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EQUICONVERGENCE THEOREMS FOR LAGUERRE SERIES

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Abstract. The Szegö equiconvergence theorem for the Laguerre series is improved. In particular, a system of exact sufficient conditions is given.

1. Introduction and statement of the results. We shall consider the expansion of a function $f \in L^1_{\mathrm{loc}}(0,\infty)$ in a Laguerre series: $f(y) \sim \sum_{n=0}^{\infty} a_n L_n(y,\alpha)$, where the coefficients a_n are defined by

$$\Gamma(\alpha+1)\binom{n+\alpha}{n} a_n = \int_0^{\infty} e^{-x} x^{\alpha} f(x) L_n(x,\alpha)\,dx, \qquad \alpha > -1\,,$$

and $L_n(x,\alpha) = (n!)^{-1} e^x x^{-\alpha} (d/dx)^n (e^{-x} x^{n+\alpha})$ are the Laguerre polynomials. In [3] Szegö proves the following equiconvergence theorem:

Theorem S. *Let the integrals*

$$\int_0^1 x^{\alpha} |f(x)|\,dx, \qquad \int_0^1 x^{\alpha/2-1/4} |f(x)|\,dx \tag{S$_1$}$$

exist. If

$$\int_n^{\infty} e^{-x/2} x^{\alpha/2-13/12} |f(x)|\,dx = o(n^{-1/2}), \qquad n \to \infty\,, \tag{S$_2$}$$

and if

$$s_n(f,x) = \sum_{k=0}^{n} a_k L_k(x,\alpha) \tag{1.1}$$

denotes the n-th partial sum of the Laguerre series of f, then

$$s_n(f,y^2) - \frac{1}{\pi}\int\limits_{y-c}^{y+c} f(x^2)\frac{\sin\sqrt{4n}(x-y)}{x-y}\,dx = o(1), \qquad n\to\infty, \tag{1.2}$$

for every $y > c > 0$, locally uniformly with respect to $y \in (c,\infty)$.

Moreover, (1.2) *is valid if* (S_2) *is replaced by*

$$\int\limits_1^\infty e^{-x/2}x^{\alpha/2-3/4}|f(x)|\,dx < \infty, \qquad \int\limits_n^\infty e^{-x}x^{\alpha-2}|f(x)|^2\,dx = o(n^{-3/2}). \tag{S_2'}$$

The goal of this paper is to improve Theorem S as follows (see Theorems 1 and 2):

THEOREM 1. *Let $h(x) = e^{-x/2}x^{\alpha/2-1/4}f(x)$, $\alpha \ge -1/2$. If*

$$\int\limits_0^1 |h(x)|\,dx < \infty, \tag{H_1}$$

$$\int\limits_1^\infty x^{-1/2}|h(x)|\,dx < \infty, \tag{H_2}$$

$$\int a(\lambda,x)(1-x/\lambda)^{-1/4}|h(x)|\,dx = o(\lambda^{1/2}), \qquad \lambda\to\infty, \tag{H_3}$$

where $a(\lambda,x)$ is the characteristic function of the interval $(\lambda/2, \lambda-\lambda^{1/3+\varepsilon})$, and

$$\int b(\lambda,x)|h(x)|\,dx = o(\lambda^{1/3}), \qquad \lambda\to\infty, \tag{H_4}$$

where $b(\lambda,x)$ is the characteristic function of $(\lambda-\lambda^{1/3+\varepsilon},\ \lambda+\lambda^{1/3+\varepsilon})$ for some $\varepsilon > 0$, then the equiconvergence result (1.2) *holds.*

R e m a r k 1. If $\alpha \ge -1/2$, then the conditions (H_1) and (S_1) coincide and as is shown in [3], p. 248, they are exact. It is easy to see that (S_2) implies (H_2)–(H_4). On the other hand, (S_2') implies (H_2), (H_3) and

$$\int b(\lambda,x)|h(x)|\,dx = o(\lambda^{1/6+\varepsilon}), \qquad \lambda\to\infty, \tag{H_4'}$$

which is more restrictive than (H_4).

R e m a r k 2. The condition (H_4) is also exact. Indeed, it is satisfied by the function $h(x) = x^{-\delta}$ for every $\delta > 0$, but not for $\delta = 0$. On the other hand, the Laguerre series of the function $f(x) = e^{x/2}x^{-\alpha/2+1/4}$ is divergent ([3], p. 267).

It turns out that for the functions $f(x)$ which are differentiable (or absolutely continuous) at infinity, we can improve the conditions (H_2) and (H_3) in such a way that they are satisfied by the function $f(x) = e^{x/2}x^{-\alpha/2+1/4-\delta}$ for every $\delta > 0$. Namely, we have

THEOREM 2. *Let the function* $g(x) = e^{-x^2/2}x^{\alpha+1/2}f(x^2)$, $\alpha \geq -1/2$, *satisfy*

$$\int_0^1 |g(x)|\,dx < \infty\,, \tag{H$_1$}$$

$$\int_N^\infty x^{-2}|g'(x)|\,dx < \infty \quad \textit{for some large } N\,, \tag{H$_2'$}$$

$$\int a(\lambda, x^2)(1 - x^2/\lambda)^{-1/4}x^{-2}|g'(x)|\,dx = o(1), \quad \lambda \to +\infty\,, \tag{H$_3'$}$$

$$\int b(\lambda, x^2)|g(x)|\,dx = o(\lambda^{1/3}), \quad \lambda \to +\infty\,. \tag{H$_4$}$$

Then the equiconvergence relation (1.2) *is valid.*

EXAMPLE 1. The function $g(x) = x^{1-\delta}$, $0 < \delta < 2$, has the properties (H$_1$), (H$_2'$), (H$_3'$), (H$_4$). The same is true for the functions $\{g(x) : g(x) = O(x^{-1+\delta})$, $x \to 0$, $\delta > 0$, $g(x) = O(x^{1-\delta})$, $x \to \infty$, $g'(x) = O(x^{1-\delta})$, $x \to \infty\}$. Therefore we have the following

COROLLARY 1. *If* $f \in L^1_{\mathrm{loc}}(0,\infty)$ *and if* $f(x) = O(x^{-\alpha/2-3/4+\delta})$, $x \to 0$, *and* $f(x) = O(e^{x/2}x^{-\alpha/2+1/4-\delta})$, $f'(x) = O(e^{x/2}x^{-\alpha/2+1/4-\delta})$, $x \to \infty$, *where* $\delta > 0$, $\alpha \geq -1/2$, *then the equiconvergence result* (1.2) *holds.* (*This is a system of exact sufficient conditions.*)

Remark 3. Theorems 1 and 2 are also true for $-1 < \alpha < -1/2$ if (H$_1$) is replaced by (S$_1$).

Let us explain briefly the main idea of the proof. We use the formula

$$s_n(f, y^2) = 2\int_0^\infty e^{-x^2/2+y^2/2}(x/y)^{\alpha+1/2}f(x^2)e(4n+4, x, y)\,dx \tag{1.3}$$

where $e(\lambda, x, y)$ is the spectral function of the operator

$$-d^2/dx^2 + x^2 + (\alpha^2 - 1/4)x^{-2} + 2 - 2\alpha\,,$$

considered as a self-adjoint operator in $L^2(0,\infty)$. Namely,

$$e(\lambda, x, y) = (e^{-x^2/2-y^2/2})\frac{(xy)^{\alpha+1/2}}{\Gamma(\alpha+1)}\sum \frac{1}{\binom{n+\alpha}{n}}L_n(x^2,\alpha)L_n(y^2,\alpha)\,. \tag{1.4}$$

Here, the sum is taken over all integers n such that $0 \leq n \leq (\lambda-4)/4$. Therefore, it suffices to know the uniform asymptotics of $e(\lambda, x, y)$ as $\lambda \to \infty$. To find it we consider the Laplace transform

$$V(p, x, y) = \int_0^\infty e^{-\lambda p}\,de(\lambda, x, y), \quad \operatorname{Re} p > 0\,, \tag{1.5}$$

and using its explicit expression ([3], p. 101), we derive the formula

$$e(\lambda,x,y)=\frac{1}{2\pi i}\int_{\varepsilon-i\pi/4}^{\varepsilon+i\pi/4}e^{\lambda p}V(p,x,y)H(\lambda,p)\,dp\,, \tag{1.6}$$

where the function $s\to H(s,p)$ is 4-periodic, $H(s,p)=2e^{(2-s)p}(\sinh 2p)^{-1}$ if $0\le s\le 4$. Next we apply the saddle-point method or the method of stationary phase.

For our purposes it is sufficient to consider the following cases: 1) $0<a_1\le x\le 2A$; 2) $2A\le x\le\sqrt{(1-\delta)\lambda}$; 3) $\sqrt{(1-\delta)\lambda}\le x\le\sqrt{(1+\delta)\lambda}$; 4) $x\ge\sqrt{(1+\delta)\lambda}$, provided that $0<a_2\le y\le A$.

THEOREM 3 (the case $0<a_1\le x\le 2A$, $0<a_2\le y\le A$). *We have the uniform asymptotics*

$$e(\lambda,x,y)=\frac{1}{2\pi}\frac{\sin\sqrt{\lambda}(x-y)}{x-y}+\frac{c_\alpha}{2\pi}\frac{\sin\sqrt{\lambda}(x+y)}{x+y}+O(\lambda^{-1/2})\,, \tag{1.7}$$

$\lambda\to\infty$, *for some constant* c_α.

It will be convenient to consider also the function

$$E(\lambda,x,y)=e(\lambda,\sqrt{\lambda}x,\sqrt{\lambda}y)\,.$$

THEOREM 4 (the case $4A^2/\lambda\le x^2\le 1-\delta$, $0<a_2\le\sqrt{\lambda}y\le A$). *For every small* $\delta>0$ *we have the uniform asymptotics*

$$E(\lambda,x,y)=F(\lambda,x,y)+c_\alpha F(\lambda,x,-y)\,, \tag{1.8}$$

$$(1-x^2)^{1/4}F(\lambda,x,y)=\lambda^{-1/2}\sum_{j=1}^{4}b_j(\lambda,x,y)\exp(i\lambda\psi_j(x,y)) \tag{1.9}$$
$$+x^{-1}O(\lambda^{-3/2}),\qquad \lambda\to+\infty\,,$$

where $|b_j|\le cx^{-1}$, $|\partial_x b_j|\le cx^{-2}$ *and*

$$|\partial_x\psi_j(x,y)|\ge cx,\qquad 1\le j\le 4\,, \tag{1.10}$$

for some constant $c>0$.

COROLLARY 2. *If* $2A\le x\le\lambda/2$, $0<a_2\le y\le A$ *then the uniform estimate* $|e(\lambda,x,y)|\le cx^{-1}$ *is valid.*

THEOREM 5 (the case $1-\delta\le x^2\le 1+\delta$, $0<a_2\le\sqrt{\lambda}y\le A$). *There exists a positive number* δ *such that we have the uniform asymptotics* (1.8) *where*

$$F(\lambda,x,y)=\lambda^{-1/3}\sum_{k\ge 0}(\alpha_{0k}(\lambda,x,y)\lambda^{-k}+\alpha_{1k}(\lambda,x,y)\lambda^{-k-1/3}),\qquad \lambda\to+\infty\,,$$

and

$$\alpha_{jk}=(a_{jk}e^{\lambda A}+b_{jk}e^{\lambda\bar{A}}\,\mathrm{Ai}^{(j)}(\lambda^{2/3}B)\,, \tag{1.11}$$

$$|a_{jk}|+|\partial_x a_{jk}|\le c,\qquad |b_{jk}|+|\partial_x b_{jk}|\le c,\qquad j=0,1\,. \tag{1.12}$$

Here $\mathrm{Ai}(s) = (2\pi)^{-1} \int \exp(i(st + t^3/3))\,dt$ *is the Airy function and*

$$A = A(x,y) = \tfrac{1}{2}(\varphi(p_+,x,y) + \varphi(p_-,x,y)), \tag{1.13}$$

$$B = B(x,y) = (\tfrac{3}{4}(\varphi(p_+,x,y) - \varphi(p_-,x,y)))^{2/3}, \tag{1.14}$$

where

$$\varphi(p,x,y) = p - \tfrac{1}{2}(x^2+y^2)\coth 2p + xy(\sinh 2p)^{-1} \tag{1.15}$$

and

$$p_\pm = it_\pm, \quad \cos 2t_\pm = -xy \pm ((1-x^2)(1-y^2))^{1/2}, \qquad 0 < t_\pm < \pi/2 \quad \textit{if } x < 1, \tag{1.16}$$

$$p_\pm = \pm\varepsilon + it, \quad \cosh 2\varepsilon = x, \quad \varepsilon > 0, \quad \cos 2t = -y, \qquad 0 < t < \pi/2 \quad \textit{if } x > 1. \tag{1.17}$$

Remark 4. The smooth functions $A(x,y)$, $B(x,y)$ satisfy

$$B(x,\pm y) > 0 \quad \text{if } x > 1, \qquad B(x,\pm y) < 0 \quad \text{if } x < 1, \tag{1.18}$$

$$B(x,\pm y) = -2^{1/3}(1-x)(1+O(1-x)) \quad \text{as } x \to 1, \tag{1.19}$$

$$\operatorname{Re} A(x,\pm y) = 0. \tag{1.20}$$

Corollary 3. *If* $1 - \delta < x^2 < 1 - \lambda^{-2/3+\varepsilon}$, $0 < a_2 \le \sqrt{\lambda}y \le A$ *for some* $\varepsilon > 0$, *then we have the uniform asymptotics* (1.8) *where*

$$F(\lambda,x,y) = \lambda^{-1/2}(1-x^2)^{-1/4}\sum_{j=1}^{4}\sum_{k\ge 0} a_{kj}(\lambda^{2/3}(1-x^2))^{-3k/2}\exp(i\lambda\psi_j),$$

$\lambda \to \infty$, *the functions* ψ_j *satisfy* (1.10) *and*

$$|a_{kj}(\lambda,x,y)| + |\partial_x a_{kj}(\lambda,x,y)| \le c.$$

Theorem 6 (the case $x^2 \ge 1+\delta$, $0 < a_2 \le \sqrt{\lambda}y \le A$). *For every small* $\delta > 0$ *the uniform estimate* $|E(\lambda,x,y)| \le c\lambda^{-1/2}\exp(-\frac{1}{2}\lambda\delta\sqrt{x^2-1})$ *holds.*

Theorems 5 and 6 imply the following

Corollary 4. *If* $x^2 \ge \lambda + \lambda^{1/3+\varepsilon}$, $\varepsilon > 0$, *and* $0 < a_2 \le y \le A$ *then we have the uniform estimate*

$$|e(\lambda,x,y)| \le c\lambda^{-1/3}\exp(-c\lambda^{1/3}(x^2/\lambda - 1)^{1/2}).$$

2. Proof of the equiconvergence theorems

Proof of Theorem 1. Let $g(x) = e^{-x^2/2}x^{\alpha+1/2}f(x^2)$. As in the proof of Theorem S ([3], p. 264), it suffices to establish a uniform estimate of the kind

$$R_n(f,y^2) = O(1)\Big(\int_0^1 |g(x)|\,dx + \int_1^\infty x^{-1}|g(x)|\,dx\Big) + o(1), \tag{2.1}$$

$n \to \infty$, where $0 < c < a_2 \le y \le A$ and

$$R_n(f, y^2) = s_n(f, y^2) - \frac{1}{\pi}\int_{y-c}^{y+c} f(x^2)\frac{\sin\sqrt{4n}(x-y)}{x-y}\,dx\,. \tag{2.2}$$

Using (1.3), (1.7) and the Riemann–Lebesgue lemma, we have

$$R_n(f, y^2) = \Big(\int_0^{a_1} + \int_{2A}^{\infty}\Big) g(x)e(4n+4, x, y)\,dx + o(1)\,, \tag{2.3}$$

$n \to \infty$, where $0 < a_2 \le y \le A$ and $a_1 = a_2 - c$. Since

$$\int_0^{a_1} |g(x)e(4n+4, x, y)|\,dx = O(1)\int_0^1 |g(x)|\,dx \quad \text{ if } \alpha \ge -1/2 \tag{2.4}$$

(see [3], p. 264), it remains to estimate the integrals

$$K_j(\lambda, y) = \int a_j(\lambda, x^2) g(x) e(\lambda, x, y)\,dx\,, \quad 1 \le j \le 4\,, \tag{2.5}$$

uniformly with respect to $y \in [a_2, A]$, $a_2 > 0$, where $x \to a_j(\lambda, x)$ is the characteristic function of the interval I_j, $1 \le j \le 4$, and $I_1 = (4A^2, \lambda/2)$, $I_2 = (\lambda/2, \lambda - \lambda^{1/3+\varepsilon})$, $I_3 = (\lambda - \lambda^{1/3+\varepsilon}, \lambda + \lambda^{1/3+\varepsilon})$, $I_4 = (\lambda + \lambda^{1/3+\varepsilon}, \infty)$. To estimate the integral K_1 we apply Corollary 2 to get

$$K_1(\lambda, y) = O(1)\int_1^{\infty} x^{-1}|g(x)|\,dx\,. \tag{2.6}$$

Further, Theorem 4 and Corollary 3 imply the estimate $|e(\lambda, x, y)a_2(\lambda, x^2)| \le c^{-1/2}(1 - x^2/\lambda)^{-1/4}$, hence (H_3) gives

$$K_2(\lambda, y) = o(1), \quad \lambda \to \infty\,. \tag{2.7}$$

Theorem 5 and (H_4) show that

$$K_3(\lambda, y) = o(1), \quad \lambda \to \infty\,. \tag{2.8}$$

Corollary 4 yields

$$|K_4(\lambda, y)| \le c\Big(\lambda^3 e^{-c\lambda^{\varepsilon/2}} \int b_1(\lambda, x^2)|g(x)|x^{-3}\,dx + \lambda^{-1/3}\int b_2(\lambda, x^2)|g(x)|\exp(-cx^{1/2})\,dx\Big)\,, \tag{2.9}$$

where $x \to b_1(\lambda, x)$ is the characteristic function of $(\lambda + \lambda^{1/3+\varepsilon}, \lambda^2)$ and $b_1 + b_2 = a_4$. Therefore (2.9) and (H_2) give

$$K_4(\lambda, y) = o(1), \quad \lambda \to \infty\,. \tag{2.10}$$

Evidently, (2.1) follows from (2.3)–(2.10). Theorem 1 is proved.

Proof of Theorem 2. We use again (2.2)–(2.4). Note first that (H_3') and (H_4) imply

$$g(x) = O(x^{5/3}), \quad x \to \infty\,. \tag{2.11}$$

Indeed, let $x_n \to \infty$ and $x_n^2 = \lambda_n + \lambda_n^{1/3}$. Then $\lambda_n - \lambda_n^{1/3} < x_n^2 - x_n^{2/3}$ for large n. Using (H_4) and the mean-value theorem, we find y_n such that $x_n^2 - x_n^{2/3} < y_n^2 < x_n^2$ and $g(y_n) = O(x_n^{5/3})$, $n \to \infty$. Further, choose λ_n so that $x_n^2 = \lambda_n - \lambda_n^{1/3}$. Since $g(x_n) - g(y_n) = \int_{y_n}^{x_n} g'(x)\,dx$ and

$$|g(x_n) - g(y_n)| \le x_n^2 \int\limits_{y_n}^{x_n} (1 - x^2/\lambda_n)^{1/4-1/4} x^{-2} |g'(x)|\,dx\,,$$

$$1 - x^2/\lambda_n < 1 - y_n^2/\lambda_n < 2\lambda_n^{-2/3} < 2x_n^{-1/3} \quad \text{ if } x_n > y_n\,,$$

we get from (H_3') the estimate $|g(x_n) - g(y_n)| \le c x_n^{5/3}$. Thus, (2.11) follows.

Now as in the proof of Theorem 1 it is sufficient to see that

$$\begin{aligned}(2.12)\qquad R_n(f, y^2) = O(1)\Big(\int\limits_0^1 |g(x)|\,dx + \int\limits_1^\infty x^{-3}|g(x)|\,dx \\ + \int\limits_N^\infty x^{-2}|g'(x)|\,dx\Big) + o(1)\,, \qquad n \to \infty\,,\end{aligned}$$

uniformly in $[a_2, A]$, $a_2 > 0$. To this end we shall estimate the integrals K_j, $1 \le j \le 4$, from (2.5). It is clear that (2.8)–(2.10) remain valid. To estimate K_1 and K_2 we consider the formulas

$$\begin{aligned}(2.13)\qquad B_j(\lambda, y) &= K_j(\lambda, \sqrt{\lambda} y) \\ &= \sqrt{\lambda} \int a_j(\lambda, \lambda x^2) g(\sqrt{\lambda} x) E(\lambda, x, y)\,dx, \qquad j = 1, 2\,.\end{aligned}$$

Using an appropriate partition of unity in the integral (1.3), we can suppose that $g(2A) = 0$. In the integral B_1, integration by parts with the help of Theorem 4 gives

$$(2.14)\qquad K_1(\lambda, y) = O(1)\Big(\int\limits_1^\infty x^{-3}|g(x)|\,dx + \int\limits_N^\infty x^{-2}|g'(x)|\,dx\Big)\,.$$

In the integral B_2 we integrate by parts, using Theorem 4 (if $1/2 < x^2 < 1 - \delta$) and Corollary 2 (if $1 - \delta < x^2 < 1 - \lambda^{-2/3+\varepsilon}$). Taking into account (2.11), we get

$$\begin{aligned}(2.15)\qquad B_2(\lambda, y) = O(1)\Big(\int\limits_1^\infty x^{-3}|g(x)|\,dx + \int\limits_N^\infty x^{-2}|g'(x)|\,dx \\ + \lambda^{-\varepsilon/4} + B(\lambda, y)\Big)\end{aligned}$$

for small $\varepsilon > 0$, where

$$(2.16)\qquad B(\lambda, y) = C(\lambda, y) + c_\alpha C(\lambda, -y)\,,$$

$$(2.17)\qquad C(\lambda, y) = \lambda^{-1} \sum_{j=1}^4 \sum_{k=0}^M \int a_2(\lambda, \lambda x^2) e^{i\lambda\psi_j} \frac{\partial}{\partial x} q(\lambda, x)\,dx\,,$$

$$q(\lambda,x)=g(\sqrt{\lambda}x)a_{kj}(\lambda,x)(1-x^2)^{-1/4}(\partial_x\psi_j)^{-1}(\lambda^{2/3}(1-x^2))^{-3k/2} \tag{2.18}$$

and $M=M(\varepsilon)$ is large enough. Since the functions $a_{kj}(\lambda,x)$, $(\partial_x\psi_j)^{-1}$ and their derivatives with respect to x are bounded when $1/2<x^2<1$, it is sufficient to estimate the integrals

$$C_1(\lambda,y)=\lambda^{-1}\int a_2(\lambda,\lambda x^2)(1-x^2)^{-5/4}|g(\sqrt{\lambda}x)|\,dx\,, \tag{2.19}$$

$$C_2(\lambda,y)=\lambda^{-1/2}\int a_2(\lambda,\lambda x^2)(1-x^2)^{-1/4}|g'(\sqrt{\lambda}x)|\,dx\,. \tag{2.20}$$

By virtue of (2.11), we have

$$C_1(\lambda,y)\le c\int\limits_{x^2<1}(1-x^2)^{-1+\delta}\,dx\lambda^{-\varepsilon/8}\qquad \text{if } \delta=\frac{3\varepsilon}{8(2-3\varepsilon)} \tag{2.21}$$

where $\varepsilon>0$ is small enough. On the other hand,

$$C_2(\lambda,y)\le c\int a_2(\lambda,x^2)(1-x^2/\lambda)^{-1/4}x^{-2}|g'(x)|\,dx\,. \tag{2.22}$$

It is not hard to see that (2.13), (2.15)–(2.22) and (H_3') yield the estimate

$$K_2(\lambda,y)=O(1)\Big(\int\limits_1^\infty x^{-3}|g(x)|\,dx+\int\limits_N^\infty x^{-2}|g'(x)|\,dx\Big)+o(1)\,, \tag{2.23}$$

$\lambda\to\infty$. Consequently, (2.12) follows from (2.3), (2.4), (2.8)–(2.10) and (2.14), (2.23). Theorem 2 is proved.

3. Proof of the asymptotics for the spectral function

Proof of Theorem 3. First we prove the formula (1.6). According to Theorem 5.1 of [3], we can write

$$V(p,x,y)=\frac{1}{2}(xy)^{1/2}e^{2p(\alpha-1)}(\sinh 2p)^{-1}e^{(1/2)(x^2+y^2)\coth 2p}i^{-\alpha}J_\alpha\Big(\frac{ixy}{\sinh 2p}\Big)$$

if $\operatorname{Re} p>0$. Notice that

$$V(p+ik\pi/2,x,y)=V(p,x,y)\,. \tag{3.1}$$

From (1.4), (1.5) it follows that

$$V(p,x,y)=p\int\limits_0^\infty e^{-\lambda p}e(\lambda,x,y)\,d\lambda,\qquad \operatorname{Re} p>0\,. \tag{3.2}$$

We want to apply the inverse Laplace formula. Since the function $\lambda\to e(\lambda,x,y)$ is only right-continuous, it is convenient to consider the Steklov average:

$$e_h(\lambda,x,y)=\frac{1}{h}\int\limits_0^h e(\lambda+\mu,x,y)\,d\mu,\qquad h>0\,.$$

Evidently $e_h(\lambda,x,y)\to e(\lambda,x,y)$ as $h\to+0$ for every fixed (λ,x,y) and (3.2) can be rewritten as follows:

$$\int\limits_0^\infty e^{-\lambda p}e_h(\lambda,x,y)\,d\lambda=\frac{e^{hp}-1}{h^2}\cdot\frac{V(p,x,y)}{p^2},\qquad h>0,\ \operatorname{Re}p>0\,.$$

Hence the inverse Laplace formula gives

$$e_h(\lambda,x,y)=\frac{1}{2\pi i}\int\limits_{\varepsilon-i\infty}^{\varepsilon+i\infty}e^{\lambda p}\frac{e^{hp}-1}{h^2}\cdot\frac{V(p,x,y)}{p^2}\,dp,\qquad \varepsilon>0\,.$$

Now the periodicity relation (3.1) and the Weierstrass theorem show that

$$e_h(\lambda,x,y)=\frac{1}{2\pi i}\int\limits_{\varepsilon-i\pi/4}^{\varepsilon+i\pi/4}e^{\lambda p}V(p,x,y)\frac{g(h,p)-g(0,p)}{h}\,dp \tag{3.3}$$

where $g(s,p)=e^{ps}f(\lambda+s,p)$ and $f(s,p)=\sum e^{isk\pi/2}(p+ik\pi/2)^{-2}$. The function $s\to f(s,p)$ is continuous, 4-periodic and

$$f(s,p)=4e^{(2-s)p}(\coth 2p+s/2-1)(\sinh 2p)^{-1}$$

for $0\le s<4$, $\operatorname{Re}p>0$. In particular, $\lim_{h\to+0}h^{-1}(g(h,p)-g(0,p))=H(\lambda,p)$ and using the Lebesgue theorem we get (1.6) from (3.3). Notice also that

$$p\to e^{\lambda p}V(p,x,y)H(\lambda,p)\quad\text{is}\quad i\pi/2\text{-periodic}\,. \tag{3.4}$$

This allows us to write

$$\begin{aligned}e(\lambda,x,y)=\frac{1}{2\pi i}\Big(&\int\limits_{\varepsilon-i\pi/4}^{\varepsilon+i\pi/4}e^{\lambda p}V(p,x,y)H(\lambda,p)\chi_1(p)\,dp\\&+\int\limits_{\varepsilon+i0}^{\varepsilon+i\pi/2}e^{\lambda p}V(p,x,y)H(\lambda,p)\chi_2(p)\,dp\Big)\end{aligned}$$

where $\chi_1(p)+\chi_2(p)=1$ on the interval $\{p=\varepsilon+it:|t|\le\pi/4\}$ and $\operatorname{supp}\chi_1(p)\subset\{p=\varepsilon+it:|t|\le\gamma<\pi/4\}$, $\chi_1(\varepsilon+it)=1$ if $|t|\le\gamma/2$ and the function $\chi_2(p)$ is $i\pi/2$-periodic. Thus

$$e(\lambda,x,y)=\frac{1}{2\pi i}\int\limits_S e^{\lambda p}V(p,x,y)H(\lambda,p)\chi(p)\,dp \tag{3.5}$$

where $S=(\varepsilon-i\pi/2,\varepsilon+i\pi/2)$ and $\chi\in C_0^\infty(S)$, $\chi(\varepsilon+it)=1$ if $|t|\le\pi/8$.

We shall now find an appropriate form of $V(p,x,y)$, separating the oscillating part. Using the formulas (1), p. 74, (6), p. 75 and (3), (4), p. 168 of [4], we can write

$$J_\alpha(z)=z^{-1/2}(e^{iz}c_\alpha^+f(-z)+e^{-iz}c_\alpha^-f(z))\qquad\text{if }\alpha\ge-1/2$$

where

$$f(z)=\begin{cases}\frac{1}{2}\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{\Gamma(\alpha+1/2)}\int_0^\infty e^{-u}u^{\alpha-1/2}\left(1-\frac{iu}{2z}\right)^{-1/2}du & \text{if } \alpha>-1/2,\\ \frac{1}{2}\left(\frac{2}{\pi}\right)^{1/2} & \text{if } \alpha=-1/2,\end{cases}\tag{3.6}$$

is a holomorphic function for $\operatorname{Re} z\neq 0$. Here $c_\alpha^- = e^{i(\pi/2)(\alpha+1/2)}$. Therefore

$$V(p,x,y)=(\sinh 2p)^{-1/2}e^{-(1/2)(x^2+y^2)\coth 2p}(e^{xy/\sinh 2p}a(p,xy) + e^{-xy/\sinh 2p}c_\alpha a(p,-xy))\tag{3.7}$$

where $c_\alpha = e^{-i(\pi/2)(\alpha+1/2)}c_\alpha^+$ and

$$a(p,xy)=\tfrac{1}{2}e^{2p(\alpha-1)}f(ixy/\sinh 2p)\,.\tag{3.8}$$

Since $-\frac{1}{2}(x^2+y^2)\coth 2p+xy/\sinh 2p=-(x-y)^2/(4p)+s(p,x,y)$ and $s(0,x,y)=0$, we have the representation

$$V(p,x,y)=p^{-1/2}(e^{-(x-y)^2/(4p)}b(p,x,y)+e^{-(x+y)^2/(4p)}c_\alpha b(p,x,-y))$$

where $b(0,x,\pm y)=1/(4\sqrt{\pi})$. Further, we have the equality

$$\frac{1}{2\sqrt{\pi p}}e^{-(x-y)^2/(4p)}=\frac{1}{2\pi i(x-y)}\int 2\xi p e^{-\xi^2p+i(x-y)\xi}\,d\xi,\qquad \operatorname{Re}p>0\,,$$

therefore

$$V(p,x,y)=W(p,x,y)+c_\alpha W(p,x,-y)\tag{3.9}$$

where

$$W(p,x,y)=\frac{p}{x-y}a(p,x,y)\int \xi e^{-\xi^2p+i(x-y)\xi}\,d\xi,\qquad \operatorname{Re}p>0\,,\tag{3.10}$$

$$a(0,x,\pm y)=1/(2\pi i)\,.\tag{3.11}$$

Now (3.5), (3.9) and (3.10) show that

$$E(\lambda,x,y)=F(\lambda,x,y)+c_\alpha F(\lambda,x,-y)\tag{3.12}$$

where

$$F(\lambda,x,y)=\frac{\sqrt{\lambda}}{x-y}\int e^{i\lambda\psi(t,\xi,x,y)}q(t,\xi,\lambda,x,y)\,dt\,d\xi\tag{3.13}$$

is an oscillating integral with respect to ξ, and

$$q(t,\xi,\lambda,x,y)=\frac{\xi}{2\pi}a(it,\sqrt{\lambda}x,\sqrt{\lambda}y)H(\lambda,it)it\chi(it)\,,\tag{3.14}$$

$$\psi(t,\xi,x,y)=(1-\xi^2)t+(x-y)\xi\,.\tag{3.15}$$

Notice that $a_1\le\sqrt{\lambda}x\le 2A$, $a_2\le\sqrt{\lambda}y\le A$ and from (3.14) and (3.6)–(3.10) it follows that

$$|\partial_t^k q|\le C_k|\xi|\,.\tag{3.16}$$

Since $|\partial_t\psi| \geq c\xi^2$ if ξ^2 is large enough, we can integrate by parts in the integral (3.13) to get

$$F(\lambda,x,y) \sim \frac{\sqrt{\lambda}}{x-y}\int e^{i\lambda\psi(t,\xi,x,y)}\kappa(\xi)q(t,\xi,\lambda,x,y)\,dt\,d\xi \tag{3.17}$$

where $\kappa \in C_0^\infty(\mathbb{R})$ is an even cut-off function and the equivalence relation "$A(\lambda,x,y) \sim B(\lambda,x,y)$" here means that $A(\lambda,x,y)-B(\lambda,x,y)=O(\lambda^{-\infty})$, uniformly with respect to x, y. Applying the method of stationary phase to the integral (3.17), we derive the uniform asymptotics

$$F(\lambda,x,y) = \lambda^{-1/2}(2\pi(x-y))^{-1}\sin\lambda(x-y) + O(\lambda^{-1/2}), \qquad \lambda\to\infty\,.$$

Together with (3.12), (3.13), this gives (1.7). Theorem 3 is proved.

Proof of Theorem 4. We start from the formulas (3.5) and (3.7). Now we use the equality

$$\int e^{-\xi^2(\sinh 2p)/2+i(x-y)\xi}\xi\,d\xi = \sqrt{2\pi}i(x-y)(\sinh 2p)^{-3/2}g(p)\,,$$

$\operatorname{Re} p>0$, where

$$g(p) = \exp\left(\frac{x^2+y^2}{2}\tanh p - \frac{x^2+y^2}{2}\coth 2p + \frac{xy}{\sinh 2p}\right).$$

Therefore we have again the representation (3.9), where

$$\begin{aligned} W(p,x,y) &= (x-y)^{-1}(\sinh 2p)a(p,x,y)\int \xi\exp(\psi(p,\xi))\,d\xi\,,\\ \psi(p,\xi) &= -\xi^2(\sinh 2p)/2 + i(x-y)\xi - (x^2+y^2)(\tanh p)/2\,,\\ a(p,x,y) &= (\exp(2p(\alpha-1)-i\pi(\alpha+1/2)/2))/(2i\sqrt{2\pi})f(ixy/\sinh 2p)\,. \end{aligned}$$

Analogously to (3.12)–(3.17) we obtain again (3.12), (3.17), where the phase function ψ now has the form

$$\psi(t,\xi,x,y) = t - \frac{\xi^2}{2}\sin 2t - \frac{x^2+y^2}{2}\tan t + (x-y)\xi \tag{3.18}$$

and

$$q(t,\xi,\lambda,x,y) = (\xi/(2\pi))a(it,\sqrt{\lambda}x,\sqrt{\lambda}y)H(\lambda,it)i(\sin 2t)\chi(it)\,,$$

so the estimate (3.16) is valid.

Applying the method of stationary phase, we see that the function (3.18) has four nondegenerate critical points: $(t_\pm,\xi_\pm)$ and $(-t_\pm,-\xi_\pm)$, where $\cos 2t_\pm = xy\pm\omega$, $\omega=((1-x^2)(1-y^2))^{1/2}$, $\xi_\pm\sin 2t_\pm = x-y$. In addition, for the Hessian ψ'' we have $\det\psi''(t_\pm,\xi_\pm)=\pm4\omega$. Thus the method of stationary phase yields the asymptotics (1.9), where $\psi_1(x,y)=\psi(t_+,\xi_+,x,y)$, $\psi_2(x,y)=(t_-,\xi_-,x,y)$, $\psi_3=-\psi_1$, $\psi_4=-\psi_2$ and $b_k(\lambda,x,y)=(x-y)^{-1}a_k(\lambda,x,y)$, $|\partial_x a_k|\leq cx^{-1}$, $1\leq k\leq 4$. Theorem 4 is proved.

Proof of Theorem 5. Starting from the formula (1.6), we use the periodicity property (3.4) and obtain the representation

$$e(\lambda, x, y) = \frac{1}{4\pi i} \int_S e^{\lambda p} V(p, x, y) H(\lambda, p)\, dp, \qquad \operatorname{Re} p > 0\,,$$

where $S = (\varepsilon - i\pi/2, \varepsilon + i\pi/2)$. Further, (3.7), (3.8) show that

$$E(\lambda, x, y) = F(\lambda, x, y) + c_\alpha F(\lambda, x, -y) \tag{3.19}$$

where

$$F(\lambda, x, y) = \int_S e^{\lambda\varphi(p,x,y)} q(p, \lambda, x, y)\, dp\,, \tag{3.20}$$

φ is given by (1.15) and

$$q(p, \lambda, x, y) = \frac{e^{2p(\alpha-1)}}{8\pi i} (\sinh 2p)^{-1/2} f(\lambda ixy/\sinh 2p) H(\lambda, p)\,. \tag{3.21}$$

To find the uniform asymptotics of the integral (3.20) as $\lambda \to \infty$, we shall apply the saddle-point method. Since $a_2 \le \sqrt{\lambda} y \le A$ the phase function $p \to \varphi(p, x, y)$ has critical points $p_\pm$ and $\overline{p}_\pm$, where $p_\pm$ are given by (1.16), (1.17). If $x = 1$, then $p_\pm = p_0 = it_0$ where $\cos 2t_0 = y$ and $0 < t_0 < \pi/2$. Hence, the critical points p_0 and $\overline{p}_0$ are degenerate and $(\partial^3\varphi/\partial p^3)(p, x, y) = 8$, $(\partial^2\varphi/\partial p\partial x)(p, x, y) = -2$ if $p = p_0$ or $p = \overline{p}_0$. Since $|x^2 - 1| < \delta$, we can choose $\delta > 0$ so that $0 < |\operatorname{Im} p| < \pi/2$ for all the critical points. Consequently, the integrand in (3.20) is holomorphic near the critical points. On the other hand, according to Lemma 2.3 of [1, p. 343], we can find a holomorphic change of variables $p = p(z, x, y)$ in a neighborhood of the points $z = 0$, $x = 1$ such that

$$\varphi(p(z, x, y), x, y) = A(x, y) - B(x, y)z + z^3/3, \qquad p(0, 1, y) = p_0\,. \tag{3.22}$$

Note also that (3.22) and (1.15) imply

$$\begin{gathered} \varphi(\overline{p(\overline{z}, x, y)}, x, y) = \overline{A(x, y)} - B(x, y)z + z^3/3\,, \\ \overline{p(0, 1, y)} = \overline{p_0}\,, \end{gathered} \tag{3.23}$$

and

$$\overline{p(\pm\sqrt{B}, x, y)} = \begin{cases} \overline{p}_\pm & \text{if } x > 1, \\ \overline{p}_\mp & \text{if } x < 1. \end{cases}$$

To use the holomorphic change of variables (3.22), (3.23) in the integral (3.20), we shall prove first that

$$F(\lambda, x, y) \sim \int_\gamma e^{\lambda\varphi(p,x,y)} q(p, \lambda, x, y)\,, \qquad \gamma = \gamma_1 \cup \gamma_2\,, \tag{3.24}$$

where γ_1 is the segment $(\varepsilon + i(t_0 - 2\varepsilon), \varepsilon + i(t_0 + 2\varepsilon))$ and γ_2 the segment $(\varepsilon - i(t_0 + 2\varepsilon), \varepsilon + i(-t_0 + 2\varepsilon))$ for $\varepsilon > 0$ small enough. The equivalence relation "$a(\lambda, x, y) \sim b(\lambda, x, y)$" here means that $a(\lambda, x, y) - b(\lambda, x, y) = O(e^{-c\lambda})$, $c > 0$. To prove (3.24), it is sufficient to use the estimate $\operatorname{Re} \varphi(p, x, y) \le -c < 0$ for

$p \in S \setminus \gamma$, which follows from the definition (1.15) for $\varepsilon > 0$ small enough. Now (3.20) and (3.22)–(3.24) yield

$$F(\lambda, x, y) \sim \sum_{j=1}^{2} e^{\lambda A_j} \int_{\gamma_j^*} e^{\lambda(-Bz+z^3/3)} q_j(z,\lambda)\, dz\,,$$

where $A_1 = A$, $A_2 = \overline{A}$ and

$$q_1(z,\lambda) = q(p(z,x,y),\lambda,x,y)\frac{\partial}{\partial z}p(z,x,y)\,,$$
$$q_2(z,\lambda) = q(\overline{p(\overline{z},x,y)},\lambda,x,y)\frac{\partial}{\partial z}\overline{p(\overline{z},x,y)}\,,$$

γ_j^* being the image of the segment γ_j. Notice that $\gamma_j^* \subset \{z : \operatorname{Re} z > 0\}$ and that the end points α_j, β_j of γ_j^* satisfy $\arg\alpha_j \in (-\pi/2, -\pi/6)$, $\arg\beta_j \in (\pi/6, \pi/2)$. Further, we use the Weierstrass preparation theorem [2]:

$$q_j(z,\lambda) = r_j + \widetilde{r}_j z + (z^2 - B)\widetilde{q}_j(z,\lambda)$$

and the following representation of the Airy function:

$$\operatorname{Ai}(s) = \frac{1}{2\pi i}\int_{\Gamma} e^{-sz+z^3/3}\, dz, \qquad \Gamma = \Gamma_1 \cup \Gamma_2, \quad \text{where}$$
$$\Gamma_1 = \{z = \varrho\exp(i\varphi_1) : \varrho \in (\infty, 0),\ \varphi_1 \in (-\pi/2, -\pi/6)\}\,,$$
$$\Gamma_2 = \{z = \varrho\exp(i\varphi_2) : \varrho \in (0, \infty),\ \varphi_2 \in (\pi/6, \pi/2)\}\,.$$

Thus we obtain the uniform asymptotics

$$F(\lambda, x, y) = \lambda^{-1/3}\sum_{k\ge 0}(\alpha_{0k}(\lambda,x,y)\lambda^{-k} + \alpha_{1k}(\lambda,x,y)\lambda^{-k-1/3})\,, \tag{3.25}$$

$\lambda \to \infty$, where the coefficients α_{jk} are given by (1.11). The remainder in the asymptotics (3.25) is estimated as in [1], p. 348.

To verify (1.12), it is sufficient to prove that

$$|q(p,\lambda,x,y)| + |\partial_x q(p,\lambda,x,y)| + |\partial_p q(p,\lambda,x,y)| \le c \tag{3.26}$$

if $|\operatorname{Re} p| \le \varepsilon$, $\varepsilon_0 \le |\operatorname{Im} p| \le \pi/2 - \varepsilon_0$, $\lambda xy \ge 1$, $x \ge \varepsilon_0$ where $\varepsilon_0 > 0$. Since $|\operatorname{Re} z| \ge c > 0$ where $z = i\lambda xy/\sinh 2p$, we can apply the asymptotics of the function $f(z)$ from (3.6), which yields the estimate $|f^{(k)}(z)| \le C_k|z|^{-k}$. Now (3.26) follows from the definition (3.21). Theorem 5 is proved.

Proof of Theorem 6. We shall use the formulas (3.19) and (3.20), where $\varepsilon = \frac{1}{2}\operatorname{arcosh} x$, $x^2 \ge 1 + \delta$. The phase function φ has critical points $p(x,y)$ and $\overline{p}(x,y)$, where $p(x,y) = \varepsilon + it$, $\cos 2t = y$, $0 < t < \pi/2$. They are nondegenerate and $\operatorname{Re}\varphi(p,x,y) < \operatorname{Re}\varphi(p(x,y),x,y)$ if $0 \le \operatorname{Im} p \le \pi/2$, $p \ne p(x,y)$, $p \in S$, and $\operatorname{Re}\varphi(p,x,y) < \operatorname{Re}\varphi(\overline{p}(x,y),x,y)$ if $-\pi/2 \le \operatorname{Im} p \le 0$, $p \ne \overline{p}(x,y)$, $p \in S$.

Applying the saddle-point method [1], we get the asymptotics

$$F(\lambda, x, y) = \sum_{k\geq 0} a_k(\lambda, x, y)\lambda^{-1/2-k}, \qquad \lambda \to \infty\,, \tag{3.27}$$

where

$$|a_k(\lambda, x, y)| \leq c_k e^{\lambda \operatorname{Re}\varphi(p(x,y),x,y)}(x^2-1)^{-1/4}\,. \tag{3.28}$$

Since $\operatorname{Re}\varphi(p(x,y),x,y) = \frac{1}{2}(\operatorname{arcosh} x - x\sqrt{x^2-1})$ and

$$x\sqrt{x^2-1} - \operatorname{arcosh} x \geq \delta\sqrt{x^2-1} \quad \text{if } x^2-1 > \delta, 0 < \delta < 1\,, \tag{3.29}$$

we obtain from (3.27)–(3.29) the estimate

$$|F(\lambda, x, y)| \leq c\lambda^{-1/2}\exp(-\tfrac{1}{2}\lambda\delta\sqrt{x^2-1})(x^2-1)^{-1/4}\,. \tag{3.30}$$

Now, Theorem 6 follows from (3.30) and (3.19).

To prove Corollary 3, it is sufficient to obtain the estimate

$$|F(\lambda, x, y)| \leq c\lambda^{-1/3}\exp(-c\lambda^{1/3}(x^2-1)^{1/2}) \quad \text{if } x^2 > 1 + \lambda^{-2/3+\varepsilon},\ \varepsilon > 0,\ a_2 \leq \sqrt{\lambda}y \leq A\,. \tag{3.31}$$

If $x^2 > 1+\delta$, (3.31) follows from (3.30). Let now $\lambda^{-2/3+\varepsilon} < x^2 - 1 < \delta$, $\varepsilon > 0$. Then we can apply Theorem 5. Using the asymptotics of the Airy function and the properties (1.13)–(1.15), (1.17), (1.18), we obtain the asymptotics (3.27) with (3.28). Hence we have again (3.30) with $\delta = \lambda^{-2/3+\varepsilon}$, and (3.31) follows.

References

[1] M. Fedoryuk, *The Saddle-Point Method*, Nauka, Moscow 1977 (in Russian).
[2] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer, 1983.
[3] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. 23, 1959.
[4] G. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 1966.