes E \to L$ is connected if its expected dimension is at least $\operatorname{rk} E - r$.

§ 1. Introduction

Let us begin by quoting from a book on topology: "The most intuitively evident topological invariant of a space is the number of connected pieces into which it falls." Thus, if one is interested in the topology of an algebraic variety, a natural first question is whether the variety is connected.

The prototype of such a result is the remarkable theorem of Lefschetz that *the solution set of a single homogeneous polynomial equation in the complex projective space* $\mathbf{P}^n$ *is connected, provided* $n \geqslant 2$. In this theorem n of course has to be at least 2, because a single polynomial on $\mathbf{P}^1$ defines a finite number of points, which is in general not a connected set. The purpose of this talk is to discuss generalizations of Lefschetz's theorem to other systems of equations on varieties possibly other than the projective space. While Sections 1 through 5 are meant to be expository, Section 6 and the Appendix contain previously unpublished results. In response to a question of Michael Schneider, I show in Section 6 that the connectedness theorems for degeneracy loci can be extended to n-ample vector bundles. In the Appendix, Joe Harris and I present a few

This paper is in final form and no version of it will be submitted for publication elsewhere.

ideas, which while not solving the odd-rank symmetric case completely, does prove the connectedness of an odd-rank symmetric degeneracy locus under a strengthened hypothesis.

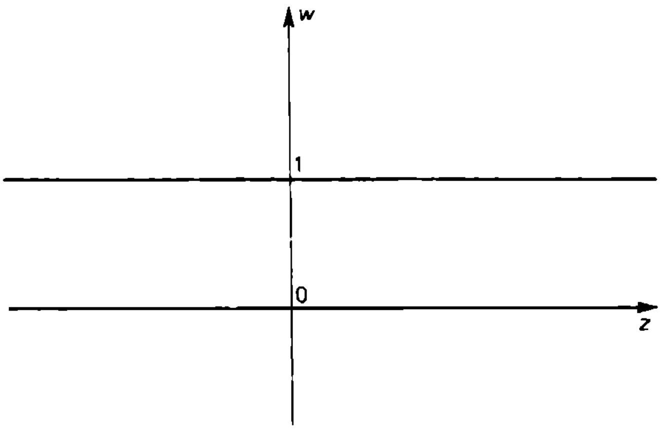
As there are many commutative algebraists in the audience, who may not be so familiar with the geometric way of thinking, I have tried to delimit the scope of my exposition by selective omission. So the theorems quoted here may not be in the generality in which they were originally proved.

To fix the terminology, a *variety* will mean a reduced quasiprojective variety over the complex numbers. The topology is the usual complex topology, unless Zariski is explicitly mentioned.

§ 2. Three naive attempts to generalize Lefschetz's theorem

We will look at three simple examples to see what hypotheses would be reasonable in any generalization of Lefschetz's connectedness theorem.

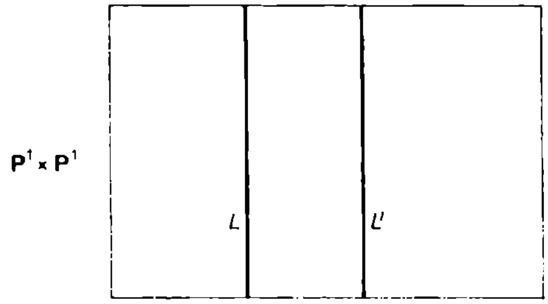
EXAMPLE 2.1. Can one replace $\mathbf{P}^n$ by $\mathbf{C}^n$? The answer is no, since it is possible to have parallel lines in $\mathbf{C}^n$.



Zero$(w(w-1))$ in $\mathbf{C}^2$ is disconnected

EXAMPLE 2.2. To replace $\mathbf{P}^n$ by some other projective variety, recall that a *hypersurface* in a complex manifold is an analytic subvariety of codimension 1. In $\mathbf{P}^n$ every hypersurface is the solution set of a single nonzero homogeneous polynomial and vice versa. So Lefschetz's theorem says precisely that every hypersurface in $\mathbf{P}^n$ is connected if $n \geqslant 2$. Is it true that every hypersurface in any smooth projective variety is connected? The answer is again no, for on any ruled surface, say $\mathbf{P}^1 \times \mathbf{P}^1$, there are parallel lines, say L, L'. Then $L \cup L'$ is a disconnected hypersurface on $\mathbf{P}^1 \times \mathbf{P}^1$.

EXAMPLE 2.3. Can one replace the single equation by more than one equation? Obviously, the number of equations cannot be arbitrary, because



not every projective variety is connected. In $\mathbf{P}^3$,

$$L_1 = \mathrm{Zero}(Z_0, Z_1) \quad \text{and} \quad L_2 = \mathrm{Zero}(Z_2, Z_3)$$

are disjoint lines. Hence, their union $L_1 \cup L_2$, which is defined by the four equations $Z_0 Z_2, Z_0 Z_3, Z_1 Z_2, Z_1 Z_3$, is not connected.

What is an "equation" on an arbitrary variety X? Let $\mathcal{O}(1)$ be the hyperplane bundle on $\mathbf{P}^n$ and let $\mathcal{O}(d)$ be the dth tensor power $\mathcal{O}(1)^{\otimes d}$. A homogeneous polynomial of degree d on $\mathbf{P}^n$ is a section of the line bundle $\mathcal{O}(d)$ on $\mathbf{P}^n$, where d is of course a positive integer. Thus, we can think of an "equation" on a variety X as a section s of a line bundle L over X. The question now is whether the zero set $Z(s)$ is connected.

From the three examples above we see that for $Z(s)$ to be connected, the following hypotheses should be assumed:

(1) The variety X is irreducible and projective (not simply quasiprojective).

(2) The line bundle L is "positive" in a suitable sense. As will be seen from the Lefschetz hyperplane theorem, the trouble with Example 2.2 is that the line bundle corresponding to the union of the two disjoint rulings is not positive.

(3) There ought be a dimension hypothesis relating the dimension of X and the number of equations.

We will find that in every connectedness theorem all three hypotheses play an essential role.

§ 3. Positivity

A line bundle L on a variety X is said to be *very ample* if it is isomorphic to the hyperplane bundle of some embedding $X \subsetneq \mathbf{P}^n$, and L is *ample* if for some positive integer m the tensor power $L^{\otimes m}$ is very ample.

THEOREM (The Lefschetz hyperplane theorem). *Let L be an ample line bundle over an irreducible smooth projective variety X and s a section of L over*

X. Then the natural map

$$H_q(Z(s); \mathbf{Z}) \to H_q(X; \mathbf{Z})$$

is an isomorphism for $q < \dim_{\mathbf{C}} X - 1$ *and a surjection for* $q = \dim_{\mathbf{C}} X - 1$. *In particular, if* $\dim_{\mathbf{C}} X - 1 \geqslant 1$, *then* $Z(s)$ *is connected.*

A nice proof of this theorem, using Morse theory, may be found in [6].

To explain the notion of ampleness for vector bundles, we first recall a classic duality between points in a vector space and linear forms on the dual projective space. Let V be a complex vector space with coordinates $z_1, \ldots, z_n$ and let V^* be the dual vector space of linear forms $\sum a_i z_i$ on V with coordinates $a_1, \ldots, a_n$. Given a point $z = (z_1, \ldots, z_n)$ in V, $z^* = \sum a_i z_i$ is a linear form on V^*, hence also a linear form on the projective space $\mathbf{P}(V^*)$. Thus, there is a one-to-one correspondence

$$\{\text{points } z \in V\} \leftrightarrow \{\text{linear forms } z^* \text{ on } \mathbf{P}(V^*)\} = \Gamma(\mathbf{P}(V^*), \mathcal{O}(1)).$$

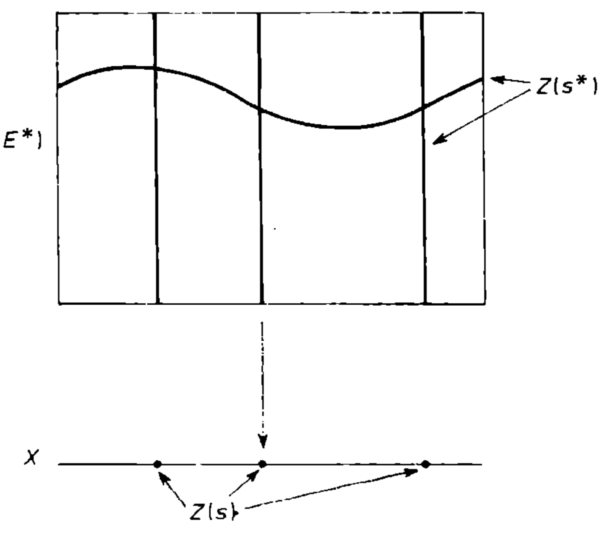
Globalizing this correspondence fiber by fiber to a vector bundle E over X, we see that a section s of E gives rise to a section s^* of $\mathcal{O}(1)$ over $\mathbf{P}(E^*)$.

DEFINITION. A vector bundle $E \to X$ is *ample* if and only if the line bundle $\mathcal{O}_{\mathbf{P}(E^*)}(1)$ over $\mathbf{P}(E^*)$ is ample.

Remark. If E is a line bundle, then $\mathbf{P}(E^*) = X$. Furthermore, $\mathcal{O}_{\mathbf{P}(E^*)}(1) \simeq E$ because for every x in X a linear form on the fiber E_x^* is a point in E_x. Thus, the line bundle E is ample as a vector bundle if and only it is ample as a line bundle.

The following proposition is a useful observation of Sommese that sometimes allows one to reduce a vector bundle problem to a line bundle problem ([8, Proof of Prop. 1.16, p. 240]).

PROPOSITION 3.1. *If E is a rank e vector bundle over X and s is a section of E, then* $\mathbf{P}(E^*) - Z(s^*)$ *is an affine-space bundle with fiber* $\mathbf{C}^{e-1}$ *over* $X - Z(s)$. *Consequently,* $\mathbf{P}(E^*) - Z(s^*)$ *and* $X - Z(s)$ *have the same cohomology.*



Proof. If $x \in Z(s)$, then $s(x)^*$ vanishes on the entire fiber E_x^*, and if $x \notin Z(s)$, then $s(x)^*$ vanishes on a hyperplane in E_x^*. So the fiber at x of $\mathbf{P}(E^*) - Z(s^*)$ above $X - Z(s)$ is $\mathbf{P}(E_x^*)$ minus a hyperplane, which is a $\mathbf{C}^{e-1}$. ∎

Using this proposition and the usual apparatus of algebraic topology, namely Lefschetz duality and the long exact sequence of a pair, the following generalization of the Lefschetz hyperplane theorem follows immediately.

THEOREM 3.2 (Griffiths [4], Sommese [8, Prop. 1.16]). *Let E be an ample vector bundle of rank e over an irreducible smooth projective variety X, and s a section of E. Then the natural map*

$$H_q(Z(s); \mathbf{Z}) \to H_q(X; \mathbf{Z})$$

is an isomorphism for $q < \dim_{\mathbf{C}} X - e$ and a surjection for $q = \dim_{\mathbf{C}} X - e$. In particular, if $\dim_{\mathbf{C}} X - e \geqslant 1$, then $Z(s)$ is connected.

§ 4. Degeneracy loci

Then notion of a *degeneracy locus* generalizes that of the zero set of a section. Let E and F be complex vector bundles over a variety X, of ranks e and f respectively, and let $u: E \to F$ be a bundle map. For each nonnegative integer r the *degeneracy locus of u of rank r* is defined to be

$$D_r(u) = \{x \in X \mid \operatorname{rk} u(x) \leqslant r\}.$$

Every section s of the vector bundle F induces a bundle map $u: \mathcal{O} \to F$ from the trivial line bundle $\mathcal{O} = X \times \mathbf{C}$ to F by setting

$$u(x, 1) = (x, s(x))$$

and extending by linearity. We then have

$$D_0(u) = Z(s),$$

showing that every zero set of a section is a degeneracy locus. Therefore, *every projective variety can be represented as a degeneracy locus in some* $\mathbf{P}^N$! For suppose a projective variety Y in $\mathbf{P}^N$ is the zero set of the homogeneous polynomials $f_1, \ldots, f_n$, where $\deg f_i = d_i$, then $(f_1, \ldots, f_n)$ defines a section s of the vector bundle $\mathcal{O}(d_1) \oplus \ldots \oplus \mathcal{O}(d_n)$ over $\mathbf{P}^N$, which in turn defines a bundle map $u: \mathcal{O} \to \mathcal{O}(d_1) \oplus \ldots \oplus \mathcal{O}(d_n)$. Then $Y = Z(s) = D_0(u)$.

Generally speaking the degeneracy loci of interest are those that have the "expected dimensions". To explain this, let $M(e, f)$ be the variety of $e \times f$ complex matrices and $D_r(M(e, f))$ the subvariety consisting of those of rank at most r. It is not difficult to show that $D_r(M(e, f))$ is an irreducible subvariety of $M(e, f)$ of codimension $(e-r)(f-r)$ [1, p. 67]. Globalizing this, we see that if E and F are vector bundles over X as before, inside the vector bundle $\operatorname{Hom}(E, F)$ is the subvariety $D_r(\operatorname{Hom}(E, F))$ whose fiber at x consists of all homomorphisms: $E_x \to F_x$ of rank $\leqslant r$, and the codimension of $D_r(\operatorname{Hom}(E, F))$

in $\mathrm{Hom}(E, F)$ is $(e-r)(f-r)$. If a bundle map $u: E \to F$ is viewed as a section s of the vector bundle $\mathrm{Hom}(E, F)$, then the degeneracy locus $D_r(u)$ is the pullback $s^{-1}(\mathrm{Hom}(E, F))$, which is clearly isomorphic to the intersection of $s(X)$ with $D_r(\mathrm{Hom}(E, F))$. If the intersection is transversal as is generically the case, it will have codimension $(e-r)(f-r)$ in X. It is in this sense that the expected dimension of $D_r(u)$ is $\dim_{\mathbf{C}} X-(e-r)(f-r)$. If u is not generic, however, then the dimension of $D_r(u)$ could very well be different from its expected dimension. As an extreme example, if u is the zero map, then $D_r(u) = X$.

Examples of degeneracy loci having the expected dimension

EXAMPLE 4.1 (Complete intersections). If F is a vector bundle of rank f over X and s is a section, then the expected dimension of the zero locus $Z(s)$ is $\dim_{\mathbf{C}} X-(1-0)(f-0) = \dim_{\mathbf{C}} X-f$. Hence, any hypersurface or complete intersection is a degeneracy locus of the expected dimension.

EXAMPLE 4.2 (Special divisors). For a generic curve C, the locus W_d^r of special divisors in the Jacobian $J(C)$ is a degeneracy locus of the expected dimension ϱ, which is called the *Brill–Noether number* of W_d^r (see [1]).

EXAMPLE 4.3 (The Segre variety). The Segre variety $S_{m,n}$ is defined to be the image of the Segre embedding of $\mathbf{P}^m \times \mathbf{P}^n$ into $\mathbf{P}^{(m+1)(n+1)-1}$ given by

$$([x_0, \dots, x_m], [y_0, \dots, y_n]) \mapsto [x_i y_j]_{0 \leqslant i \leqslant m, 0 \leqslant j \leqslant n}.$$

Let z_{ij}, $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant n$, be the homogeneous coordinates on $\mathbf{P}^{(m+1)(n+1)-1}$. The matrix $z = [z_{ij}]$ defines a bundle map u: $\mathcal{O}^{\oplus(n+1)} \to \mathcal{O}(1)^{\oplus(m+1)}$ over $\mathbf{P}^{(m+1)(n+1)-1}$ by

$$u(z, a) = (z, z \cdot a)$$

for $a \in \mathbf{C}^{(n+1)}$. Then the Segre variety $S_{m,n}$ is the rank 1 degeneracy locus $D_1(u)$. Its expected dimension is $(m+1)(n+1)-1-mn = m+n$. Hence the dimension of $S_{m,n}$ is equal to its expected dimension.

EXAMPLE 4.4 (Determinantal varieties). Instead of the zero set of a collection of polynomials, we can consider the varieties defined by rank conditions on an $f \times e$ matrix $[u_{ij}(z)]$ of homogeneous polynomials:

$$D_r(u) = \{z \in \mathbf{P}^n \mid \mathrm{rk}\,[u_{ij}(z)] \leqslant r\}.$$

Without some restrictions on the degrees d_{ij} of the u_{ij}'s, this $D_r(u)$ is not well-defined, since for a nonzero complex number λ

$$\mathrm{rk}\,[u_{ij}(\lambda z)] = \mathrm{rk}\,[\lambda^{d_{ij}} u_{ij}(z)] \neq \mathrm{rk}\,[u_{ij}(z)].$$

We will assume that there are two sets of integers: the *row degrees* $d_1, \dots, d_f$, and the *column degrees* $d'_1, \ldots, d'_e$, such that $d_{ij} = d_i + d'_j \geqslant 0$. Then

$$\operatorname{rk}[u_{ij}(\lambda z)] = \operatorname{rk}[\lambda^{d_{ij}} u_{ij}(z)] = \operatorname{rk}[\lambda^{d_i} \lambda^{d'_j} u_{ij}(z)] = \operatorname{rk}[u_{ij}(z)],$$

and $D_r(u)$ is a well-defined subvariety of P^n. It is called a *determinantal variety*, because it is defined by the $(r+1)\times(r+1)$ minors of $[u_{ij}(z)]$.

Note that if the entries of $[u_{ij}(z)]$ in any given row has the same degree, then the degree condition is automatically satisfied, by taking all d'_j zero; similarly for the columns. In particular, there are no degree restrictions on a $1\times e$ or an $f\times 1$ matrix $[u_{ij}(z)]$.

The matrix $[u_{ij}(z)]$ defines a bundle map $u\colon \mathcal{O}(-d_1)\oplus \ldots \oplus \mathcal{O}(-d_e) \to \mathcal{O}(d'_1)\oplus \ldots \oplus \mathcal{O}(d'_f)$ by matrix multiplication:

$$\left(z, \begin{bmatrix} a_1 \\ \vdots \\ a_e \end{bmatrix}\right) \mapsto \left(z, [u_{ij}(z)] \begin{bmatrix} a_1 \\ \vdots \\ a_e \end{bmatrix}\right).$$

So a determinantal variety defined by the matrix $[u_{ij}(z)]$ is a degeneracy locus of the bundle map u.

THEOREM 4.5 (Fulton–Lazarsfeld [2]). *Let X be an irreducible variety, E, F vector bundles of ranks e, f respectively, and $u\colon E \to F$ a bundle map. If* $\operatorname{Hom}(E, F)$ *is ample and* $\dim_{\mathbf{C}} X - (e-r)(f-r) \geqslant 1$, *then $D_r(u)$ is connected.*

Symmetric degeneracy loci

In practice one sometimes encounters bundle maps satisfying symmetry conditions. If E is a vector bundle and L a line bundle, then a bundle map u: $E\otimes E \to L$ is *symmetric* if it is symmetric on each fiber; similarly for skew-symmetric maps.

EXAMPLE 4.6 (Symmetric determinantal varieties). A *symmetric determinantal variety* is a determinantal variety defined by a symmetric matrix $[u_{ij}(z)]$, where of course the row degrees $d_1, \ldots, d_e$ coincide with the column degrees. Such a matrix defines a symmetric bundle map $u\colon E\otimes E \to \mathcal{O}$, where $E = \mathcal{O}(-d_1)\oplus \ldots \oplus \mathcal{O}(-d_e)$. Therefore, a symmetric determinantal variety of the matrix $[u_{ij}(z)]$ is a symmetric degeneracy locus of the bundle map u.

EXAMPLE 4.7 (Quadrics of rank $\leqslant r$ in $\mathbf{P}^{e-1}$). The set of all quadrics in $\mathbf{P}^{e-1}$ is parametrized by the projective space of all symmetric $e\times e$ matrices:

$$\mathbf{P}\{[z_{ij}] \mid 1 \leqslant i, j \leqslant e,\ z_{ij} = z_{ji}\}.$$

Hence, the space of all quadrics in $\mathbf{P}^{e-1}$ is isomorphic to $\mathbf{P}^n$, where $n = \binom{e+1}{2} - 1$. Inside this $\mathbf{P}^n$, the space of all quadrics of rank $\leqslant r$ is the subvariety

$$D_r = \{[z_{ij}] \in \mathbf{P}^n \mid \operatorname{rk}[z_{ij}] \leqslant r\},$$

which has dimension $n-\binom{e-r-1}{2}$. We would like to say that D_r is connected whenever $\dim_{\mathbf{C}} D_r \geqslant 1$, but this does not follow from the Fulton–Lazarsfeld theorem, which implies the connectedness of D_r only if $n-(e-r)^2 \geqslant 1$.

Since a symmetric bundle map $u\colon E \otimes E \to L$ over X may also be viewed as a bundle map: $E \to E^* \otimes L$, u is a section of $\operatorname{Hom}(E, E^* \otimes L)$. By the Fulton–Lazarsfeld theorem, if $\dim_{\mathbf{C}} X-(e-r)^2 \geqslant 1$, then $D_r(u)$ is connected. This, however, is not the correct theorem for a symmetric degeneracy locus, since if $u\colon E \otimes E \to L$ is symmetric, then u is never a generic section of $\operatorname{Hom}(E, E^* \otimes L)$, so that $D_r(u)$ will in general not have dimension $\dim_{\mathbf{C}} X-(e-r)^2$. Indeed, one can show without much difficulty that the expected dimension of a symmetric $D_r(u)$ is $\dim_{\mathbf{C}} X-\binom{e-r+1}{2}$.

In [3, Remark (2), p. 50] Fulton and Lazarsfeld stated the following conjectures.

CONJECTURE 4.8. *Let $u\colon E \otimes E \to L$ be a symmetric bundle map over an irreducible variety X. If* $(\operatorname{Sym}^2 E^*) \otimes L$ *is ample and* $\dim_{\mathbf{C}} X-\binom{e-r+1}{2} \geqslant 1$, *then $D_r(u)$ is connected.*

CONJECTURE 4.9. *Let $u\colon E \otimes E \to L$ be a skew-symmetric bundle map over an irreducible variety X, and r an even integer. If $(\bigwedge^2 E) \otimes L$ is ample and* $\dim_{\mathbf{C}} X-\binom{e-r}{2} \geqslant 1$, *then $D_r(u)$ is connected.*

§ 5. The connectedness of symmetric degeneracy loci: even ranks

A complete proof of Conjectures 4.8 and 4.9 for r even may be found in [9]. What follows is an outline of the main ideas of the proof for the symmetric case when r is even, say $r = 2p$.

For simplicity we assume that X is smooth and L is the trivial line bundle $\mathbf{C}$, so u is a symmetric bundle map: $E \otimes E \to \mathbf{C}$. The first step is to represent the degeneracy locus $D_{2p}(u)$ as the image of a zero locus on a Grassmann bundle. This is a construction of Pragacz [7], and it is based on the following characterization of the rank of a symmetric bilinear form.

PROPOSITION 5.1. *Let W be a complex vector space of dimension e. A symmetric bilinear map $\phi\colon W \times W \to \mathbf{C}$ has rank $\leqslant 2p$ if and only if it has an isotropic subspace of dimension $e-p$. (An isotropic subspace is a subspace V such that $\phi|_{V \times V} = 0$.)*

Let $\pi\colon G(e-p, E) \to X$ be the Grassmann bundle of $(e-p)$-dimensional subspaces of fibres of E. On the Grassmann bundle there is a tautological exact sequence

$$0 \to S \to \pi^* E \to Q \to 0,$$

where S and Q are the universal sub- and quotient bundles respectively. The inclusion $S \hookrightarrow \pi^* E$ induces naturally a surjection $\pi^* \operatorname{Sym}^2 E^* \to \operatorname{Sym}^2 S^*$, which may be interpreted as the restriction of a quadratic form on the bundle E to the subbundle S. We view the symmetric bundle map $u\colon E\otimes E\to\mathbf{C}$ as a section of the vector bundle $\mathrm{Sym}^2 E^*$. It pulls back to a section $\pi^* u$ of $\pi^*\mathrm{Sym}^2 E^*$ over $G(e-p, E)$, and by composition with the restriction we get a section t of $\mathrm{Sym}^2 S^*$:

$$t(x, V\subset E_x) = u(x)|_{V\times V}.$$

We therefore have the following diagram:

$$\begin{tikzcd}
P-Z(t) \subset \arrow[dd, "\text{same cohomology}"'] \arrow[rrrrddd, bend left, "h"] & P = P(\mathrm{Sym}^2 S) \arrow[dd] & & & \\
& & \pi^*\mathrm{Sym}^2 E^* \arrow[r] \arrow[dl, "\pi^* u"'] & \mathrm{Sym}^2 S^* & \\
G-Z(t) \subset & G = G(e-p, E) \arrow[urr, dashed] \arrow[ddr, "\pi"'] & & \mathcal{O}(1) \arrow[d, shift right] & \\
& & \mathrm{Sym}^2 E^* \arrow[d, shift right] & P(\mathrm{Sym}^2 E) = P' \arrow[u, shift right, "u^*"'] \arrow[dl] & \supset P'-Z(u^*) = U \\
& & D_{2p}(u) \subset X \arrow[u, shift right, "u"'] & &
\end{tikzcd}$$

PROPOSITION 5.2. *The degeneracy locus $D_{2p}(u)$ is the image of the zero locus $Z(t)$.*

Proof. Observe that $t(x, V\subset E_x) = u(x)|_{V\times V} = 0$ if and only if V is an isotropic subspace for $u(x)$. Therefore,

$$\begin{aligned}
x\in\pi(Z(t)) \quad & \text{iff } u(x) \text{ has an isotropic subspace of dimension } e-p,\\
& \text{iff } \operatorname{rk} u(x)\leqslant e-p,\\
& \text{iff } x\in D_{2p}(u) \text{ by Prop. 5.1.} \ \blacksquare
\end{aligned}$$

Thus, to prove the connectedness of the degeneracy locus $D_{2p}(u)$ it suffices to prove the connectedness of the zero locus $Z(t)$. Since t is a section of $\mathrm{Sym}^2 S^*$, if $\mathrm{Sym}^2 S^*$ were ample, the Lefschetz-type theorem (3.2) for an ample vector bundle would apply. Unfortunately, $\mathrm{Sym}^2 S^*$ is not ample, and so we try to prove the connectedness of $Z(t)$ by checking the following cohomological criterion.

PROPOSITION 5.3. *Let M be a connected compact orientable manifold of real dimension n and A a subset of M. Then*

(i) *A is nonempty if and only if $H^n(M-A;\mathbf{Z}) = 0$.*
(ii) *A is connected if $H^n(M-A;\mathbf{Z}) = H^{n-1}(M-A;\mathbf{Z}) = 0$.*

Let $G = G(e-p, E)$. To compute $H^*(G-Z(t);\mathbf{Z})$, we apply the duality construction of Section 3 and consider $Z(t^*)$ in $P := \mathbf{P}(\mathrm{Sym}^2 S)$. Since

$\mathbf{P}(\mathrm{Sym}^2 S) - Z(t^*)$ is an affine space bundle over $G - Z(t)$, by Proposition 3.1

$$H^*(G - Z(t); \mathbf{Z}) \simeq H^*(\mathbf{P}(\mathrm{Sym}^2 S) - Z(t^*); \mathbf{Z}).$$

It is easily shown that the natural map $\mathbf{P}(\mathrm{Sym}^2 S) \to \mathbf{P}(\mathrm{Sym}^2 E)$ induces a map h: $\mathbf{P}(\mathrm{Sym}^2 S) - Z(t^*) \to \mathbf{P}(\mathrm{Sym}^2 E) - Z(u^*)$ and that the fiber of h at $(x, \phi \in \mathrm{Sym}^2 E_x)$ is the Grassmannian $G(e - p - \mathrm{rk}\,\phi, e - \mathrm{rk}\,\phi)$. Since u^* is by definition a section of the ample line bundle $\mathcal{O}(1)$ over $P(\mathrm{Sym}^2 E)$, the complement $\mathbf{P}(\mathrm{Sym}^2 E) - Z(u^*)$ is an affine variety; hence, by a standard theorem its cohomology vanishes above the complex dimension of the variety. With the map h: $\mathbf{P}(\mathrm{Sym}^2 S) - Z(t^*) \to \mathbf{P}(\mathrm{Sym}^2 E) - Z(u^*)$, the cohomology of $\mathbf{P}(\mathrm{Sym}^2 S) - Z(t^*)$ can then be computed by applying the following cohomology comparison theorem. Denote by $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$ the set of natural numbers.

LEMMA 5.4. *Let h: $M \to Y$ be a surjective proper morphism from any variety M to an affine variety Y. Suppose there is a strictly increasing function d: $\mathbf{N} \to \mathbf{N}$ and a sequence of closed subvarieties*

$$\ldots \subset Y_{k+1} \subset Y_k \subset \ldots \subset Y_0 = Y$$

such that for all x in $Y_k - Y_{k+1}$,

$$d(k) = \dim_{\mathbf{C}} h^{-1}(x).$$

Define

$$R = \max_{k \geq 0} \{\dim_{\mathbf{C}} Y_k + 2d(k)\}.$$

Then $H^q(M; \mathbf{Z}) = 0$ for all $q > R$.

In the present situation R turns out to be

$$R = \dim_{\mathbf{C}} X + \binom{e-1}{2} - 1 - p.$$

A straightforward computation shows that

$$2\dim_{\mathbf{C}} G - 1 > R \Leftrightarrow \dim_{\mathbf{C}} X \geq \binom{e-2p+1}{2} + 1.$$

Hence,

$$H^q(\mathbf{P}(\mathrm{Sym}^2 S) - Z(t^*); \mathbf{Z}) = H^q(G - Z(t); \mathbf{Z}) = 0$$

for $q = 2\dim_{\mathbf{C}} G$, $2\dim_{\mathbf{C}} G - 1$. By the cohomological criterion for connectedness, $Z(t)$ is connected. As the continuous image of a connected set, $D_{2p}(u)$ is also connected.

§ 6. Extension to n-ample vector bundles

The concept of n-ampleness was introduced by Sommese [8]. It is a useful generalization of ampleness, for many of the classical cohomology theorems for ample vector bundles, such as the Lefschetz hyperplane theorem, the hard

Lefschetz theorem, and the Kodaira–Nakano–Le Potier vanishing theorem, all have analogues for n-ample vector bundles, usually with the dimension shifted by n. In this section we prove that the connectedness theorems for degeneracy loci can be extended to n-ample vector bundles.

DEFINITION. A line bundle L on a projective variety X is called *n-ample* if for some positive integer m, $L^{\otimes m}$ is spanned by global sections and the associated map $\iota_{L^{\otimes m}}$: $X \to \mathbf{P}^N$ given by a basis of global sections of $L^{\otimes m}$ has at most n-dimensional fibers. A vector bundle E on X is *n-ample* if the tautological line bundle $\mathcal{O}_{\mathbf{P}(E^*)}(1)$ on $\mathbf{P}(E^*)$ is n-ample.

Remark 6.1. *A line bundle L on X is* 0*-ample if and only if it is ample.*

Proof. ($\Leftarrow$) Clear.

($\Rightarrow$) Suppose L is 0-ample. By definition $\iota_{L^{\otimes m}}$: $X \to \mathbf{P}^N$ is finite onto its image for some positive integer m. Let $f = \iota_{L^{\otimes m}}$. Then $L^{\otimes m} = f^* \mathcal{O}_{f(X)}(1)$. Since the pullback of an ample bundle under a finite surjective morphism remains ample ([5, Prop. 1.6, p. 84]), $L^{\otimes m}$ is ample. Therefore, L is ample. ∎

DEFINITION. We call a divisor D on a projective variety X *n-ample* if it is the zero set of a section of an n-ample line bundle on X.

Since every affine variety is the complement of an ample divisor in a projective variety, the complement of an n-ample divisor generalizes the notion of an affine variety.

A sheaf $\mathcal{F}$ of abelian groups over a variety X is *constructible* if X is a disjoint union of locally closed subsets over each of which $\mathcal{F}$ is locally constant. Constructible sheaves arise naturally in the computation of cohomology groups, because under a morphism of varieties the direct image sheaves of a constructible sheaf are again constructible. Another property of constructible sheaves is that the cohomology of a constructible sheaf on an affine variety vanishes above the complex dimension of the variety (for a proof, see [1, p. 315]). We can generalize this to complements of n-ample divisors.

THEOREM 6.2. *Let D be an n-ample divisor on a projective variety X, and $\mathcal{F}$ a constructible sheaf on $X-D$. Then $H^q(X-D, \mathcal{F}) = 0$ for $q > \dim_{\mathbf{C}} X + n$.*

Proof (Joe Harris). Suppose D is the zero set of a section s of the n-ample line bundle L. For some positive integer m the map $f = \iota_{L^{\otimes m}}$: $X \to \mathbf{P}^N$ has at most n-dimensional fibers. Since $D = \mathrm{Zero}(s) = \mathrm{Zero}(s^{\otimes m})$, D is the inverse image under f of a hyperplane section H of $f(X)$. Hence, f maps $X-D$ to $f(X)-H$ with at most n-dimensional fibers. Applying the Leray spectral sequence to f: $X-D \to f(X)-H$, we see that

$$H^j(f(X)-H, R^i f_* \mathcal{F}) \Rightarrow H^{i+j}(X-D, \mathcal{F}).$$

Since the fibers of f are at most n-dimensional, $R^i f_* \mathcal{F} = 0$ for $i > 2n$. Since

$f(X)-H$ is affine and $R^i f_* \mathscr{F}$ is constructible, $H^j(f(H)-H, R^i f_* \mathscr{F}) = 0$ for $j > \dim_{\mathbf{C}} f(X)-H = \dim_{\mathbf{C}} f(X) = \dim_{\mathbf{C}} X - n$. Therefore, if $i+j > 2n+\dim_{\mathbf{C}} X - n = \dim_{\mathbf{C}} X + n$, then $i > 2n$ or $j > \dim_{\mathbf{C}} X + n$, so that $H^j(f(X)-H, R^i f_* \mathscr{F}) = 0$. By Leray's spectral sequences,

$$H^{i+j}(X-D, \mathscr{F}) = 0 \quad \text{for } i+j > \dim_{\mathbf{C}} X + n. \ \blacksquare$$

The cohomology comparison lemma (5.4) can now be generalized to the complement of an n-ample divisor. Denote by $\mathbf{N}$ the set of natural numbers $\{0, 1, 2, 3, \ldots\}$.

LEMMA 6.3. *Let $h\colon M \to Y$ be a surjective proper morphism where M is any variety, and Y is the complement of an n-ample divisor in a projective variety. Suppose there is a strictly increasing function $d\colon \mathbf{N} \to \mathbf{N}$ and a sequence of closed subvarieties*

$$\ldots \subset Y_{k+1} \subset Y_k \subset Y_{k-1} \subset \ldots \subset Y_0 = Y$$

such that for all x in $Y_k - Y_{k+1}$,

$$d(k) = \dim_{\mathbf{C}} h^{-1}(x).$$

Define

$$R = \max_{k \geqslant 0} \{\dim_{\mathbf{C}} Y_k + 2d(k)\}.$$

Then

$$H^q(M, \mathscr{F}) = 0 \quad \textit{for all } q > R+n.$$

Proof. Since the restriction of an n-ample line to a closed subvariety is again n-ample, all the Y_k's are complements of n-ample divisors. The rest of the proof is identical to [9, § 4], except one uses Theorem 6.2 instead of the vanishing theorem of a constructible sheaf on an affine variety; hence, the shift by n in the conclusion. $\blacksquare$

Following the same set-up as in Section 5 we can now prove the following connectedness theorems.

THEOREM 6.4. *Let X be an irreducible variety, E, F vector bundles of rank e, f respectively, and L a line bundle over X.*

(a) *Suppose $u\colon E \to F$ is a bundle map and r a nonnegative integer. If $\operatorname{Hom}(E, F)$ is n-ample and $\dim_{\mathbf{C}} X - (e-r)(f-r) \geqslant n+1$, then $D_r(u)$ is connected.*

(b) *Suppose $u\colon E \otimes E \to L$ is a symmetric bundle map and r is even. If $(\operatorname{Sym}^2 E^*) \otimes L$ is n-ample and $\dim_{\mathbf{C}} X - \binom{e-r+1}{2} \geqslant n+1$, then $D_r(u)$ is connected.*

(c) *Suppose $u\colon E \otimes E \to L$ is a skew-symmetric bundle map and r is even. If $(\bigwedge^2 E^*) \otimes L$ is n-ample and $\dim_{\mathbf{C}} X - \binom{e-r}{2} \geqslant n+1$, then $D_r(u)$ is connected.*

Proof. The proof is exactly the same as for ample vector bundles, but the use of Lemma 6.3 instead of Lemma 5.4 entails a shift in the dimension by n. We will sketch a proof of Part (a) only. By taking the transpose of u if necessary, we may assume that $f \geqslant e$. Let $\pi\colon G(e-r,\, E) \to E$ be the Grassmann bundle of $(e-r)$-dimensional subspaces of the fibers E_x of the bundle E, and let S be the universal subbundle over $G := G(e-r,\, E)$. As in Section 5, we have the following diagram

$$\begin{tikzcd}
P\text{-}Z(t^*) \arrow[r, phantom, "\subset"] \arrow[dd, "\text{some cohomology}"'] \arrow[dddrrr, bend left, "h"] & P = \mathbf{P}(\mathrm{Hom}(\pi^*F, S)) \arrow[dd] & & \\
 & \pi^*\mathrm{Hom}(E, F) \arrow[r] \arrow[d, shift right] & \mathrm{Hom}(S, \pi^*F) & \\
G\text{-}Z(t) \arrow[r, phantom, "\subset"] & G = G(e-r, E) \arrow[u, shift right, "\pi^*u"'] \arrow[ur, dashed, "t"] \arrow[ddr, "\pi"'] & & \\
 & & \mathrm{Hom}(E, F) \arrow[d, shift right] & \mathbf{P}(\mathrm{Hom}(F, E)) = P' \supset P'\text{-}Z(u^*) \arrow[dl] \\
 & & D_r(u) \subset X \arrow[u, shift right, "u"'] &
\end{tikzcd}$$

In this diagram the bundle $\mathrm{Hom}(F,\, E)$ is the dual of $\mathrm{Hom}(E,\, F)$, and the map from $\mathbf{P}(\mathrm{Hom}(\pi^* F,\, S))$ to $\mathbf{P}(\mathrm{Hom}(F,\, E))$ is the obvious one induced by the inclusion $S \subsetneq \pi^* E$. The fiber of h at $(x,\, \phi \in \mathrm{Hom}(F_x,\, E_x))$ is the Grassmannian $G(e-p-\mathrm{rk}\,\phi,\, E_x/\mathrm{im}\,\phi)$. Stratify $P' - Z(u^*)$ by

$$Y_k = \mathbf{P}\big(D_{e-r-k}(\mathrm{Hom}(F,\, E))\big) - Z(u^*).$$

If $\phi \in Y_k - Y_{k+1}$, then $\mathrm{rk}\,\phi = e-r-k$, and $d(k) = kr$. Since

$$\dim_{\mathbf{C}} Y_k = \dim_{\mathbf{C}} P' - (r+k)(f-e+r+k),$$

it is easily checked that

$$R = \dim_{\mathbf{C}} X + (f+r)(e-r) - 1.$$

Hence,

$$2\dim_{\mathbf{C}} G - 1 > R + n \Leftrightarrow \dim_{\mathbf{C}} X - (e-r)(f-r) \geqslant n+1.$$

By Lemma 6.3, $H^q\big(\mathbf{P}(\mathrm{Hom}(\pi^* F,\, S)) - Z(t^*);\, \mathbf{Z}\big) = H^q\big(G - Z(t);\, \mathbf{Z}\big) = 0$ for $q = 2\dim_{\mathbf{C}} G,\ 2\dim_{\mathbf{C}} G - 1$. As in Section 5, it follows that $Z(t)$ and therefore $D_r(u)$ is connected. ∎

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