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MORE HAMMOCKS

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Let A be a representation-directed algebra. If e is a primitive idempotent of A, the subset of the Auslander–Reiten quiver Γ_A given by the indecomposable modules X with $Xe \neq 0$ is a hammock. In this paper we generalize this result. Let e_a and e_b be two primitive idempotents in A and let $w \in e_a A e_b$ be a nonzero element. We show that the subset of Γ_A given by the indecomposable modules X with $Xw \neq 0$ is a hammock. We also give various descriptions of the corresponding hammock function.

1. Introduction and the results

For the general notation we refer to [R]. Let k be a field and let A be the factor algebra of the path algebra of the quiver $\Delta = (\Delta_0, \Delta_1)$ by an admissible ideal I. Assume that A is representation-directed. Denote by Γ_A the Auslander–Reiten quiver of A. By mod-A we denote the category of finitely generated right A-modules and by ind-A the full subcategory of indecomposables. By $D = \mathrm{Hom}_k(-, k)$ we denote the usual duality. For any $X \in$ mod-A we denote by $[X]$ the isomorphism class of X; thus $(\Gamma_A)_0 = \{[X] \mid X \in \mathrm{ind}\, A\}$. For any $a \in \Delta_0$ let e_a be the corresponding primitive idempotent. We denote by $P_A(a) = P(a) = e_a A$ the indecomposable projective right A-module corresponding to a, and by $I_A(a) = I(a) = D(Ae_a)$ the corresponding indecomposable injective A-module. Let w be a nonzero element in $e_a A e_b$. Since $e_a A e_b \cong \mathrm{Hom}_A(P(b), P(a))$, we also consider w as a map, which is just left multiplication by the element w. Define $C(w) = \mathrm{Coker}\, w$. Note that $P(b) \to P(a) \to C(w) \to 0$ is a projective presentation of $C(w)$.

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THEOREM 1. *The full translation subquiver of Γ_A given by $H_w = \{[X] \mid X \cdot w \neq 0\}$ is a hammock with hammock function h_w given by*

$$h_w[X] = \dim \operatorname{Hom}(P(a), X) - \dim \operatorname{Hom}(C(w), X).$$

If we apply the Nakayama functor $\nu_A = D\operatorname{Hom}_A(-, A)$ to our projective presentation of $C(w)$, we obtain an injective presentation $0 \to \tau C(w) \to I(b) \to I(a)$ for the Auslander–Reiten translate $\tau C(w)$ of $C(w)$. Denote by $S(a)$ the simple A-module corresponding to a. Let $M(w)$ be an A-module satisfying $\operatorname{top} M(w) \cong S(a)$ and $\operatorname{soc} M(w) \cong S(b)$. Clearly, $M(w)$ is indecomposable. It follows that $M(w)$ is uniquely determined up to isomorphism, and isomorphic to the image of any nonzero map in $\operatorname{Hom}(P(a), I(b))$.

We define a quasi-poset structure on mod-A. For any two modules X and Y, we set $X \leqslant Y$ if and only if there is a chain of submodules $V \subseteq U \subseteq Y$ such that $X \cong U/V$. Clearly, $X \leqslant Y$ and $Y \leqslant X$ is equivalent to $X \cong Y$, hence this defines a poset structure on $(\Gamma_A)_0$.

THEOREM 2. *If X is an A-module, then*

$$\begin{aligned} h_w[X] &:= \dim \operatorname{Hom}(P(a), X) - \dim \operatorname{Hom}(C(w), X) \\ &= \dim \operatorname{Hom}(X, I(b)) - \dim \operatorname{Hom}(X, \tau C(w)) \\ &= \operatorname{rank} \operatorname{Hom}(w, X) \\ &= \operatorname{rank} \operatorname{Hom}(X, \nu_A w) \\ &= \min\{\dim \operatorname{Hom}(P(a), X), \dim \operatorname{Hom}(P(b), X)\} \\ &= \max\{l \mid l \cdot \operatorname{\mathbf{dim}} M(w) \leqslant \operatorname{\mathbf{dim}} X\} \\ &= \max\{l \mid M(w)^l \leqslant X\}. \end{aligned}$$

Here, $\operatorname{\mathbf{dim}} X$ denotes the dimension vector of X.

Using these characterizations of h_w, we obtain the following consequence.

COROLLARY. *Let $w = w_1 w_2 w_3$, where $w_1 \in e_a A e_c$, $w_2 \in e_c A e_d$, $w_3 \in e_d A e_b$. Then $H_w = H_{w_1} \cap H_{w_3}$. In particular, $H_w = H_{e_a} \cap H_{e_b}$.*

Proof. Clearly, $H_w \subseteq H_{w_1} \cap H_{w_3} \subseteq H_{e_a} \cap H_{e_b}$. Conversely, if $X \in$ ind-A, with $[X] \in H_{e_a} \cap H_{e_b}$, then $\dim \operatorname{Hom}(P(a), X) \geqslant 1$, $\dim \operatorname{Hom}(P(b), X) \geqslant 1$, thus $[X] \in H_w$ according to Theorem 2.

For $w = e_a$ we obtain in this way some of the results of [RV].

2. Proof of Theorem 1

We recall the definition of a hammock. A finite directed translation quiver H is a *hammock* provided that there is a function h: $H_0 \to \mathbb{N}$ such that

(a) there is a source ω in H and $h(\omega) = 1$,

(b) for every vertex $x \in H_0$ we have $h(x) \geqslant 1$,

(c) for every vertex $x \neq \omega$

$$h(x)+h(\tau x) = \sum_{y \to x} h(y),$$

where by abuse of notation we set $h(\tau x) = 0$ if x is a projective vertex,

(d) for every injective vertex q

$$h(q) \geqslant \sum_{q \to y} h(y).$$

Such a function is uniquely determined and is called the *hammock function* [Br], [RV].

LEMMA 1. *The following conditions are equivalent for $X \in$ ind-A*:

(a) $[X] \in H_w$.
(b) $X_w := \mathrm{Hom}(w, X) \neq 0$.
(c) $\dim \mathrm{Hom}(P(a), X) - \dim \mathrm{Hom}(C(w), X) \geqslant 1$.
(d) $\mathrm{Hom}(P(a), X) \cdot w \neq 0$.

Proof. (a)$\Leftrightarrow$(b). By definition of H_w.

(b)$\Leftrightarrow$(c). The sequence

$$0 \to \mathrm{Hom}(C(w), X) \to \mathrm{Hom}(P(a), X) \xrightarrow{X_w} \mathrm{Hom}(P(b), X)$$

is exact, hence $X_w \neq 0$ if and only if $\mathrm{Hom}(C(w), X)$ is a proper subspace of $\mathrm{Hom}(P(a), X)$.

(b)$\Leftrightarrow$(d). Trivial.

LEMMA 2. *Neither $[C(w)]$ nor $[\tau C(w)]$ is in H_w.*

Proof. Notice that $X \cdot w \neq 0$ implies that both $X \cdot e_a \neq 0$ and $X \cdot e_b \neq 0$. Since $C(w) \cdot e_b = 0$ and $\tau C(w) \cdot e_a = 0$, they are not in H_w.

Now we prove Theorem 1.

Let $\omega = [P(a)]$. Clearly, $h_w(\omega) = 1$, thus $\omega \in H_w$. This is property (a) with $h = h_w$.

Let $X \in$ ind-A. By Lemma 1, we have $[X] \in H_w$ if and only if $h_w[X] \geqslant 1$; this shows property (b).

Now we want to show property (c): Assume $[X] \in H_w$. Let $Y \to X$ be a right almost split map. Since $C(w)$ does not belong to H_w, the induced map $\mathrm{Hom}(C(w), Y) \to \mathrm{Hom}(C(w), X)$ is surjective. First, let X be projective. Then $Y \to X$ is a monomorphism, thus $\mathrm{Hom}(C(w), Y) \to \mathrm{Hom}(C(w), X)$ is bijective. If $X \not\cong P(a)$, then similarly we have a bijection $\mathrm{Hom}(P(a), Y) \to \mathrm{Hom}(P(a), X)$. It follows in this case that $h_w[X] = h_w[Y]$. Second, let X be nonprojective, and let $0 \to \tau X \to Y \to X \to 0$ be an Auslander–Reiten sequence. Applying

$\operatorname{Hom}(C(w), -)$ and $\operatorname{Hom}(P(a), -)$ we obtain the exact sequences

$$0 \to \operatorname{Hom}(C(w), \tau X) \to \operatorname{Hom}(C(w), Y) \to \operatorname{Hom}(C(w), X) \to 0,$$

$$0 \to \operatorname{Hom}(P(a), \tau X) \to \operatorname{Hom}(P(a), Y) \to \operatorname{Hom}(P(a), X) \to 0.$$

Therefore

$$(*) \qquad h_w[X] + h_w[\tau X] = h_w[Y].$$

Now, write $Y = \bigoplus_{i=1}^{s} Y_i$ with indecomposable modules Y_i, and assume $h_w[Y_i] \neq 0$ for $1 \leqslant i \leqslant r$. Then

$$h_w[Y] = \sum_{i=1}^{r} h_w[Y_i] = \sum_{[Y'] \to [X]} h_w[Y'].$$

This proves (c).

For the proof of (d), let $Q \in \text{ind-}A$ such that $[Q] \in H_w$ is an injective vertex. First, assume that Q is not an injective module. Thus $Q \cong \tau X$ for some $X \in \text{ind-}A$, and $h_w[X] = 0$. Note that $X \ncong C(w)$, since $\tau C(w) \notin H_w$ by Lemma 2. The additivity property $(*)$ gives the desired inequality (even equality). So assume now that $Q = I(c)$ for some vertex c of Δ. We apply $\operatorname{Hom}(P(a), -)$ to $0 \to \operatorname{soc} I(c) \to I(c) \to I(c)/\operatorname{soc} I(c) \to 0$ and obtain a short exact sequence

$$0 \to \operatorname{Hom}(P(a), \operatorname{soc} I(c)) \to \operatorname{Hom}(P(a), I(c)) \to \operatorname{Hom}(P(a), I(c)/\operatorname{soc} I(c)) \to 0;$$

thus $\dim \operatorname{Hom}(P(a), Q) \geqslant \dim \operatorname{Hom}(P(a), Q/\operatorname{soc} Q)$. Now we apply $\operatorname{Hom}(C(w), -)$ and obtain

$$0 \to \operatorname{Hom}(C(w), \operatorname{soc} I(c)) \to \operatorname{Hom}(C(w), I(c)) \to \operatorname{Hom}(C(w), I(c)/\operatorname{soc} I(c)).$$

Observe that $\operatorname{soc} I(c) \cong S(c)$, so $\operatorname{Hom}(C(w), \operatorname{soc} I(c)) \neq 0$ would imply that $c = a$, but either $[I(a)] \notin H_w$ or $w = e_a$, and in that case $C(w) = C(e_a) = 0$. Thus $\operatorname{Hom}(C(w), \operatorname{soc} I(c)) = 0$. Hence we have

$$\dim \operatorname{Hom}(C(w), I(c)) \leqslant \dim \operatorname{Hom}(C(w), I(c)/\operatorname{soc} I(c)),$$

whence we conclude

$$\begin{aligned} h_w[Q] &= \dim \operatorname{Hom}(P(a), Q) - \dim \operatorname{Hom}(C(w), Q) \\ &\geqslant \dim \operatorname{Hom}(P(a), Q/\operatorname{soc} Q) - \dim \operatorname{Hom}(C(w), Q/\operatorname{soc} Q) \\ &= h_w[Q/\operatorname{soc} Q] = \sum_{[Q] \to [Y]} h_w[Y]. \end{aligned}$$

3. Proof of Theorem 2

There exists an invertible natural transformation α_P: $\operatorname{Hom}(P, -) \to D\operatorname{Hom}(-, \nu P)$ for any projective A-module P (see e.g. [G]). For $X \in \text{mod-}A$ we therefore obtain the following commutative diagram, where the rows are exact sequences:

$$\begin{array}{ccccc} 0\to \operatorname{Hom}(C(w),X)\to & \operatorname{Hom}(P(a),X) & \to & \operatorname{Hom}(P(b),X) & \\ & \downarrow\cong & & \downarrow\cong & \\ & D\operatorname{Hom}(X,I(a)) & \to & D\operatorname{Hom}(X,I(b)) & \to D\operatorname{Hom}(X,\tau C(w))\to 0 \end{array}$$

In particular, this yields the following exact sequence:

$$0\to \operatorname{Hom}(C(w),X)\to \operatorname{Hom}(P(a),X) \to D\operatorname{Hom}(X,I(b))\to D\operatorname{Hom}(X,\tau C(w))\to 0.$$

Thus

$$\begin{aligned}\dim\operatorname{Hom}(P(a),X)-\dim\operatorname{Hom}(C(w),X) &= \dim\operatorname{Hom}(X,I(b))-\dim\operatorname{Hom}(X,\tau C(w))\\ &= \operatorname{rank}\operatorname{Hom}(w,X)=\operatorname{rank}\operatorname{Hom}(X,\nu_A w).\end{aligned}$$

Since A is representation-directed, $\operatorname{Hom}(w, X)$ is either injective or surjective. Thus

$$h_w[X] = \min\bigl(\dim\operatorname{Hom}(P(a),X),\ \dim\operatorname{Hom}(P(b),X)\bigr).$$

This shows the first four equalities of Theorem 2. Since Δ is directed, we have $\dim\operatorname{Hom}(P(a), I(b)) \leqslant 1$. Hence any two nonzero maps have the same image in $I(b)$, which we define to be $M(w)$. Obviously $\operatorname{top} M(w) \cong S(a)$ and $\operatorname{soc} M(w) \cong S(b)$, and hence $M(w)$ is indecomposable. According to [BS], any indecomposable A-module is uniquely determined by its top and its socle. For $M(w)$, this is immediate, as the following proof shows.

Lemma 3. *If* $M\in \text{mod-}A$ *satisfies* $\operatorname{top} M \cong S(a)$ *and* $\operatorname{soc} M \cong S(b)$, *then* $M \cong M(w)$.

Proof. Consider the map $M \twoheadrightarrow S(a)$; we obtain a surjective map $P(a)\twoheadrightarrow M$ by lifting the map $M\twoheadrightarrow S(a)$. Similarly, we obtain an injective map $M \hookrightarrow I(b)$. Thus the composition is a nonzero map in $\operatorname{Hom}(P(a), I(b))$ and the image is M, thus $M \cong M(w)$.

Proposition. *Let* $X\in \text{ind-}A$ *be indecomposable. The following are equivalent*:

(a) $l \leqslant \min\bigl(\dim\operatorname{Hom}(P(a),X),\ \dim\operatorname{Hom}(P(b),X)\bigr)$.
(b) $l\cdot \underline{\dim}\, M(w) \leqslant \underline{\dim}\, X$.
(c) $M(w)^l \leqslant X$.

Proof. (c)$\Rightarrow$(b). Obviously $M(w)^l \leqslant X$ implies $\underline{\dim}\, M(w)^l \leqslant \underline{\dim}\, X$ and $\underline{\dim}\, M(w)^l = l\cdot \underline{\dim}\, M(w)$.

(b)$\Rightarrow$(a) follows from the fact that

$$\dim\operatorname{Hom}(P(a), M(w)) = 1 = \dim\operatorname{Hom}(M(w), I(b)).$$

(a)$\Rightarrow$(c). Since A is representation-directed, $\operatorname{Hom}(w, X) = w^*$ is either mono or epi. Assume w^* is mono. Let $d = \dim \operatorname{Hom}(P(a), X)$, and let $f_1, \ldots, f_d$ be a basis of $\operatorname{Hom}(P(a), X)$. Since w^* is mono, the set $w^*f_1, \ldots, w^*f_d$ is linearly independent. Hence there are maps $g_1, \ldots, g_d \in \operatorname{Hom}(X, I(b))$ such that $g_i \circ w^* f_j$ is zero if $i \neq j$ and nonzero if $i = j$. Define maps $f\colon P(a)^d \to X$ by $f = (f_1, \ldots, f_d)$ and $g\colon X \to I(b)^d$ by $g = \begin{bmatrix} g_1 \\ \vdots \\ g_d \end{bmatrix}$. Observe that $g_i \circ w^* f_j = g_i f_j w = 0$ implies that $g_i f_j = 0$ by injectivity of w^*. Thus

$$gf = \begin{bmatrix} g_1 f_1 & & 0 \\ & \ddots & \\ 0 & & g_d f_d \end{bmatrix},$$

where all $g_i f_i$ are nonzero maps in $\operatorname{Hom}(P(a), I(b))$, so $\operatorname{Im}(gf) \cong M(w)^d$. Since g maps $\operatorname{Im} f$ onto $\operatorname{Im}(gf)$, thus $M(w)^d \cong \operatorname{Im}(gf) \cong \operatorname{Im} f / (\operatorname{Im} f \cap \operatorname{Ker} g)$. Since $\operatorname{Im} f \cap \operatorname{Ker} g \subseteq \operatorname{Im} f \subseteq X$ is a chain of submodules, we see by definition of $\leqslant$ that $M(w)^d \leqslant X$.

If w^* is epi, we apply a similar argument.

COROLLARY.

$$\begin{aligned} h_w[X] &= \min\{\dim \operatorname{Hom}(P(a), X), \dim \operatorname{Hom}(P(b), X)\} \\ &= \max\{l \mid l \cdot \underline{\dim}\, M(w) \leqslant \underline{\dim}\, X\} \\ &= \max\{l \mid M(w)^l \leqslant X\}. \end{aligned}$$

This proves the remaining equalities of Theorem 2.

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