TOPICS IN ALGEBRA
BANACH CENTER PUBLICATIONS, VOLUME 26, PART 1
PWN-POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

A GENERALIZED PICARD GROUP FOR PRIME RINGS

S. MONTGOMERY

University of Southern California
Los Angeles, California, U.S.A.

Let R be a prime ring. The generalized Picard group is defined to be $p(R) = X\text{-inn}\,R/\text{Inn}\,R$, the quotient group of X-inner automorphisms of R modulo the subgroup of inner automorphisms. X-inner automorphisms were introduced in ring theory in 1975 [Kh75] and about the same time in C^*-algebras. They have had many uses in studying fixed rings and noncommutative Galois theory, ideals in crossed products, and in computing automorphism groups of some specific rings. Thus there are a number of papers in the literature describing X-inner automorphisms.

In this paper we look more specifically at the quotient group $p(R)$. First we discuss why it is an analog of the classical Picard group (or more precisely the central Picard group). We then give a number of examples of $p(R)$, which can be extracted from the known results on X-inner automorphisms. We then return to $p(R)$ and examine how it behaves on the ring $M_n(R)$ of $n \times n$ matrices over R. Finally, we include a discussion of the symmetric ring of quotients and the relationship between X-inner automorphisms and the "partly inner" automorphisms used in C^*-algebras.

The author would like to thank R. M. Guralnick for some helpful suggestions in § 3, particularly concerning the Proposition and Corollary. She also acknowledges support from NSF Grant DMS 87-00641.

§ 1. X-inner automorphisms and invertible ideals

We first recall the definition of X-inner automorphisms. Let $Q_l(R)$ denote the left Martindale quotient ring of R; that is, $Q_l(R) = \varinjlim_{0 \neq I} \text{Hom}({}_R I, R)$, the direct

---

This paper is in final form and no version of it will be submitted for publication elsewhere.

limit over the nonzero ideals I of R, where $\mathrm{Hom}({}_RI, R)$ denotes the left R-module maps from I to R. The ring R imbeds into $Q_l(R)$ via $r \to r_R$, right multiplication by r on $I = R$. If $\sigma \in \mathrm{Aut}\, R$, then σ has a unique extension to $Q_l(R)$. We say σ is *X-inner* if σ becomes inner on $Q_l(R)$.

In fact the definition of X-inner is left-right symmetric (since σ is X-inner if and only if there exist $0 \neq a, b, c, d \in R$ such that $arb = cr^{\sigma}d$, for all $r \in R$ [MP 84]), and thus one could have used the right Martindale quotient ring $Q_r(R)$, or the *symmetric* quotient ring $Q_s(R) = Q_l(R) \cap Q_r(R)$, which is what we shall use here. We will write $Q_s(R) = Q(R)$, or just Q when there is no ambiguity about which ring is involved. The center C of Q is called the *extended center* of R, and is known to be a field.

One reference for $Q_l(R)$ and X-inner automorphisms is [M80]. $Q_s(R)$ will be discussed further in Section 4.

We require a few more facts. If σ is X-inner on R, induced by $q \in Q$, then $qR = Rq$, that is, q is R-normalizing. Conversely, if $0 \neq x \in Q$ is an R-normalizing element, then x is a unit in Q and so determines a unique X-inner automorphism σ_x, via $x\sigma_x(r) = rx$, for all $r \in R$ [M80, p. 43].

We now consider ideals. An *R-ideal of Q* is an R-R-subbimodule I of Q; I is *invertible* if there exists an R-ideal J of Q such that $IJ = JI = R$. The set $\mathscr{I}(R)$ of all invertible R-ideals of Q is a group under multiplication. An R-ideal I is *principal* if there exists $x \in Q$ such that $I = xR = Rx$; if $x \neq 0$, then by the above remark $x^{-1} \in Q$ and so I is invertible. Let $\mathscr{P}(R)$ denote the subgroup of $\mathscr{I}(R)$ of all nonzero principal ideals. $\mathscr{C}(R)$ denotes the subgroup of central principal ideals; that is, $I \in \mathscr{C}(R)$ if $I = cR$, some $c \in C$.

Our first lemma is essentially due to K. A. Brown. Although he states it in the situation when R is Noetherian and Q is the classical quotient ring, the same proof works [B, Lemma 4.2].

LEMMA 1. *For any prime ring R, $p(R) \cong \mathscr{P}(R)/\mathscr{C}(R)$.*

Proof. Define $\varphi\colon \mathscr{P}(R) \to X\text{-inn}\, R/\mathrm{Inn}\, R = p(R)$ by $\varphi(Rx) = \sigma_x \mathrm{Inn}\, R$, where $Rx = xR$ and σ_x denotes conjugation by x as above. φ is well defined, for if $Rx = Ry$, then $x = uy$ for some unit u of R, and so $\sigma_x \equiv \sigma_y \pmod{\mathrm{Inn}\, R}$. Clearly φ is onto. Also $\mathrm{Ker}\, \varphi = \mathscr{C}(R)$, for

$$\begin{aligned} I = Rx \in \mathrm{Ker}\, \varphi &\Leftrightarrow \sigma_x \in \mathrm{Inn}\, R \\ &\Leftrightarrow x = cu, \text{ for some } c \in C \text{ and some unit } u \text{ of } R \\ &\Leftrightarrow I = Rx = Rc \in \mathscr{C}(R). \end{aligned}$$

The lemma motivates our calling $p(R)$ a generalized Picard group, although it is actually closer to the central Picard group [R, p. 320], or central class group [B]. The p also suggests that we are considering principal ideals. As is true classically, a ring R will be considered good if $p(R)$ is "small"; that is, $p(R)$ is finite or at least abelian.

§ 2. Examples

As mentioned in the introduction, many examples of $p(R)$ can be obtained from known results on X-inner automorphisms. We survey many of those results here.

A. PI rings and Azumaya algebras

If R is prime PI with center Z, and $g \in \operatorname{Aut} R$ fixes Z, then g fixes the center of $Q = RZ^{-1}$ and so is inner on Q by the Skolem–Noether theorem. That is, g is X-inner. Conversely, any $g \in X\text{-inn}\, R$ fixes Z. Thus $X\text{-inn}\, R = \operatorname{Aut}_Z R$, and so $p(R) = \operatorname{Aut}_Z R/\operatorname{Inn} R = \operatorname{Out}_Z R$.

If R is in addition an Azumaya algebra of rank n^2 over Z, then $\operatorname{Out}_Z R$ is abelian and n-torsion, by work of Rosenberg–Zelinsky and Knus–Ojanguren (see [KO]). If also R is affine over a field k of characteristic 0, it is known that $\operatorname{Out}_Z R$ is finite.

For general PI rings, however, $\operatorname{Out}_Z R$ is not necessarily well behaved, and can even be a simple group. A subgroup of $\operatorname{Out}_Z R = p(R)$ is better behaved: let $\operatorname{Loc\,Inn} R$ denote the "locally inner" automorphisms of R: that is, all $\sigma \in \operatorname{Aut}_Z R$ such that σ becomes inner on every localization of R at a maximal ideal of Z. Then $\operatorname{Loc\,Inn} R/\operatorname{Inn} R$ is always abelian [C].

B. Free algebras and free products

If R is a free algebra of rank $\geqslant 2$ over a field k, then it is known that $Q(R) = R$ [Kh78]. Consequently $X\text{-inn}\, R \subseteq \operatorname{Inn} R = \langle 1 \rangle$ and so $p(R) = \langle 1 \rangle$.

More generally, free products (or coproducts) have been studied in the papers [MaM], [LiMa], [Ma]. The best result so far is that if D is a division ring and R_1, R_2 are von Neumann finite D-rings, then every X-inner automorphism of $R = R_1 \amalg_D R_2$ is inner, except for a few special cases [Ma] (the only exception when D is a field and the R_i are domains occurs if $[R_1{:}D] = [R_2{:}D] = 2$ [MaM]). Thus $p(R) = \langle 1 \rangle$ for such rings.

We do not know whether or not $Q(R) = R$ in this situation (excluding the special cases).

C. Enveloping algebras of Lie algebras and Ore extensions

Let L be a Lie algebra over a field k and $R = U(L)$ its universal enveloping algebra. An element $q \in Q(R)$ is called a semi-invariant for the action of $\operatorname{ad} L$ if there exists $\lambda \in L^*$, the dual of L, such that $\operatorname{ad} x(q) = [x, q] = \lambda(x)q$, for all $x \in L$. If q is a semi-invariant, then q is R-normalizing, so determines an X-inner automorphism σ given by $\sigma(x) = q^{-1}xq = x + \lambda(x)$, for all $x \in L$. Conversely, it is shown in [M81] that any X-inner automorphism is of this form. Since $U(L)$ has no nontrivial units, $\operatorname{Inn} R = \langle 1 \rangle$, and thus $p(R) = X\text{-inn}\, R$ can be identified with an additive subgroup of L^*. Therefore $p(R)$ is abelian; if also k has characteristic 0, $p(R)$ is torsion-free.

For any domain R, let $S = R[x; d]$ be an Ore extension for d a derivation of R. In [M83], it is proved that $p(S)$ is an extension of a subquotient of $C - \{0\}$ by a subgroup of $p(R)$. In particular, $p(S)$ is abelian provided $p(R) = 1$. A result is also proved for iterated Ore extensions, giving another proof that $p(R)$ is abelian for $R = U(L)$, when L is a solvable Lie algebra.

D. Group algebras and crossed products

If $R = kG$ is a prime group algebra over the field k, let W denote the set of X-inners of kG which normalize the group k^*G of trivial units. Then [MP82] it can be proved that $W/W \cap \operatorname{Inn} R$ is a periodic abelian group. If G is a right ordered group, then $W = X\text{-inn}\, R$ and thus $p(R)$ is periodic abelian. If also $\Delta(G) = 1$, where Δ is the f.c. subgroup of G, then $p(R) = 1$. In particular, this applies to the case when G is a free group, although the fact that $p(R) = 1$ for $R = kG$, G free, can also be obtained from the free product results [MaM].

The description of W is also of interest. [MP81] proves that $\sigma \in W \Leftrightarrow \sigma = \sigma_1\sigma_2\sigma_3$, where $\sigma_1 \in \operatorname{Inn} G$, σ_3 fixes a subgroup H of G of finite index, and σ_2 is X-inner of "scalar type", that is, there exists a linear character $\lambda\colon G \to k$ such that $\sigma_2(g) = \lambda(g)g$, for all $g \in G$. Such an automorphism is X-inner if and only if there exists $\alpha \in kG$ such that $\alpha^{-1}g\alpha = \lambda(g)g$, or $g\alpha g^{-1} = \lambda(g)\alpha$, for all g; this says that α is a semi-invariant for the action of $\operatorname{Ad} G$, and is analogous to what happened for $U(L)$.

In a related result, [B] considers group rings RG for R a commutative Noetherian UFD and G a dihedral-free polycyclic-by-finite group with $\Delta^+(G) = 1$. He proves that for such group rings,

$$p(RG) \cong H^1(G/C_G(\Delta), R^* \times \Delta)$$

where R^* denotes the units of R. As a consequence, he obtains a different proof of a result in [MP83].

X-inners of crossed products $R*G$ are studied in [MP86]. When R and $R*G$ are prime, results are obtained about the group $\mathscr{X}$ of X-inners of $R*G$ which normalize both R and the group of trivial units of $R*G$; again an appropriate quotient of $\mathscr{X}$ is a torsion abelian group. A major special case occurs when $R = U(L)$, an enveloping algebra. Here the skew group ring $H = U(L)G$ is actually a Hopf algebra, provided G acts as automorphisms of L. As for enveloping algebras and group algebras, the semi-invariants for the action of $\operatorname{ad} H$ on itself give examples of X-inner automorphisms of H.

E. Generic matrices and their trace rings

Let $R = k\{X_1, \ldots, X_d\}$ be the ring of d generic $n \times n$ matrices, where $d, n \geqslant 2$. It was proved independently in [LvKh] and [M81] that X-inn R is trivial; thus $p(R) = \langle 1 \rangle$.

More generally, if T is the trace ring of R, then again X-inn T is trivial and so $p(T) = \langle 1 \rangle$, with one exception: the case $n = d = 2$. In this case $X\text{-inn}\, T \cong \mathbf{Z}_2$; the unique nontrivial automorphism is given by $X_i^\sigma = -X_i + (\operatorname{tr} X_i)I$, $i = 1, 2$. This result is due to [L] in characteristic 0 and to [GM] in characteristic $p \neq 0$.

F. Finally, $p(R)$ can be arbitrary. In [MP83] it is shown that for any group G, there exists a subring R of some group algebra such that $p(R) \cong G$.

§ 3. $p(R)$, module isomorphisms, and matrices

In this section we return to our general consideration of $p(R)$ and its properties. We first show that $\mathscr{P}(R)$ can be viewed as ideals module-isomorphic to R.

LEMMA 2. *Assume $a \in Q$ such that $I = aR$ is an invertible R-ideal of Q and such that a is right regular on R. Then $aR = Ra$, and thus $a^{-1} \in Q$ and $I \in \mathscr{P}(R)$.*

Proof. Since $a \in Q$, there exists an ideal K of R such that $0 \neq aK \subseteq R$. Moreover, since I is an R-ideal, $R(aK) = (Ra)K \subseteq (aR)K = aK$, and thus aK is an ideal of R. Define f: $aK \to R$ by $f(ak) = k$; f is well defined since a is right regular. Clearly f is a right R-map, and thus determines an element $\tilde{f} \in Q_r(R)$. Letting a_L denote left multiplication by a, both $f \circ a_L$ and $a_L \circ f$ are the identity on aK. Thus in Q_r, $\tilde{f} = (\tilde{a}_L)^{-1}$. But $a \to a_L$ is just the imbedding of R in Q_r. Thus $a^{-1} \in Q_r$.

Now $Ra \subseteq aR$ implies $a^{-1}R \subseteq Ra^{-1}$, and thus $J = Ra^{-1}$ is an R-ideal of Q_r with $JI = R$. But I is invertible in Q, and so in Q_r; it follows that $J = I^{-1}$. Thus $J \subseteq Q = Q_s$ and in particular $a^{-1} \in Q$. Finally, $R = IJ = aRa^{-1}$, and so $aR = Ra$.

The lemma raises an interesting question: namely, what happens if I is not assumed to be invertible? That is, if R is prime and $a \in Q$ such that $Ra \subseteq aR$, when must a be R-normalizing? R. Guralnick has proved that a is normalizing if R is a finite module over its center. We do not know what happens when R is an affine PI algebra.

PROPOSITION. *For any $I \in \mathscr{I}(R)$, the following are equivalent:*

(1) *$I \cong R$ as right R-modules.*
(2) *$I \cong R$ as left R-modules.*
(3) *$I \in \mathscr{P}(R)$.*

Proof. Clearly (3) implies (1) and (2), since if $I = Ra = aR$ for some unit $a \in Q$, $I \cong R$ both as a right and left R-module.

(1) $\Rightarrow$ (3). Let $\varphi\colon R \to I$ be a right R-module isomorphism and let $a = \varphi(1) \in I \subseteq Q$. Since φ is one-to-one, a is right regular, and since φ is onto, $I = aR$. Thus by Lemma 2, $a^{-1} \in Q$ and $aR = Ra$, proving (3). (2) $\Rightarrow$ (3) uses the left version of Lemma 2.

We now proceed to matrix rings. For convenience we will write $S = M_n(R)$, the ring of $n \times n$ matrices over R. The proof of the next lemma is straightforward.

LEMMA 3. *Let $S = M_n(R)$, for R a prime ring.*

(1) $Q(S) = M_n(Q(R))$.

(2) *I is an S-ideal of $Q(S) \Leftrightarrow I = M_n(J)$, for J an R-ideal of Q.*

(3) *If $I = M_n(J)$, then $I \in \mathscr{I}(S) \Leftrightarrow J \in \mathscr{I}(R)$.*

For any R-module V, $V^{(n)}$ denotes the direct sum of n copies of V.

COROLLARY. *For any $I \in \mathscr{I}(R)$, the following are equivalent:*

(1) *$I^{(n)} \cong R^{(n)}$ as right R-modules.*

(2) *$I^{(n)} \cong R^{(n)}$ as left R-modules.*

(3) $M_n(I) \in \mathscr{P}(M_n(R))$.

Proof. From the Morita correspondence, $I^{(n)} \cong R^{(n)}$ as right R-modules if and only if $M_n(I) \cong M_n(R)$ as right $M_n(R)$-modules; similarly on the left. The corollary now follows from the Proposition.

We may now define $\mathscr{I}^n(R)$ to be the set of all $I \in \mathscr{I}(R)$ which satisfy any of the equivalent conditions of the corollary. It is easy to see that $\mathscr{I}^n(R)$ is a subgroup of $\mathscr{I}(R)$. Note that $\mathscr{I}^1(R) = \mathscr{P}(R)$.

THEOREM. $p(M_n(R)) \cong \mathscr{I}^n(R)/\mathscr{C}(R)$.

Proof. By Lemma 1, $p(M_n(R)) \cong \mathscr{P}(M_n(R))/\mathscr{C}(M_n(R))$. Since the center of $M_n(Q)$ is just scalar matrices over $C = C(R)$, $M_n(I) \in \mathscr{C}(M_n(R))$ if and only if $I \in \mathscr{C}(R)$. The theorem now follows from the corollary and the definition of $\mathscr{I}^n(R)$.

We note an easy consequence of the theorem: $p(M_n(R))$ will be abelian whenever $\mathscr{I}^n(R)$ is abelian. More generally, we are interested in the relationship between $p(R)$ and $p(M_n(R))$. Since $p(R) \cong \mathscr{I}^1(R)/\mathscr{C}(R)$, it is clearly a subgroup of $p(M_n(R))$. This could of course be seen directly; X-inners of R extend to X-inners of $S = M_n(R)$ by conjugating by the appropriate "scalar" matrix. Identifying X-inn R with its image in X-inn S, X-inn $R \cap \operatorname{Inn} S = \operatorname{Inn} R$. Thus $p(R) = X\text{-inn}\, R/\operatorname{Inn} R \subset X\text{-inn}\, S/\operatorname{Inn} S = p(S)$. One might hope that in general $p(R)$ would be a normal subgroup of $p(M_n(R))$. However, this is false.

EXAMPLE. *Let R have $\mathbf{Z}$-basis* $\{1, x, y, xy \mid x^2 = -1,\ y^2 = -5,\ xy$

$= -xy\}$; *thus R is an order in a generalized quaternion (division) algebra D and R has center $\mathbf{Z}$. Then $\mathrm{Out}_{\mathbf{Z}}R$ is not normal in $\mathrm{Out}_{\mathbf{Z}}M_2(R)$.*

Proof (sketch). There are several ways to see this. First is a direct computation, which was done jointly with D. S. Passman. We show first that $\mathrm{Aut}_{\mathbf{Z}}R = \langle \sigma_{1+x}, \sigma_y \rangle$, where σ_{1+x} denotes conjugation by $1+x$ and σ_y by y. Thus $\mathrm{Aut}_{\mathbf{Z}}R \cong \mathbf{D}_4$, the dihedral group of 8 elements. Also $\mathrm{Inn}\, R = \langle \sigma_x \rangle$ and $\sigma_x = (\sigma_{1+x})^2$; consequently $\mathrm{Out}_{\mathbf{Z}}R \cong \mathbf{Z}_2 \times \mathbf{Z}_2$.

Letting $S = M_2(R)$, extend $\mathrm{Aut}_{\mathbf{Z}}R$ to $\mathrm{Aut}\, S$. We then show that the element

$$A = \begin{pmatrix} 2 & -1+y \\ 1+y & -2 \end{pmatrix}$$

is S-normalizing, so $\sigma_A \in \mathrm{Aut}_{\mathbf{Z}}S$, and that $\sigma_A \sigma_{1+x} \sigma_{A^{-1}} = \sigma_B$, where

$$B = \begin{pmatrix} 1+x(-4+y) & -2xy \\ 2xy & 1+x(-4-y) \end{pmatrix}.$$

So far the arguments are relatively straightforward. However, to finish the proof, a much more tedious computation is needed to show that $\sigma_B \notin \mathrm{Aut}\, R \cdot \mathrm{Inn}\, S$, and so in $\mathrm{Out}_{\mathbf{Z}}S$, conjugation by the image of σ_A does not normalize $\mathrm{Out}_{\mathbf{Z}}S$.

An alternative to the last computation has been suggested by D. Estes and R. Guralnick. Since $\mathbf{Z}$ is a PID and S is a $\mathbf{Z}$-order, it follows from known results on orders that for any n,

$$\mathrm{Out}_{\mathbf{Z}}M_n(R) \subset \prod_{p \in \mathbf{Z}} \mathrm{Out}\, R_{(p)},$$

where p denotes a prime in $\mathbf{Z}$ and $R_{(p)}$ the localization at p. One can check that $\mathrm{Out}\, R_{(p)} = \langle 1 \rangle$ if $p \neq 2, 5$, that $\mathrm{Out}\, R_{(2)} \cong \mathbf{Z}_2$ and that $\mathrm{Out}\, R_{(5)} \cong S_3$. Thus $\mathrm{Out}_{\mathbf{Z}}S \subseteq \mathbf{Z}_2 \times S_3$. In fact $\mathrm{Out}_{\mathbf{Z}}S = \mathbf{Z}_2 \times S_3$, since $\mathrm{Out}_{\mathbf{Z}}R \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \subset \mathrm{Out}_{\mathbf{Z}}S$ and $\sigma_A \notin \mathrm{Out}_{\mathbf{Z}}R$. Clearly $\mathbf{Z}_2 \times \mathbf{Z}_2$ is not normal in $\mathbf{Z}_2 \times S_3$.

§ 4. Quotient rings, inner automorphisms, and C^*-algebras

In this section we discuss in more detail the symmetric ring of quotients and the relationship of X-inner automorphisms to the analogous definition for C^*-algebras.

We first review the multiplier algebra $M(A)$ of an algebra A [H]. $M(A)$ is defined to be the set of pairs (f, g), where f (respectively g) is a left (right) A-module map of A to itself, satisfying $f(a)b = ag(b)$, for all $a, b \in A$. $M(A)$ is an algebra under composition of maps. Notice that A imbeds in $M(A)$ via $a \to (a_R, a_L)$, where a_R (resp. a_L) denotes right (left) multiplication by a.

Although the ring $Q_s(R)$ was defined as $Q_l(R) \cap Q_r(R)$ in [Kh77], an equivalent formulation with the flavor of multipliers can be given [P]: let I, J be nonzero ideals of R, and say $f\colon {}_RI \to {}_RR$ and $g\colon J_R \to R_R$. The pair (f, g) is called *balanced* if $f(a)b = ag(b)$, for all $a \in I$, $b \in J$. Then $Q_s(R)$ may be defined as the direct limit of the balanced pairs (f, g) over pairs I, J of ideals of R (in fact, one may assume $I = J$ since $I \cap J \neq 0$). R imbeds in $Q_s(R)$ as above, via $r \to (r_R, r_L)$.

It is this point of view which is close to that used in C^*-algebras. If A is a prime C^*-algebra, consider the set $\mathscr{S}$ of nonzero closed ideals of A. Then $M^\infty(A)$ is defined to be the direct limit over $I \in \mathscr{S}$ of the multiplier algebras $M(I)$; this definition is due to G. Pedersen (see [E]). At first glance $M^\infty(A)$ appears to differ from $Q_s(A)$ in two major ways: first, only closed ideals are used, and second, the maps f, g are I-maps from I to I rather than A-maps from A to I. However, the second difference can be eliminated. For, since any closed ideal $I \neq 0$ is itself a C^*-algebra, $I^2 = I$. It is then easy to see that the two kinds of maps are the same. It follows that the only difference between $M^\infty(A)$ and $Q_s(A)$ is that only closed ideals are used in $M^\infty(A)$; in fact, $M^\infty(A)$ imbeds into $Q_s(A)$.

We may now define the C^*-analog of X-inner automorphisms. A (continuous) automorphism σ of a (prime) C^*-algebra A is called *partly inner* if σ becomes inner when extended to $M^\infty(A)$. Such automorphism are used in [Ri] to study prime crossed products, obtaining analogs of the algebraic results.

References

[B] K. A. Brown, *Class groups and automorphism groups of group rings*, Glasgow Math. J. 28 (1986), 79–86.

[C] D. T-M. Chi, *Automorphisms of prime Azumaya algebras*, Comm. Algebra 12 (1984), 1207–1212 (see also Ph.D. Thesis, University of Southern California, 1984).

[E] G. A. Elliott, *Automorphisms determined by multipliers on ideals of a C^*-algebra*, J. Funct. Anal. 23 (1976), 1–10.

[GM] R. M. Guralnick and S. Montgomery, *Normalizing elements of the trace ring of generic matrices*, Comm. Algebra 17 (1989), 1805–1813.

[H] G. Hochschild, *Cohomology and representations of associative algebras*, Duke Math. J. 14 (1947), 921–948.

[KO] M.-A. Knus and M. Ojanguren, *Théorie de la Descente et Algèbres d'Azumaya*,

[Kh75] V. K. Kharchenko, *Generalized identities with automorphisms*, Algebra i Logika 14 (1975), 215–237 (in Russian).

[Kh77] —, *Galois theory of semiprime rings*, ibid. 16 (1977), 313–363 (in Russian).

[Kh78] —, *Algebras of invariants of free algebras*, ibid. 17 (1978), 478–487 (in Russian). Lecture Notes in Math. 389, Springer, Berlin 1974.

[L] L. Le Bruyn, *Trace rings of generic matrices are unique factorization domains*, Glasgow Math. J. 28 (1986), 11–13.

[LiMa] A. Lichtman and W. S. Martindale, *The normal closure of the coproduct of domains over a division ring*, Comm. Algebra 13 (1985), 1643–1664.

[LvKh] I. V. L'vov and V. K. Kharchenko, *Normal elements of the algebra of generic matrices are central*, Sibirsk. Mat. Zh. 23 (1) (1982), 193–195 (in Russian).

[Ma] W. S. Martindale, *The normal closure of the coproduct of rings over a division ring*, Trans. Amer. Math. Soc. 293 (1986), 303–317.

[MaM] W. S. Martindale and S. Montgomery, *The normal closure of coproducts of domains*, J. Algebra 82 (1983), 1–17.

[M80] S. Montgomery, *Fixed Rings of Finite Automorphism Groups of Associative Rings*, Lecture Notes in Math. 818, Springer, Berlin 1980.

[M81] —, *X-inner automorphisms of filtered algebras*, Proc. Amer. Math. Soc. 83 (1981), 263–268.

[M83] —, *X-inner automorphisms of filtered algebras II*, ibid. 87 (1983), 569–575.

[MP81] S. Montgomery and D. S. Passman, *X-inner automorphisms of group rings*, Houston J. Math. 7 (1981), 395–402.

[MP82] —, —, *X-inner automorphisms of group rings II*, ibid. 8 (1982), 537–544.

[MP84] —, —, *Outer Galois theory of prime rings*, Rocky Mountain J. Math. 14 (1984), 305–318.

[MP86] —, —, *X-inner automorphisms of crossed products and semi-invariants of Hopf algebras*, Israel J. Math. 55 (1986), 33–57.

[P] D. S. Passman, *Computing the symmetric ring of quotients*, J. Algebra 105 (1987), 207–235.

[Pe] G. Pedersen, *C*-Algebras and their Automorphism Groups*, Academic Press, New York 1979.

[R] I. Reiner, *Maximal Orders*, Academic Press, New York 1975.

[Ri] M. A. Rieffel, *Actions of finite groups on C*-algebras*, Math. Scand. 47 (1980), 157–176.