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ON THE MULTIPLICATIVE SUBGROUP OF A GROUP RING

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Let R be a subring of the field of complex numbers and let $G \subseteq \mathrm{Gl}_n(F)$ be a linear group over a commutative ring F with torsion-free additive group. In this paper we describe some properties of torsion subgroups of groups of normalized units in the group ring RG.

Let G be a group and let R be a subring of the field of complex numbers. (For all unexplained notions and notation, see [7].) Bass [1], Bovdi [2], May [6] and Passman [7] showed independently that the trace of a torsion unit in RG is 0, but with some restriction on the group and the ring. In fact, they used the following condition:

1) *If e is an idempotent of $\mathbf{C}G$ and $\operatorname{tr} e = m/n$ where m and n are coprime, then $p^{-1} \notin R$ for all primes p dividing n.*

It is easy to see that condition 1) implies the following:

2) *If there exists an element of G of prime order p, then $p^{-1} \notin R$.*

Cliff and Sehgal [4] showed that if G is a polycyclic-by-finite group then conditions 1) and 2) are equivalent. Strojnowski [9] extended this result to the linear group $\mathrm{Gl}_n(F)$, where F is a commutative ring with torsion-free additive group. So we have:

THEOREM 1. *Let R be a domain of characteristic 0. Let $G \subseteq \mathrm{Gl}_n(F)$ be a group such that for every element g of G of prime order p, $p^{-1} \notin R$. Let $u \in V(RG)$ be a torsion unit. Then $\operatorname{tr} u = 0$.*

This paper is in final form and no version of it will be submitted for publication elsewhere.

In this paper we show the following consequences of this result:

THEOREM 2. *Let R be a domain of characteristic* 0. *Let $G \subseteq \mathrm{Gl}_n(F)$ be a group such that for every element g of G of prime order p, $p^{-1} \notin R$. Suppose there exists a unit in $V(RG)$ of prime power order p^n. Then there exists an element of G of order p^n.*

THEOREM 3. *Let R be a domain of characteristic* 0 *and let G be a finitely generated subgroup of the linear group* $\mathrm{Gl}_n(F)$, *where F is a field of characteristic* 0. *If G has an element of prime order p, assume that p is a nonunit of R. Let U be a torsion subgroup of $V(RG)$. Then*:

1) *The subgroup U is finite and $|U|$ divides $[G:H]$, for every torsion-free subgroup H of G of finite index.*

2) *There exists a subgroup H of G of finite index such that the elements of U are RH-independent.*

The proof of Theorem 2 is the same as Bovdi's proof of the polycyclic-by-finite version of the theorem (see [3]). Since it is not easy to get this book, we give a sketch of proof.

Let $\mathrm{tr}_{p,n}\colon RG \to R$ be the trace map defined by

$$\mathrm{tr}_{p,n} \sum_{g \in G} r_g g = \sum_{o(g) = p^n} r_g,$$

where p is a prime and $n \geqslant 0$. Then $\mathrm{tr}\, u = \mathrm{tr}_{p,0}\, u$ and $\mathrm{tr}\, u^{p^n} \equiv \sum_{i \leqslant n} \mathrm{tr}_{p,i}\, u \pmod{pR}$. Assume $u \in V(RG)$ is of prime power order p^n. Then

$$\mathrm{tr}_{p,n} u \equiv \mathrm{tr}\, u^{p^n} - \mathrm{tr}\, u^{p^{n-1}} \pmod{pR},$$
$$\mathrm{tr}_{p,n} u \equiv 1 \pmod{pR}.$$

Hence there exists an element of G of order p^n.

To prove Theorem 3 we need the following facts.

LEMMA 4. *Let h be an element of G of infinite order. Then* $\mathrm{tr}_h u = 0$.

Proof. In the ring $\mathbf{C}G$ the unit u is a linear combination $u = \sum_{i=0}^{n-1} \varepsilon^{n-i} e_i$, where ε is an nth root of unity and $e_i = n^{-1} \sum_{j=0}^{n-1} (\varepsilon^i u)^j$. Since e_i's are idempotents, $\mathrm{tr}_h e_i = 0$ and so $\mathrm{tr}_h u = 0$.

LEMMA 5. *Let H be a torsion-free normal subgroup of G. Let tr_H be the trace map defined by*

$$\mathrm{tr}_H \sum_{g \in G} r_g g = \sum_{g \in H} r_g.$$

If $u \in V(RG)$ is of finite order, then $\mathrm{tr}_H u = 0$.

Proof. By Lemma 4, $\mathrm{tr}_H u = \mathrm{tr}\, u$ and by Theorem 1, $\mathrm{tr}\, u = 0$.

LEMMA 6. *Let H be a torsion-free normal subgroup of G. Let U be a torsion subgroup of $V(RG)$ and suppose there is a finite sum $\sum_{u \in U} r_u u \in I(R, H)RG$ with $r_u \in R$. Then $r_u = 0$ for all u.*

Proof. Suppose $r_v \neq 0$. Then $x = \sum_{u \in U} r_u uv^{-1} \in I(R, H)RG$. Now $0 = \mathrm{tr}_H x = r_v + \sum_{u \neq v} r_u \mathrm{tr}_H uv^{-1} = r_v$, a contradiction.

As an immediate consequence we obtain:

COROLLARY 7. *The elements of U are R-independent.*

COROLLARY 8. *Let U be a torsion subgroup of $V(RG)$. Then U is finite.*

Proof. By Kargapolov's theorem [5] the group G contains a torsion-free normal subgroup H of finite index. Let φ: $RG \to R(G/H)$ be the canonical homomorphism. Then, by Lemma 6, the elements $\{\varphi(u) \mid u \in U\}$ are R-independent. Hence $|U| \leqslant [G:H] < \infty$.

LEMMA 9. *Let G be a group and let $u_1, \ldots, u_n$ be elements of $\mathbf{C}G$. Let H be a subgroup of G such that the elements of the set $\{g \mid g \in \mathrm{supp}\, u_i$ for some $i \leqslant n\}$ are in different right cosets of H in G. Then the elements u_i are $\mathbf{C}$-independent if and only if they are $\mathbf{C}H$-independent.*

Proof. The implication $\Leftarrow$ is obvious.
$\Rightarrow$ Let $\sum_{i=1}^{n} r_i u_i = 0$, where $r_i \in \mathbf{C}H$, and assume by contradiction that $1 \in \mathrm{supp}\, r_1$. Since the elements u_i are $\mathbf{C}$-independent, there exists $g \in G$ such that $g \in \mathrm{supp} \sum_{i=1}^{n} (\mathrm{tr}\, r_i) u_i$. Then there exist i and j such that $g \in \mathrm{supp}\, u_i$ and $hg \in \mathrm{supp}\, u_j$, for some $h \neq 1$, $h \in H$. This contradicts our assumption that the elements lie in different cosets.

Proof of Theorem 3. 1) Let $\mathbf{P} = (|U|^{-1} \sum_{u \in U} u)\mathbf{C}G$ be a projective $\mathbf{C}G$-module and $\mathbf{P}_H$ its restriction to a $\mathbf{C}H$-module. Then the trace $T_1(\mathbf{P}) = 1/|U|$, since $u \neq 1 \Rightarrow \mathrm{tr}\, u = 0$. Furthermore, $T_h(\mathbf{P}_H) = 0$ for $h \in H \backslash \{1\}$. Consequently,

$$\dim_{\mathbf{C}}(\mathbf{P}_H \otimes_{\mathbf{C}H} \mathbf{C}) = T_1(\mathbf{P}_H \otimes_{\mathbf{C}H} \mathbf{C}) = T_1(\mathbf{P}_H) = [G:H] T_1(\mathbf{P}) = [G:H]/|U|,$$

where the third equality follows from [7, Lemma 13.4.13]. Hence $|U|$ divides $[G:H]$.

2) Since the group G is residually finite, there exists a subgroup H of G of finite index such that the elements of the set $\{g \mid g \in \mathrm{supp}\, u$ for some $u \in U\}$ lie in different right cosets of H in G. Then by Lemma 9 and Corollary 7, H is the required group.

THEOREM 10. *Let G be a polycyclic-by-finite group and H its poly-$\mathbf{Z}$ normal subgroup. Let R be a domain of characteristic 0 such that for every element g of G of prime order p, p is a nonunit in R. Let U be a torsion subgroup of $V(RG)$. Then the elements of U are RH-independent.*

Proof. By Lemma 6 the elements of $\{u+I(R, H)RG \mid u \in U\}$ are R-independent. Hence the elements of U are RH-independent, since H satisfies condition $E(R)$, by [8, Corollary 1.4].

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