TOPICS IN ALGEBRA
BANACH CENTER PUBLICATIONS, VOLUME 26, PART 1
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

# RIGID TRANSJECTIVE MODULES

DIETER HAPPEL

*Fakultät für Mathematik, Universität Bielefeld*
*Bielefeld, F.R.G.*

LUISE UNGER

*Fachbereich Mathematik, Universität Paderborn*
*Paderborn, F.R.G.*

Let $A$ be a basic and connected finite-dimensional hereditary algebra over an algebraically closed field $k$ (i.e. $A$ is given as a path algebra $k\vec{\Delta}$ of a finite and connected quiver $\vec{\Delta}$ without oriented cycles). We denote by $\operatorname{mod} A$ the category of finitely generated left $A$-modules. Either $A$ or its opposite algebra $A^*$ is a one-point extension algebra of a basic, finite-dimensional hereditary algebra $B$. We will assume that $A$ is of the form $B[R]$ where $R$ is an indecomposable $B$-module. In this situation we have a canonical embedding of $\operatorname{mod} B$ into $\operatorname{mod} A$. It is well known that all but finitely many isomorphism classes of indecomposable $B$-modules will be regular $A$-modules via this embedding (cf. [Ri2] and [U]). In this note we study the full subcategory $\mathcal{N}$ of $\operatorname{mod} B$ given by the indecomposable $B$-modules which are not regular $A$-modules. In other words, we are interested in the indecomposable preprojective or preinjective $B$-modules which still have this property when considered as $A$-modules. We will call modules of this type rigid transjective modules. Note that we deviate from previous use of this terminology.

Suppose that the algebra $B$ is of the form $k\vec{\Delta}^0$ for some finite quiver $\vec{\Delta}^0$ without oriented cycles. Then we show that the indecomposable modules in $\mathcal{N}$ can be identified with a full subtranslation quiver $\mathcal{M}$ of a transjective component of the quiver of the derived category of bounded complexes of $B$-modules, which is of the form $\mathbf{Z}\vec{\Delta}^0$. Note that $\mathcal{M}$ is a finite translation quiver.

---

This paper is in final form and no version of it will be submitted for publication elsewhere.

For the convenience of the reader we will recall in Section 1 some of the necessary background needed in the later developments. In Section 2 we investigate the translation quiver $\mathcal{M}$. Finally, Section 3 contains some examples.

We point out that these investigations were inspired by Ringel's work on wing modules (see Chapter 3 of [Ri1]).

Unless stated otherwise we follow the terminology of [Ri1].

## 1. Preliminaries

### 1.1. One-point extensions

We recall the definition of a one-point extension algebra. For more details and the representation-theoretic tools available in this context we refer to [Ri1]. Let $B$ be a finite-dimensional $k$-algebra and let $R$ be in $\operatorname{mod} B$. The *one-point extension algebra* $B[R]$ of $B$ by $R$ is by definition the finite-dimensional $k$-algebra

$$B[R] = \begin{pmatrix} B & R \\ 0 & k \end{pmatrix}$$

with multiplication

$$\begin{pmatrix} b & r \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} b' & r' \\ 0 & \lambda' \end{pmatrix} = \begin{pmatrix} bb' & br' + r\lambda' \\ 0 & \lambda\lambda' \end{pmatrix}$$

where $b$, $b' \in B$, $r$, $r' \in R$ and $\lambda$, $\lambda' \in k$.

There is the dual concept of a one-point coextension (cf. [Ri1]).

Let $B$ be a finite-dimensional $k$-algebra and let $R \in \operatorname{mod} B$. We denote by $A$ the one-point extension of $B$ by $R$. Let $\omega \in A$ be a primitive idempotent such that $R = \operatorname{rad} A\omega$. If $A$ is hereditary then clearly $B$ is hereditary and $R$ is a projective $B$-module.

We have a full exact embedding of $\operatorname{mod} B$ into $\operatorname{mod} A$. And we identify $\operatorname{mod} B$ with the full subcategory containing the $A$-modules $X$ such that $\operatorname{Hom}_A(A\omega, X) = 0$.

### 1.2. Derived categories

Let $A$ be a finite-dimensional $k$-algebra. An $A$-module ${}_AM$ is called a *tilting module* if (i) $\operatorname{pd}_A M \leqslant 1$, (ii) $\operatorname{Ext}^1_A(M, M) = 0$, (iii) there exists an exact sequence

$$0 \to {}_AA \to M^0 \to M^1 \to 0$$

with $M^0$, $M^1 \in \operatorname{add} M$ where $\operatorname{add} M$ denotes the additive category generated by the module $M$.

Let $D^{\mathrm{b}}(A)$ be the derived category of bounded complexes over $\operatorname{mod} A$. Recall that $\operatorname{mod} A$ is fully embedded into $D^{\mathrm{b}}(A)$ by sending a module ${}_AX$ to a complex which is concentrated in degree zero. We will identify $\operatorname{mod} A$ with its image in $D^{\mathrm{b}}(A)$. Let $T$ be the translation functor on $D^{\mathrm{d}}(A)$. Then the following formula is useful:

$$\operatorname{Hom}_{D^{\mathrm{b}}(A)}(X, T^i Y) \simeq \operatorname{Ext}^i_A(X, Y) \quad \text{for } X, Y \in \operatorname{mod} A \text{ and } i \in \mathbf{Z}.$$

Let $A = B[R]$ be a hereditary finite-dimensional $k$-algebra. Then it is straightforward to see that the embedding of $\operatorname{mod} B$ into $\operatorname{mod} A$ extends to a full and faithful exact functor of triangulated categories between $D^{\mathrm{b}}(B)$ and $D^{\mathrm{b}}(A)$. Moreover, it is easily seen that $D^{\mathrm{b}}(B)$ can be identified with the full subcategory of $D^{\mathrm{b}}(A)$ containing the complexes $X^{\cdot}$ such that $\operatorname{Hom}_{D^{\mathrm{b}}(A)}(T^i A\omega, X^{\cdot}) = 0$ for all $i \in \mathbf{Z}$ where $R = \operatorname{rad} A\omega$.

Let ${}_AM$ be a tilting module and $C = \operatorname{End}_A M$. Then there exists a triangle equivalence $F\colon D^{\mathrm{b}}(A) \to D^{\mathrm{b}}(C)$ such that $F({}_AM) = {}_CC$. In fact, $F$ is the right derived functor of $\operatorname{Hom}_A(M, -)$ (cf. [H]).

We finally recall from [H] some of the results on the structure of $D^{\mathrm{b}}(A)$ if $A$ is a finite-dimensional hereditary $k$-algebra. Most important to us is that an indecomposable object $X^{\cdot}$ in $D^{\mathrm{b}}(A)$ is isomorphic to $T^i X$ for some indecomposable $A$-module $X$ and for some $i \in \mathbf{Z}$. Let $A = k\vec{\Delta}$ be representation-infinite. Then the components of the quiver of $D^{\mathrm{b}}(A)$ are either of the form $\mathbf{Z}\vec{\Delta}$ or of the same form as the regular components of the Auslander–Reiten quiver of $A$. The quiver of $D^{\mathrm{b}}(A)$ has the structure of a stable translation quiver. We also recall that the translation $\tau$ is induced by an equivalence, again denoted by $\tau$, on $D^{\mathrm{b}}(A)$. Let $\mathscr{C}_A[0]$ be the component containing the indecomposable projective $A$-modules. Then $\mathscr{C}_A[0]$ is of the form $\mathbf{Z}\vec{\Delta}$.

## 2. Rigid transjective modules

We keep the notation from Section 1.

Let $A = k\vec{\Delta}$, where $\vec{\Delta}$ is a finite and connected quiver without oriented cycles. Assume that we can write $A$ in the form $A = B[R]$, where $R = \operatorname{rad} A\omega$ for some primitive idempotent $\omega \in A$. Let $B = k\vec{\Delta}^0$. An indecomposable $A$-module ${}_AX$ is called *transjective* if ${}_AX$ is either preprojective or preinjective. A transjective $B$-module ${}_BY$ is called *rigid* if ${}_AY$ is transjective. If $B$ is representation-infinite then clearly an indecomposable preprojective $B$-module ${}_BY$ is not preinjective when considered as an $A$-module. Dually, in this case an indecomposable preinjective $B$-module ${}_BY$ is not preprojective when considered as an $A$-module.

Let $\mathscr{N}_p$ (resp. $\mathscr{N}_q$) be the full subcategory of $\operatorname{mod} A$ containing the rigid preprojective (resp. preinjective) $B$-modules. Let $\mathscr{N} \subset \operatorname{mod} B$ be the union of $\mathscr{N}_p$ and $\mathscr{N}_q$. As mentioned in the introduction, this category contains up to isomorphism only finitely many indecomposable modules.

We will now give a different description of $\mathscr{N}$. Let

$$\mathscr{M} = \{X^{\cdot} \in \mathscr{C}_A[0] \mid \operatorname{Hom}_{D^{\mathrm{b}}(A)}(A\omega, T^i X^{\cdot}) = 0 \text{ for all } i \in \mathbf{Z}\}.$$

We will consider $\mathcal{M}$ also as a full subcategory of $D^b(B)$ (cf. 1.2) and clearly $\mathcal{M} \subset \mathscr{C}_B[0] \simeq \mathbf{Z}\vec{\Delta}^0$. Let $X^\cdot \in \mathcal{M}$. Then either $X^\cdot$ is a rigid preprojective $B$-module or $TX^\cdot$ is a rigid preinjective $B$-module. In other words, $\mathrm{Ob}(\mathcal{M})$ is the union of $\mathrm{Ob}(\mathcal{N}_p)$ and $T^-\mathrm{Ob}(\mathcal{N}_q)$ (where $\mathrm{Ob}(\mathscr{C})$ denotes the class of objects in a given category $\mathscr{C}$). Observe that an indecomposable projective $B$-module is always rigid.

Let $\mathscr{S}$, $\mathscr{S}'$ be complete slices of $\mathbf{Z}\vec{\Delta}$ such that there exists a path from a vertex in $\mathscr{S}$ to a vertex in $\mathscr{S}'$ and there is no vertex $p$ of $\mathbf{Z}\vec{\Delta}$ with $p \notin \mathscr{S} \cup \mathscr{S}'$ admitting a path to a vertex in $\mathscr{S}$ and being the endpoint of a path starting at a vertex of $\mathscr{S}'$. Then we denote by $[\mathscr{S}, \mathscr{S}']$ the full subtranslation quiver of $\mathbf{Z}\vec{\Delta}$ containing the vertices $p$ admitting a path to a vertex in $\mathscr{S}'$ and being the endpoint of a path starting at a vertex of $\mathscr{S}$. We call $\mathscr{S}$ (resp. $\mathscr{S}'$) the *left* (resp. *right*) *border complete slice* of the *interval* $[\mathscr{S}, \mathscr{S}']$.

PROPOSITION. *Let $A = B[R]$ be as above. Then there exist complete slices $\mathscr{S}$, $\mathscr{S}'$ of $\mathscr{C}_B[0]$ such that $\mathcal{M} = [\mathscr{S}, \mathscr{S}']$.*

*Proof.* Let $X^\cdot \in \mathcal{M}$. First we assume that $X^\cdot \in \mathcal{N}_p$. If $X^\cdot$ is not $B$-projective then $Y^\cdot \in \mathcal{M}$ whenever there exists an arrow $Y^\cdot \to X^\cdot$ in $\mathbf{Z}\vec{\Delta}^0$, since otherwise there would exist in $\mathrm{mod}\,A$ a nonzero map from a regular $A$-module to a preprojective $A$-module, an absurdity. Next suppose that $X^\cdot = Be$ for some primitive idempotent $e \in B$. Let $Y^\cdot \in \mathbf{Z}\vec{\Delta}^0$ such that there exists an arrow $Y^\cdot \to X^\cdot$. If $Y^\cdot$ is $B$-projective then clearly $Y^\cdot \in \mathcal{M}$. Otherwise $TY^\cdot$ is $B$-injective. If $Y^\cdot \in \mathcal{M}$ then $Z^\cdot \in \mathcal{M}$ whenever there exists an arrow $Y^\cdot \to Z^\cdot$ in $\mathbf{Z}\vec{\Delta}^0$. This shows the assertion in this case.

If $TX^\cdot \in \mathcal{N}_q$ then we infer by duality that $Y^\cdot \in \mathcal{M}$ whenever there exists an arrow $X^\cdot \to Y^\cdot$ in $\mathbf{Z}\vec{\Delta}^0$.

Thus there exist complete slices $\mathscr{S}$, $\mathscr{S}'$ of $\mathbf{Z}\vec{\Delta}^0$ such that $\mathcal{M} = [\mathscr{S}, \mathscr{S}']$.

We call $\mathcal{M}$ the *translation quiver of the rigid transjective B-modules.*

From now on we assume that $A = k\vec{\Delta} = B[R]$, and that the underlying graph $\Delta$ of $\vec{\Delta}$ is a tree. Let $B = k\vec{\Delta}^0$. Then $R = \mathrm{rad}\,A\omega$ is an indecomposable $B$-projective module, which is still projective when considered as an $A$-module.

Next we show that $\mathscr{S}$ and $\mathscr{S}'$ are independent of the orientation of $\vec{\Delta}^0$.

For this let $A' = k\vec{\Delta}' = B'[R']$ with $B' = k\vec{\Delta}^{0\prime}$ and $R' = \mathrm{rad}\,A'\omega'$ be given with the property that $\Delta = \Delta'$ and $\Delta^0 = \Delta^{0\prime}$. Let $\mathcal{M} = [\mathscr{S}, \mathscr{S}'] \subset \mathscr{C}_B[0]$ be the translation quiver of the rigid $B$-modules and $\mathcal{M}' = [\mathscr{T}, \mathscr{T}'] \subset \mathscr{C}_{B'}[0]$ the translation quiver of the rigid $B'$-modules defined above.

THEOREM. *$\mathcal{M}$ and $\mathcal{M}'$ are isomorphic translation quivers. In particular, $\mathscr{S} = \mathscr{T}$ and $\mathscr{S}' = \mathscr{T}'$.*

*Proof.* It suffices to prove the assertion for a fixed orientation of $\vec{\Delta}$. Assume that the vertex $\omega$ (corresponding to the primitive idempotent $\omega$ with $R = \mathrm{rad}\,A\omega$) is the only source of $\vec{\Delta}$. Then there exists a multiplicity-free tilting module ${}_AM$ such that $A' = \mathrm{End}_A M$. We can even assume that $A\omega$ is a direct summand of ${}_AM$ and that ${}_AM$ contains an additional projective direct summand. Then we have a decomposition ${}_AM = A\omega \oplus \tilde{M}$. Then $\tilde{M}$ is a tilting module in $\mathrm{mod}B$ such that $\mathrm{End}_B \tilde{M} = B'$ and that $\tilde{M}$ contains a projective direct summand. Then $\mathrm{Hom}_B(\tilde{M}, -)$ induces a triangle equivalence $D^b(B) \to D^b(B')$ and an isomorphism of translation quivers $f\colon \mathscr{C}_B[0] \to \mathscr{C}_{B'}[0]$.

Observe that the indecomposable summands of $\tilde{M}$ are contained in $\mathscr{M}$.

Let $X^\cdot \in \mathscr{M}$. We first show that $f(X^\cdot) \in \mathscr{M}'$ whenever $X^\cdot \in \mathscr{S} \cup \mathscr{S}'$.

If $X = X^\cdot \in \mathscr{S}'$, then the $B$-module ${}_BX$ is a torsion module in the torsion theory induced by $\tilde{M}$ on $\mathrm{mod}B$. Thus $f(X^\cdot) = \mathrm{Hom}_B(\tilde{M}, X) = {}_{B'}Y$. Suppose that ${}_{B'}Y$ is not $A'$-preprojective. Then there exist infinitely many nonisomorphic indecomposable $A'$-modules $Z'$ such that $\mathrm{Hom}_{A'}(Z', Y) \neq 0$. Since almost all $A'$-modules are torsion-free with respect to the torsion theory induced by $M$ on $\mathrm{mod}A'$, there exist infinitely many nonisomorphic indecomposable $A'$-modules $Z'$ of the form $\mathrm{Hom}_A(M, Z)$ with $\mathrm{Hom}_{A'}(Z', Y) \neq 0$. Thus $\mathrm{Hom}_A(Z, X) \neq 0$ for infinitely many nonisomorphic indecomposable $A$-modules $Z$, a contradiction.

If $X^\cdot \in \mathscr{S}$, then either $X = X^\cdot$ is $B$-projective or the $B$-module $Y = TX^\cdot$ is preinjective. The second case follows in the same way as above. In case $X$ is $B$-projective, clearly $X$ is not a direct summand of $\tilde{M}$. So $X$ is torsion-free with respect to the torsion theory induced by $\tilde{M}$ on $\mathrm{mod}B$. But then $X$ is also torsion-free with respect to the torsion theory induced by $M$ on $\mathrm{mod}A$. In particular, $\mathrm{Ext}^1_A(M, X)$ is $A'$-preinjective, but $\mathrm{Ext}^1_A(M, X) = \mathrm{Ext}^1_B(\tilde{M}, X)$ and $f(X^\cdot) = T^- \mathrm{Ext}^1_B(\tilde{M}, X)$ shows the assertion in this case.

Thus $f(\mathscr{M})$ is contained in $\mathscr{M}'$.

Analogously we can disprove the assumption that the above inclusion is proper. This shows that $\mathscr{M}$ and $\mathscr{M}'$ are isomorphic as translation quivers.

COROLLARY. *Let $A = B[R]$ and let $\mathscr{M} = [\mathscr{S}, \mathscr{S}']$ be as above. Then $\mathscr{S}^* = \mathscr{S}'$, where $\mathscr{S}^*$ denotes the opposite quiver of $\mathscr{S}$.*

*Proof.* Let $A = B[Ba]$ for a primitive idempotent $a$ in $B$ and let $A' = [D(aB)]B$, where $D$ denotes the standard duality with respect to the ground field. Let $\mathscr{M} = [\mathscr{S}, \mathscr{S}']$ be the translation quiver of the rigid transjective $B$-modules defined above. Using one-point coextensions by injective $B$-modules we may similarly define

$$\mathscr{M}' = \{X^\cdot \in \mathscr{C}'_A[0] \mid \mathrm{Hom}_{D^b(A')}(T^i X^\cdot, D(\omega' A')) = 0 \text{ for all } i \in \mathbf{Z}\}.$$

Note that $\omega'$ denotes the idempotent in $A'$ corresponding to the coextension vertex. We will call $\mathscr{M}'$ also the translation quiver of the rigid transjective $B$-modules, for it is easily seen that $\mathscr{M}$ and $\mathscr{M}'$ are isomorphic as translation quivers using arguments similar to the proof of the theorem above. Note that we consider $\mathcal{M}'$ also as a subtranslation quiver of $\mathscr{C}_B[0]$. So in particular if $\mathcal{M}' = [\mathscr{T}, \mathscr{T}']$ we infer that $\mathscr{S}' = \mathscr{T}'$. On the other hand, we may also consider $A^* = [D(aB^*)]B^*$ the opposite algebra of $A$. Then let $\mathcal{M}'' = [\mathscr{V}, \mathscr{V}']$ be the translation quiver of the rigid transjective $B^*$-modules in the sense just defined. Then $\mathcal{M}' \simeq \mathcal{M}''$ by the dual of the theorem above, hence $\mathscr{V}' = \mathscr{S}'$. But $\mathscr{V}' = \mathscr{S}^*$ by duality, thus $\mathscr{S}' = \mathscr{T}' = \mathscr{S}^*$.

We point out that for the proof of the theorem and the corollary we did not actually need the rather strong assumption that $\Delta$ is a tree. We only used the fact that $A' = \operatorname{End}_A M$, where ${}_A M$ is a tilting module such that $A\omega$ and an indecomposable $B$-projective module are direct summands of ${}_A M$.

## 3. Examples

We keep the notation from the previous sections.

Let $A = k\vec{\Delta}$ be representation-tame, where $\Delta$ is a finite tree. Thus $\Delta$ is of the form $\tilde{\mathbf{D}}_n$, $\tilde{\mathbf{E}}_6$, $\tilde{\mathbf{E}}_7$ or $\tilde{\mathbf{E}}_8$. Moreover, we assume that $A = B[R]$, where $R$ is the indecomposable projective $B$-module dominating the wing module of $B$ [Ri1]. Let $b$ be the branching point of $\Delta$. Then it follows from [Ri1] that the right border complete slice $\mathscr{S}'$ of $\mathcal{M}$ has the subspace orientation with $b$ being the unique sink.

If $A = k\vec{\Delta}$ is wild, then the right border complete slice $\mathscr{S}'$ of $\mathcal{M}$ usually has more than one sink. For example this happens if $\Delta = T_{134}$ (in the notation of [U]). There does not exist an algebra $B$ and an indecomposable $B$-module $R$ such that $A = B[R]$ and the right border complete slice $\mathscr{S}'$ of $\mathcal{M}$ has a unique sink.

## References

[H] D. Happel, *On the derived category of a finite-dimensional algebra*, Comment. Math. Helv. 62 (1987), 339–389.

[Ri1] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, Heidelberg 1984.

[Ri2] —, *The regular components of the Auslander–Reiten quiver of a tilted algebra*, Chinese Ann. Math. 9B(1) (1988), 1–18.

[U] L. Unger, *Lower bounds for faithful, preinjective modules*, Manuscripta Math. 57 (1986), 1–31.